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# A bicharacteristic formulation of the ideal MHD equations

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**Abstract.** On a characteristic surface $\Omega$ of a hyperbolic system of first-order equations in multi-dimensions $(\mathbf{x}, t)$, there exits a compatibility condition which is in the form of a transport equation along a bicharacteristic on $\Omega$. This result can be interpreted also as a transport equation along rays of the wavefront $\Omega_t$ in $\mathbf{x}$-space associated with $\Omega$. For a system of quasi-linear equations, the ray equations (which has two distinct parts) and the transport equation form a coupled system of underdetermined equations. As an example of this bicharacteristic formulation, we consider two-dimensional unsteady flow of an ideal magnetohydrodynamics gas with a plane aligned magnetic field. For any mode of propagation in this two-dimensional flow, there are three ray equations: two for the spatial coordinates $x$ and $y$ and one for the ray diffraction. In spite of little longer calculations, the final four equations (three ray equations and one transport equation) for the fast magneto-acoustic wave are simple and elegant and cannot be derived in these simple forms by use of a computer program like REDUCE.

## 1. Introduction

The ideal magentohydrodynamic (MHD) equations for an unsteady compressible medium in multi-dimensions form a beautiful system of equations. But the system is quite complex and sustains anisotropic waves. The system in three-space dimensions has eight modes of propagation: two fast waves, two Alfven waves, two slow waves and two waves convected with the fluid, we shall explain the last two in Sec. 2. Though the system is hyperbolic in general, there are situations when strict hyperbolicity breaks down since the characteristic velocities coalesce (two of them become equal). There is a vast literature in journals and books (e.g. Biskamp 1997; Courant and Hilbert 1962; Jeffrey 1966; Kulikovskiy and Lyubimov 1965) on this subject. The basic theory of MHD equations is about 60 year old and may be treated as a classical. Courant and Hilbert (1962) describes and also gives references to many classical work, especially on the geometry of characteristic and ray surfaces. However, there appears to be no analysis of bicharacteristic formulation in a more general form, i.e. the derivation of the ray velocity, diffraction (i.e. the rate of rotation) of the rays and transport equation along a bicharacteristic curve taking full nonlinear equations without making any perturbations. We concentrate in this paper on the bicharacteristic formulation of all modes of propagation as long as they do not coalesce with any other mode. We also discuss only the unsteady planar magnetic field-aligned MHD

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flow in two-space dimensions in $(x, y)$-plane (with $z$-component of the fluid velocity $q_z = 0$ and that of the magnetic field $B_z = 0$). Though this makes the calculation a bit simpler, the expressions are still quite long. A more general case will make the expressions bigger. One may use a program like REDUCE, Computer Algebra System, to do these calculations. However, REDUCE would not recognise special relations of various terms and the expressions will be unduly long and simple results of this paper will be lost. The characteristics for steady two-dimensional MHD flow were obtained by Kogan (1960). The classical work on this aspect is that of Sears (1960) (see also Webb et al. 2005a, b). However, our aim in this paper is to discuss characteristics of the unsteady MHD flow. Webb et al. (2005a) derived equations for linear, non-WKB (Wentzel-Kramer-Brillouin) waves in a non-uniform background plasma flow and it turns out that the Cauchy-characteristic manifolds for the non-WKB linear waves correspond to the dispersion equation for the nonlinear waves in WKB limit. In our work, we take the full MHD equation without any perturbation. We shall comment on this later in Secs 2 and 3 after we have derived the wave velocities.

This work was inspired by the work of Poedts and De Sterck and their collaborators (De Sterck (1999), De Sterck et al. (1998, 1999), De Sterck and Poedts (1999a,b)), where a successful attempt has been made to numerically simulate experimentally observed dimple in the bow shock geometry formed by the interaction of magnetically dominated solar wind with Earth's magnetosphere. While going through their characteristic analysis, it was felt that a full bicharacteristic analysis based on the theory of a hyperbolic system may be worth. Present work is a result of this feeling.

In the derivation of the characteristic partial differential equation for the system of MHD equations in Sec. 2, we have followed the usual formula for a hyperbolic system. However, for this system, one can use the Lagrangian approach of Webb and Zank (2007) (see Appendix). In this approach, the Eulerian position of the fluid element $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$, where $\mathbf{x}_0$ is the Lagrangian fluid label, satisfies a system of second-order wave equations. An interesting point is that the characteristic equation of this second-order system give the usual MHD characteristic equation or the dispersion relation for the Alfven wave mode, and the fast and slow magneto-acoustic modes.

In Sec. 2, we present the MHD equations in two-space dimensions and basic properties of the waves. We shall also present a full discussion of the two waves convected with the fluid and arguments for considering a general procedure for discussion of the bicharacteristic formulation. Then in Sec. 3, we present bicharacteristic formulation of a general quasi-linear system of $n$ equations in $m + 1$ independent variables $(\mathbf{x}, t)$, $\mathbf{x} = (x_1, x_2, \ldots x_m)$ for a mode of propagation corresponding to a simple real characteristic velocity $c$. Such a general presentation in more than two independent variables is valuable for future analysis of the MHD equations in three-space dimensions (or their extensions). In this section we derive the ray equations (the ray velocity is more commonly known as phase velocity in MHD) and the transport equation along a ray (or the bicharacteristic curve in space-time). For these derivations, we need not take a hyperbolic system but only assumption we require is that the characteristic velocity $c$ is simple and real. Sections 4, 5 and 6 are devoted to the derivation of the components of the ray velocity, the diffraction equation and the transport equation along rays for the fast magneto-acoustic wave. In the Appendix, we give a list of symbols and some results used in this paper.

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## 2. MHD equations for a polytropic gas with special reference to a two-dimensional flow with plane aligned magnetic field

Let $\rho$ be the mass density, $\mathbf{q}$ the velocity and $p$ the pressure of a plasma and $\mathbf{B}$ the magnetic field vector. We assume that the plasma behaves as an ideal and polytropic gas. For an ideal polytropic gas $p = A(S)\rho^{\gamma}$, where $S$ is entropy and $\gamma$ is a constant (Courant and Friedrichs 1948, Sec. 3). Here $\rho$ and $p$ are independent since $S$ is not assumed to be a constant. For such a gas the total energy density $e$ per unit mass of the plasma is given by

$$e = \frac{p}{(\gamma-1)\rho} + \frac{1}{2}q^2 + \frac{1}{\rho}B^2, \quad B^2 = |\mathbf{B}|^2. \tag{2.1}$$

The unit of magnetic field is so chosen that the magnetic permeability $\mu = 1$.

The ideal MHD equations (with dissipative fluxes set equal to zero) in conservation form (with source terms on the right-hand side) are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{q}\\ \mathbf{B}\\ e\end{bmatrix} + \nabla\begin{bmatrix}\rho\mathbf{q}\\ \rho(\mathbf{qq}) + \mathbf{I}(p+\frac{1}{2}B^2) - (\mathbf{BB})\\ (\mathbf{qB}) - (\mathbf{Bq})\\ (e+p+\frac{1}{2}B^2)\mathbf{q} - \langle\mathbf{q},\mathbf{B}\rangle\mathbf{B}\end{bmatrix} = -\begin{bmatrix}0\\ \mathbf{B}\\ \mathbf{q}\\ \langle\mathbf{q},\mathbf{B}\rangle\end{bmatrix}\langle\nabla,\mathbf{B}\rangle, \tag{2.2}$$

where $\mathbf{I}$ is identity matrix, $\langle,\rangle$ represent inner product and the outer product $\mathbf{qB}$ of two vectors represents the matrix

$$\mathbf{qB} = \begin{bmatrix} q_1B_1 & q_1B_2 & q_1B_3\\ q_2B_1 & q_2B_2 & q_2B_3\\ q_3B_1 & q_3B_2 & q_3B_3\end{bmatrix}. \tag{2.3}$$

The equations (2.2) have to be supplemented by the divergence free condition $\langle\nabla,\mathbf{B}\rangle = 0$ as an initial condition.

The system (2.2) differs from the usual MHD equations by the presence of additional term proportional to $\langle\nabla,\mathbf{B}\rangle$ on the right-hand side. There are many theoretical and computational reasons (De Sterck 1999) which require that the right-hand side (though in reality zero) should not be set equal to zero in equations above. Retaining these terms make the non-relativistic equations (2.2) to be *Galilean invariant* and *symmetrizable* (Godunov 1972). Further, inclusion of the right-hand side term assures that *small* $\langle\nabla,\mathbf{B}\rangle$ *errors are consistently accounted for in a numerically stable way* and do not lead to accumulation of inaccuracies (Powell et al. 1999). The paper of Powell et al. contains a full discussion of MHD equations and also presents numerical results. Some important references on various theoretical and computational aspects of including the additional term proportional to $\langle\nabla,B\rangle$ are Morrison and Greene (1980) and Janhunen (2000).

The system (2.2) contains equations for eight dependent variables: $\rho$, three in $\mathbf{q}$, $p$ and three in $\mathbf{B}$ in three-space dimensions. The system has eight real eigenvalues and these characteristic velocities of a wavefront $\Omega_t$ correspond to two MHD fast waves, two MHD slow waves, two waves convected with the plasma (i.e. the fluid) and two MHD normal waves (Alfven waves). We shall give expressions of the eigenvalues in a particular case, which we consider below.

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In order to simplify the algebraic calculations in this paper, we discuss waves in a two-dimensional flow in $(x, y)$-plane ($q_3 = 0$) and with plane aligned magnetic field ($B_3 = 0$). In this case, the differential form of the conservation equations (2.2) reduce to

$$A\mathbf{u}_t + B^{(1)}\mathbf{u}_x + B^{(2)}\mathbf{u}_y = 0, \tag{2.4}$$

where

$$\mathbf{u} = (\rho, q_1, q_2, B_1, B_2, p)^T, \ A = I = \textit{identity matrix} \tag{2.5}$$

and

$$B^{(1)} = \begin{bmatrix} q_1 & \rho & 0 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 & \frac{B_2}{\rho} & \frac{1}{\rho} \\ 0 & 0 & q_1 & 0 & -\frac{B_1}{\rho} & 0 \\ 0 & 0 & 0 & q_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & q_1 & 0 \\ 0 & a^2\rho & 0 & 0 & 0 & q_1 \end{bmatrix}, \quad B^{(2)} = \begin{bmatrix} q_2 & 0 & \rho & 0 & 0 & 0 \\ 0 & q_2 & 0 & -\frac{B_2}{\rho} & 0 & 0 \\ 0 & 0 & q_2 & \frac{B_1}{\rho} & 0 & \frac{1}{\rho} \\ 0 & -B_2 & B_1 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_2 & 0 \\ 0 & 0 & a^2\rho & 0 & 0 & q_2 \end{bmatrix}. \tag{2.6}$$

The eigenvalues of the system (2.4) are the roots of the six degree equation in $c$

$$\det\,[n_1 B^{(1)} + n_2 B^{(2)} - cI] = 0, \ |\mathbf{n}| = 1. \tag{2.7}$$

A little long calculation reduces this equation to

$$s^2\left\{s^4 - (a^2 + a_A^2)s^2 + a^2 a_{An}^2\right\} = 0, \ s = n_1 q_1 + n_2 q_2 - c, \tag{2.8}$$

where

$$a^2 = \frac{\gamma p}{\rho}, \ a_A^2 = \frac{B^2}{\rho} = \frac{B_1^2 + B_2^2}{\rho}, \ a_{An}^2 = \frac{B_N^2}{\rho} = \frac{(n_1 B_1 + n_2 B_2)^2}{\rho}. \tag{2.9}$$

Here $a$ is the usual sound speed of a polytropic gas and $a_A$ is the Alfven wave speed. The six eigenvalues are

$$c_{1,2} = q_n \pm a_{fn}, \ c_{3,4} = q_n \pm a_{sn}, \ c_5 = c_6 = q_n, \tag{2.10}$$

where

$$q_n = n_1 q_1 + n_2 q_2, \tag{2.11}$$

and

$$a_{fn}^2 = \frac{1}{2}\left[a^2 + a_A^2 + \left\{(a^2 + a_A^2)^2 - 4a^2 a_{An}^2\right\}^{1/2}\right], \tag{2.12}$$

$$a_{sn}^2 = \frac{1}{2}\left[a^2 + a_A^2 - \left\{(a^2 + a_A^2)^2 - 4a^2 a_{An}^2\right\}^{1/2}\right]. \tag{2.13}$$

Here $a_{fn}$ and $a_{sn}$ are fast and slow magneto-acoustic wave speeds. Thus, a two-dimensional plane aligned MHD flow of a compressible fluid has six anisotropic waves. Forward-facing fast wave $c_1$, slow wave $c_3$; backward-facing fast wave $c_2$, slow wave $c_4$ and two waves $c_5$, $c_6$ convected with the fluid velocity $\mathbf{q}$. Let us define tangential magnetic speed $a_{At}$ by

$$a_{At} := \frac{B_T}{\rho} := \frac{n_1 B_2 - n_2 B_1}{\rho}, \tag{2.14}$$

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then

$$a_A^2 = a_{An}^2 + a_{At}^2, \quad a_{An} = \frac{n_1 B_1 + n_2 B_2}{\sqrt{\rho}}, \quad a_{At} = \frac{n_1 B_2 - n_2 B_1}{\sqrt{\rho}}. \tag{2.15}$$

We now prove a well-known result.

THEOREM 2.1. *In general*

$$a_{fn} \geqslant max\ \{a, a_{An}\} \geqslant\ min\ \{a, a_{An}\} \geqslant a_{sn}. \tag{2.16}$$

*Equalities in this relation are true only if* $a_{At} = 0$. *For* $a_{At} = 0$,

$$a_{fn} = a \quad when \quad a > a_{An}, \tag{2.17}$$

$$a_{fn} = a_{An} \quad when \quad a_{An} > a \tag{2.18}$$

*and similar result for* $a_{sn}$.

*Proof.* Using (2.15), we get

$$\left(a^2 + a_A^2\right)^2 - 4a^2 a_{An}^2 = \left(a^2 - a_{An}^2\right)^2 + 2a_{At}^2\left(a^2 + a_{An}^2\right) + a_{At}^4$$

$$\geqslant \left(a^2 - a_{An}^2\right)^2 \quad \text{(equality if } a_{At} = 0\text{)}.$$

Hence, when $a > a_{An}$, from (2.12)

$$a_{fn}^2 \geqslant \frac{1}{2}\left[\left(a^2 + a_A^2\right) + \left(a^2 - a_{An}^2\right)\right]$$

$$= a^2 + \frac{1}{2}\left(a_A^2 - a_{An}^2\right) \geqslant a^2. \tag{2.19}$$

Similarly, when $a < a_{An} < a_A$

$$a_{fn}^2 \geqslant \frac{1}{2}\left[\left(a^2 + a_A^2\right) + \left(a_{An}^2 - a^2\right)\right] = \frac{1}{2}\left(a_A^2 + a_{An}^2\right)$$

$$= \frac{1}{2}\left(2a_{An}^2 + a_{At}^2\right) \geqslant a_{An}^2. \tag{2.20}$$

Equality in (2.19) and (2.20) follow when $a_{At} = 0$. Hence the first inequality in (2.16) and the (2.17)–(2.18) follow. Similarly, we can prove the result *min* $\{a, a_{An}\} \geqslant a_{sn}$ in (2.16).

Let us discuss the main spirit of this paper for a simple convective modes $c_5$ and $c_6$ in (2.10), which in terms of the phase function $\varphi$ satisfying

$$c = -\frac{\varphi_t}{|\nabla\varphi|}, \quad \mathbf{n} = \frac{\nabla\varphi}{|\nabla\varphi|} \tag{2.21}$$

gives the characteristic or eikonal equation

$$F := \varphi_t + q_1\varphi_x + q_2\varphi_y = 0. \tag{2.22}$$

This is one of the simplest Hamilton–Jacobi equations from which we can easily write the Hamilton canonical equation for $\varphi_t = -\omega$, $\mathbf{k} = \nabla\varphi$ and from which we can can derive the ray equations

$$\frac{Dx}{Dt} = q_1, \qquad \frac{Dy}{Dt} = q_2, \tag{2.23}$$

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$$\frac{Dn_1}{Dt} = n_2\left(n_1\frac{\partial q_1}{\partial\lambda} + n_2\frac{\partial q_2}{\partial\lambda}\right) \text{ or } \frac{D\theta}{Dt} = -\left(n_1\frac{\partial q_1}{\partial\lambda} + n_2\frac{\partial q_2}{\partial\lambda}\right), \tag{2.24}$$

where $q_1$ and $q_2$ are the components of the ray velocity and $\theta$ is the angle which ray makes with the $x$-axis i.e. $n_1 = \cos\theta$, $n_2 = \sin\theta$, $D/Dt$ is the convective derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_1\frac{\partial}{\partial x} + q_2\frac{\partial}{\partial y} \tag{2.25}$$

and

$$\frac{\partial}{\partial\lambda} = n_1\frac{\partial}{\partial y} - n_2\frac{\partial}{\partial x}. \tag{2.26}$$

Corresponding to the double eigenvalue $c = q_n = n_2q_1 + n_2q_2$, the two left eigenvectors $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$ of the system (2.4), given by $\mathbf{l}(n_1B^{(1)} + n_2B^{(2)} - cA) = 0$, are

$$\mathbf{l}^{(1)} = (-a^2, 0, 0, 0, 0, 1),\ \mathbf{l}^{(2)} = (0, 0, 0, n_1, n_2, 0). \tag{2.27}$$

Pre-multiplying (2.4) by $\mathbf{l}^{(1)}$ and $\mathbf{l}^{(2)}$, we get two independent compatibility conditions

$$\frac{Dp}{Dt} - a^2\frac{D\rho}{Dt} = 0 \text{ or } \frac{DS}{Dt} = 0 \tag{2.28}$$

and

$$n_1\frac{DB_1}{Dt} + n_2\frac{dB_2}{Dt} = B_2\frac{\partial q_1}{\partial\lambda} - B_1\frac{\partial q_2}{\partial\lambda}. \tag{2.29}$$

Using the expressions $B_N = (n_1B_1 + n_2B_2)$ and $B_T = n_1B_2 - n_2B_N$, we can write (2.29) as

$$\frac{DB_N}{Dt} = (B_2 - n_1B_T)\frac{\partial q_1}{\partial\lambda} - (B_1 + n_1B_T)\frac{\partial q_2}{\partial\lambda}. \tag{2.30}$$

Equations (2.23), (2.24), (2.28) and (2.30) represent the bicharacteristic formulation of the two convective waves mentioned in the first paragraph of the introduction. Let $\Omega_t$ be a curve convected with the fluid and let the unit normal to this curve be denoted by $(n_1, n_2)$. Then $\Omega_t$ is a wavefront moving with the fluid. The ray velocity vector corresponding to $\Omega_t$ is $(q_1, q_2)$ and $\partial/\partial\lambda$ represents a tangential derivative i.e. spatial rate of change along $\Omega_t$. In a non-uniform motion of the fluid, the time rate of change of the normal direction is given by (2.24) giving a diffraction of rays. Equation (2.28) shows that the entropy $S$ on $\Omega_t$ is convected with the fluid and remains constant along rays (2.23) – a result very well known. The equation (2.30) is a very interesting result. It shows that the normal component $B_N$ of the magnetic field is also convected with the fluid along $\Omega_t$ but it decays or amplifies along a ray depending on a combination of the gradients of $q_1$ and $q_2$ along the wavefront $\Omega_t$. Note that the coefficients of $\partial q_1/\partial\lambda$ and $\partial q_2/\partial\lambda$ in (2.30) are linear combinations of $B_1$ and $B_2$. These result are true not only for the two-dimensional flow with plane aligned magnetic field but also for the more general two-dimensional flow, where we get the same equations (2.22)–(2.30).

We derived the wave velocities (2.10) without making any perturbation procedure on a given state (either constant state or space and time-dependent state). The usual WKB approximation for small-amplitude nonlinear theory, leads to a dispersion relation which is the same as the dispersion relation for the linear equations. The results are valid as long as the linear rays do not deviate much from the exact nonlinear rays. But even in a small-amplitude nonlinear theory a significant

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diffraction of rays takes place over longer distances or in the regions of high gradients (Prasad 2000, chapters 4 and 6 in Prasad 2001). This leads to a significant change in the geometry of a weakly nonlinear wavefront and the amplitude distribution on the nonlinear wavefront. One of the very remarkable results is that the linear caustic may be resolved and the wavefront may take a shape without folds but with special singularity *kink* (called shock–shock by Whitham 1974). We shall point out this result, which is probably not known in MHD, again in the next section. This physically realistic and remarkable effect in a genuinely nonlinear characteristic field is seen through a very special perturbation scheme used in (Prasad 2000, chapter 4 in Prasad 2001), which captures the first-order wave amplitude in the characteristic equation, which is missed in the formal perturbation scheme by Choquet-Bruhat (1969). We usually call our theory, obtained by this special perturbation scheme, as 'weakly nonlinear ray theory' (WNLRT).

In the derivation of equations (2.22)–(2.29), we have used the MHD equations with full nonlinearity. This captures the ray diffraction terms on the right-hand side of (2.24) with full nonlinearity. In the usual WKB limit or Choquet-Bruhat (1969) approach, the right-hand side of (2.24) will be replaced by the quantities in the base state on which perturbation has been made. This will miss the nonlinear diffraction of the rays. But no nonlinear effects will appear in the convective waves even in our WNLRT since the convective modes are not genuinely nonlinear. A simple check for it would be to first find out the right eigenvectors $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$ of the matrix $n_1B^{(1)}+n_2B^{(2)}-q_nI$. These are $\mathbf{r}^{(1)} = [1,0,0,0,0,0]^T$ and $\mathbf{r}^{(2)} = [0,0,0,n_1,n_2,0]^T$. Then in terms of characteristic variables $w_1$ and $w_2$, the small amplitude perturbation in the dependent variable $\mathbf{u}$ in (2.5) over constant base state $\mathbf{u_0}$ is given approximately by

$$\mathbf{u} = \mathbf{u}_0 + \epsilon\{\mathbf{r}^{(1)}w_1 + \mathbf{r}^{(2)}w_2\}, \tag{2.31}$$

where $\epsilon$ is small, giving $q_1 = q_{10}$ and $q_2 = q_{20}$ respectively. Therefore, in WNLRT the operator $D/Dt$ and the right-hand sides of (2.23), (2.24), (2.29) and (2.30) become independent of $w_1$ and $w_2$. Thus nonlinear effects do not appear in the convective modes even in WNLRT.

We have derived the ray equations (2.23) and (2.24) from the equation (2.22), which is itself obtained by a very simple procedure of taking a factor, namely $\varphi_t + q_1\varphi_z + q_2\varphi_y$, of the characteristic determinant, i.e. $det(B^{(1)}\varphi_x + B^{(2)}\varphi_y + A\varphi_t)$ and equating it to zero. We can surely do it for the other modes, say the fast MHD wave. This will amount to considering the characteristic equation

$$(\varphi_t + \mathbf{q}.\nabla\varphi)^4 - (a^2 + a_A^2)(\varphi_t + \mathbf{q}.\nabla\varphi)^2|\nabla\varphi|^2 + a^2\langle\mathbf{B}, \nabla\varphi\rangle^2|\nabla\varphi|^2 = 0 \tag{2.32}$$

as done by Webb et al. (2005a), equation (4.20). For the forward-facing fast wave, this would lead to dealing with the eikonal equation

$$\varphi_t + q_n + a_{fn}(\nabla\varphi, a, a_A, B_1, B_2) = 0. \tag{2.33}$$

This Hamilton–Jacobi equation would give correct ray equations but let us note that $a_{fn}$ contains two square roots and complicated terms containing $\rho, p, B_1, B_2$ and $\nabla\varphi$. Hence writing the Hamilton canonical equations leading to ray equations and the ray diffraction equations would not be very much simpler than the general procedure used by us in the Secs 4 and 5. In the derivation of the expression (2.12)

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for $a_{fn}(\rho, B_1, B_2, p, \mathbf{n})$ we have used $n_1^2 + n_2^2 = 1$ in many steps, and in using the eikonal equation in the form

$$\omega := -\varphi_t = \Omega := \langle \mathbf{k}.\mathbf{q} \rangle + |\mathbf{k}| a_{fn}(\rho, B_1, B_2, p, \mathbf{n}), \tag{2.34}$$

one needs to be extremely careful to restore all terms involving $\mathbf{k} = \nabla\varphi$ in it. Only then we can use the ray equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{q} + \mathbf{n} a_{fn} + (\mathbf{I} - \mathbf{nn}) \cdot \nabla_{\mathbf{n}} a_{fn} \tag{2.35}$$

of Webb et al. (2005a). There is a typographical error in the equation (4.27) in the paper of Webb et al., where the term $1/k$ should not be there.

We have completed discussion of the two convective modes out of the six modes listed in (2.10). The bicharacteristic formulation (2.23)–(2.30) leads to a good understanding of the waves convected with the fluid. Taking up all the four remaining modes will be labour intensive, will be similar to just one of them and will make the paper unduly long. Hence in this and the Secs 4–6, we shall concentrate on the forward-facing fast magneto-acoustic wave assuming the eigenvalue

$$c_1 = q_n + a_{fn} \text{ to be simple i.e, } B_T = n_1 B_2 - n_2 B_1 \neq 0. \tag{2.36}$$

We first calculate the right and left eigenvectors corresponding to $c_1$.

**Calculation of right eigenvector**. The right eigenvector $\mathbf{r} = [r_1, r_2, r_3, r_4, r_5, r_6]^T$ is given by $(n_1 B^{(1)} + n_2 B^{(2)} - c_1 I) \cdot \mathbf{r} = 0$ i.e. the six linear equations:

$$\begin{aligned}
&-a_{fn} r_1 + n_1 \rho r_2 + n_2 \rho r_3 = 0,\\
&-a_{fn} r_1 - (n_2 B_2/\rho) r_4 + (n_1 B_2/\rho) r_5 + (n_1/\rho) r_6 = 0,\\
&-a_{fn} r_3 + (n_2 B_1/\rho) r_4 - (n_1 B_1/\rho) r_5 + (n_2/\rho) r_6 = 0,\\
&-n_2 B_2 r_2 + n_2 B_1 r_3 - a_{fn} r_4 = 0,\\
&n_1 B_2 r_2 - n_1 B_1 r_3 - a_{fn} r_4 = 0,\\
&n_1 \rho a^2 r_2 + n_2 \rho a^2 r_3 - a_{fn} r_6 = 0.
\end{aligned} \tag{2.37}$$

If we choose the component $r_1$, say $r_1 = \rho/a_{fn}$, the solution of (2.37) is unique. However, depending on different ways we handle these equations, we get different expressions for $r_2, r_3, r_4, r_5$ and $r_6$. But, it is possible to prove (sometimes using many steps and using the equation $a_{fn}^4 - (a^2 + a_A^2) a_{fn}^2 + a^2 a_{An}^2 = 0$ satisfied by $a_{fn}$, see (2.8)) that all the expressions are equivalent. Now we write a simpler and elegant form or $\mathbf{r}$:

$$\begin{aligned}
&r_1 = \frac{\rho}{a_{fn}},\ r_6 = \frac{a^2 \rho}{a_{fn}},\\
&r_2 = -\frac{1}{B_T}\left(B_2 - n_2 \frac{a^2}{a_{fn}^2} B_N\right),\ r_3 = -\frac{1}{B_T}\left(B_1 - n_1 \frac{a^2}{a_{fn}^2} B_N\right),\\
&r_4 = -n_2 \frac{1}{a_{fn} B_T}\left(B^2 - \frac{a^2}{a_{fn}^2} B_N^2\right),\quad r_5 = n_1 \frac{1}{a_{fn} B_T}\left(B^2 - \frac{a^2}{a_{fn}^2} B_N^2\right).
\end{aligned} \tag{2.38}$$

Note that $r_1$ and $r_6$ are written in one line.

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**Calculation of the left eigenvector**. Similarly, we can calculate the left eigenvector $\mathbf{l} = [l_1, l_2, l_3, l_4, l_5, l_6]$ corresponding to the fast mode $c_1$ by solving

$$\mathbf{l}\left(n_1 B^{(1)} + n_2 B^{(2)} - c_1 I\right) = 0. \tag{2.39}$$

We give here a simple form of $\mathbf{l}$:

$$\begin{aligned}
&l_1 = 0\,, \quad l_6 = \frac{1}{\rho a_{fn}},\\
&l_2 = \frac{1}{B_T}\left(B_2 - n_2\frac{a^2}{a_{fn}^2}B_N\right),\ l_3 = -\frac{1}{B_T}\left(B_1 - n_1\frac{a^2}{a_{fn}^2}B_N\right),\\
&l_4 = -n_2\frac{1}{\rho a_{fn}B_T}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right),\ l_5 = n_1\frac{1}{\rho a_{fn}B_T}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right).
\end{aligned} \tag{2.40}$$

In this and subsequent sections we shall freely use the relations (2.9) and (2.15). We calculate here an expression for $\mathbf{l}A\mathbf{r}$:

$$\begin{aligned}
K &:= \mathbf{l}A\mathbf{r}\\
&= \frac{a^2}{a_{fn}^2} + \frac{1}{a_{At}^2}\left\{a_A^2 - \frac{2a^2a_{An}^2}{a_{fn}^2} + \frac{a^4a_{An}^2}{a_{fn}^4}\right\} + \frac{1}{a_{At}^2a_{fn}^2}\left(a_A^2 - \frac{a^2a_{An}^2}{a_{fn}^2}\right)^2\\
&= \frac{2a^2}{a_{fn}^2} + \frac{1}{a_{At}^2}\left\{a_A^2 - \frac{a^2(a_{An}^2 + a_{At}^2)}{a_{fn}^2} - \frac{a^2a_{An}^2}{a_{fn}^2} + \frac{a^4a_{An}^2}{a_{fn}^4}\right\} + \frac{1}{a_{At}^2a_{fn}^2}\left(a_A^2 - \frac{a^2a_{An}^2}{a_{fn}^2}\right)^2.
\end{aligned} \tag{2.41}$$

THEOREM 2.2. *$K$ has another equivalent expression*

$$\begin{aligned}
K &:= \mathbf{l}A\mathbf{r}\\
&= \frac{2a^2}{a_{fn}^2} + \frac{2}{a_{At}^2a_{fn}^2}\left(a_A^2 - \frac{a^2}{a_{fn}^2}a_{An}^2\right)^2.
\end{aligned} \tag{2.42}$$

*Proof*. The expression in the curly bracket in (2.41) is

$$G := \frac{1}{a_{fn}^4}\left\{a_A^2a_{fn}^4 - a^2a_A^2a_{fn}^2 - a^2a_{An}^2a_{fn}^2 + a^4a_{An}^2\right\}.$$

After adding an subtracting two terms

$$\begin{aligned}
G &= \frac{1}{a_{fn}^4}\left\{a_A^2a_{fn}^4 - a^2a_A^2a_{fn}^2 - a_A^4a_{fn}^2 + a^2a_A^2a_{An}^2 + a_A^4a_{fn}^2\right.\\
&\quad \left. - a^2a_A^2a_{An}^2 - (1/a_{fn}^2)\left(a^2a_{An}^2a_{fn}^4 - a^4a_{An}^2a_{fn}^2\right)\right\},\\
&= \frac{1}{a_{fn}^4}\left\{a_A^2\left(a_{fn}^4 - (a^2 + a_A^2)a_{fn}^2 + a^2a_{An}^2\right) + a_A^4a_{fn}^2\right.\\
&\quad \left. - a^2a_A^2a_{An}^2 - (1/a_{fn}^2)a^2a_{An}^2\left(a_{fn}^4 - a^2a_{fn}^2\right)\right\}.
\end{aligned} \tag{2.43}$$

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The first term in the curly bracket vanishes due to the equation satisfied by $a_{fn}$. For the same reason $a_{fn}^4 - a^2 a_{fn}^2 = a_A^2 a_{fn}^2 - a^2 a_{An}^2$. Hence

$$G = \frac{1}{a_{fn}^6}\left\{a_A^4 a_{fn}^4 - 2a^2 a_A^2 a_{An}^2 a_{fn}^2 + a^4 a_{An}^4\right\}$$

$$= \frac{1}{a_{fn}^6}\left(a_A^2 a_{fn}^2 - a^2 a_{An}^2\right)^2. \tag{2.44}$$

Substituting this expression for the curly bracket in (2.41), we find that $K$ in (2.41) reduces to $K$ in (2.42) showing the equivalence. Before we take up the bicharacteristic formulation of the fast wave, we present a systematic derivation of ray and diffraction equations and the compatibility condition for a general hyperbolic system.

## 3. Bicharacteristic formulation for a general system of quasi-linear equations

Consider a quasi-linear system of $n$ first-order partial differential equations in $m+1$ independent variables $(\mathbf{x}, t) = (x_1, x_2, \ldots, x_m, t)$ :

$$A(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_t + B^{(\alpha)}(\mathbf{u}, \mathbf{x}, t)\mathbf{u}_{x_\alpha} + \mathbf{C}(\mathbf{u}, \mathbf{x}, t) = 0, \tag{3.1}$$

where $\mathbf{u} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B^{(\alpha)} \in \mathbb{R}^{n\times n}$ and $\mathbf{C} \in \mathbb{R}^n$. We follow a summation convention in which a repeated suffix $\alpha, \beta, \gamma$ will imply sum over the range $1, 2, \ldots, m$ and a repeated suffix $i, j, k$ over the range $1, 2, \ldots, n$.

Let us assume that the system (3.1) has a real and simple characteristic velocity $c$ satisfying the characteristic equation

$$\det[n_\alpha B^{(\alpha)} - cA] = 0, \quad |\mathbf{n}| = 1. \tag{3.2}$$

Let $\mathbf{l}$ and $\mathbf{r}$ be the left and right eigenvectors corresponding to this eigenvalue i.e.

$$\mathbf{l}(n_\alpha B^{(\alpha)} - cA) = 0, \quad (n_\alpha B^{(\alpha)} - cA)\mathbf{r} = 0. \tag{3.3}$$

Then the components $\chi_\alpha$ of bicharacteristic or the ray velocity $\boldsymbol{\chi}$ corresponding to the eigenvalue $c$ are given by the lemma on bicharacteristic directions (Courant and Hilbert 1962)

$$\chi_\alpha = \frac{\mathbf{l}B^{(\alpha)}\mathbf{r}}{\mathbf{l}A\mathbf{r}}. \tag{3.4}$$

It is simple to see the relation

$$c = n_\alpha \chi_\alpha = \frac{\mathbf{l}(n_\alpha B^{(\alpha)})\mathbf{r}}{\mathbf{l}A\mathbf{r}}. \tag{3.5}$$

A characteristic surface $\Omega$ is a $a$ surface in space-time. A wavefront $\Omega_t$ is a projection on $\mathbf{x}$-space of a section of $\Omega$ by $t =$ constant plane, $\mathbf{n}$ is the unit normal of $\Omega_t$ and the eigenvalue $c$ represents the local normal velocity of a wave front $\Omega_t$ in $\mathbf{x}$-space (Prasad 2001, 2007). A ray is a projection on $\mathbf{x}$-space of a bicharacteristic curve in space-time. Since the ideal MHD equations are homogeneous equations of first order, ray velocity and phase velocity are same. Hence in MHD, one uses 'phase velocity' for the ray velocity used here.

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THEOREM 3.1. *The ray equations of the system (3.1) corresponding to the eigenvalue c are given by*

$$\frac{dx_\alpha}{dt} = \frac{\mathbf{l}B^{(\alpha)}\mathbf{r}}{\mathbf{l}A\mathbf{r}} = \chi_\alpha, \tag{3.6}$$

$$\frac{dn_\alpha}{dt} = -\frac{1}{\mathbf{l}A\mathbf{r}}\mathbf{l}\left\{n_\beta\left(n_\gamma\frac{\partial B^{(\gamma)}}{\partial \eta^\alpha_\beta} - c\frac{\partial A}{\partial \eta^\alpha_\beta}\right)\right\}\mathbf{r} = \psi_\alpha, \text{ say} \tag{3.7}$$

*where*

$$\frac{\partial}{\partial \eta^\alpha_\beta} = n_\beta\frac{\partial}{\partial x_\alpha} - n_\alpha\frac{\partial}{\partial x_\beta}. \tag{3.8}$$

*Further, the system (3.1) implies a compatibility condition on the characteristic surface Ω of (3.1) in the form*

$$\mathbf{l}A\frac{d\mathbf{u}}{dt} + \mathbf{l}(B^{(\alpha)} - \chi_\alpha A)\frac{\partial \mathbf{u}}{\partial x_\alpha} + \mathbf{l}C = 0. \tag{3.9}$$

**Note 2.1:** The theorem was first stated by Prasad (1993), see also Sec. 2.4 in (Prasad 2001). A complete proof of the theorem is available in (Prasad 2007).

This theorem forms a basis of the derivation of all results in subsequent sections of this paper. It is worth looking in detail the structure of various terms appearing in the compatibility condition (3.9). We first notice that $d/dt$ represents the time rate of change along a bicharacteristic curve in space-time or time rate of change as we move with the ray velocity $\boldsymbol{\chi}$ in $\mathbf{x}$-space i.e.

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \chi^{(\alpha)}\frac{\partial}{\partial x_\alpha}. \tag{3.10}$$

This operator represents differentiation in a tangential direction on the characteristic surface $\Omega$ in space-time. Each operator

$$\tilde{\partial}_j := s^{(\alpha)}_j\frac{\partial}{\partial x_\alpha} := l_i\left(B^{(\alpha)}_{ij} - \chi_\alpha A_{ij}\right)\frac{\partial}{\partial x_\alpha} \tag{3.11}$$

for $j = 1, 2, \ldots, n$ operating on $u_j$ in the second term in the characteristic compatibility condition (3.9) is a tangential derivative on the characteristic surface $\Omega$. We also notice that

$$n_\alpha s^{(\alpha)}_j = l_i A_{ij}(c - n_\alpha\chi_\alpha) = 0, \text{ for each } j, \tag{3.12}$$

and hence the inner product $\langle \mathbf{n}, \tilde{\partial}_j\rangle$ of $\mathbf{n}$ with the row vector operator $\tilde{\partial}_j$ vanishes showing that $\tilde{\partial}_j$, for each $j$, is tangential derivative not only on $\Omega$ but also on the wavefront $\Omega_t$ in $\mathbf{x}$-space.

Let us introduce a row vector

$$\mathbf{s}^{(\alpha)} = (s^{(\alpha)}_1, s^{(\alpha)}_2, \ldots, s^{(\alpha)}_n) = \mathbf{l}(B^{(\alpha)} - \chi_\alpha A) \tag{3.13}$$

then from (3.4)

$$\langle \mathbf{s}^{(\alpha)}, \mathbf{r}\rangle = s^{(\alpha)}_j r_j = 0, \ \alpha = 1, 2, \ldots, m. \tag{3.14}$$

Let $\mathbf{L}$ be the projection of the gradient $\nabla$ in $\mathbf{x}$ space onto the tangent plane of $\Omega_t$ i.e.

$$\mathbf{L} = \nabla - \mathbf{n}\langle \mathbf{n}, \nabla\rangle, \tag{3.15}$$

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then we can verify that the components of $\mathbf{L}$ are

$$L_{(\alpha)} = n_\beta \frac{\partial}{\partial \eta_\beta^{(\alpha)}}. \tag{3.16}$$

$\mathbf{L}$ has $m$ components. Let $D_1, D_2, \ldots, D_{m-1}$ be the $m-1$ independent components of $\mathbf{L}$. Now we express $\tilde{\partial}_j$ as

$$\tilde{\partial}_j := s_j^{(\alpha)} \frac{\partial}{\partial x_\alpha} = \sum_{p=1}^{m-1} d_{pj} D_p \tag{3.17}$$

and introduce

$$\mathbf{d}_p = (d_{p1}, d_{p2}, \ldots, d_{pn}), \ p = 1, 2, \ldots, m-1. \tag{3.18}$$

Now the characteristic compatibility condition (3.9) becomes

$$\mathbf{l} A \frac{d\mathbf{u}}{dt} + \sum_{p=1}^{m-1} \mathbf{d}_p D_p \mathbf{u} = 0 \tag{3.19}$$

or written explicitly

$$l_{ij} \ A_{ij} \frac{du_j}{dt} + \sum_{p=1}^{m-1} (d_{p1} D_p u_1 + d_{p2} D_p u_2 + \ldots + d_{pn} D_p u_n) + \mathbf{l} C = 0, \tag{3.20}$$

which, unlike (3.9), contains only the $m-1$ independent tangential derivatives $D_1, \ldots, D_{m-1}$ on $\Omega_t$.

We note that $|\mathbf{n}| = 1$ so that only $m-1$ of the components of $\mathbf{n}$ are independent and hence we can show that only $m-1$ of the equations (3.7) are independent. For a linear system, the equations (3.6) and (3.7) decouple from the compatibility condition (3.9) or (3.20). In this case the right-hand side of (3.6) and (3.7) are functions of $\mathbf{x}$ and $t$ only and hence the rays can be traced independent of a solution of the system (3.1). However, for a quasi-linear system such as the MHD equations, $A$ and $B^{(\alpha)}$ depend also on $\mathbf{u}$ and hence the expression $\psi_\alpha$ in (3.7) contains variables $\mathbf{x}, t, \mathbf{u}$ and $\partial \mathbf{u}/\partial \eta_\beta^\alpha$. In this case the system (3.6)–(3.9) in $2m + n - 1$ quantities ($m$ components of $\mathbf{x}$, $m-1$ independent components of $\mathbf{n}$ and $n$ components of $\mathbf{u}$) is an under determined unless $n = 1$ ($n = 1$ corresponds to a single partial differential equation). However, if a small-amplitude high frequency approximation is made leading to the weakly nonlinear ray theory or shock ray theory (Prasad 2000, Prasad 2001), it is possible to express the $n$-dependent variables in $\mathbf{u}$ in terms of a scalar small-amplitude $w$ on the wavefront $\Omega_t$ and we get $2m$ equations for $\mathbf{x}$, $m-1$ component of $\mathbf{n}$ and $w$. The high-frequency approximation, which we refer here, is not the usual perturbation procedure used by Choquet-Bruhat (1969) but a special perturbation scheme which captures the small amplitude $w$ on the wavefront $\Omega_t$ in the characteristic (or eikonal) equation, when the characteristic field is genuinely nonlinear. In this case, we get a beautiful theory to trace the successive positions of a nonlinear wavefront or a shock front in which the geometry of the rays and the amplitude $w$ on the front mutually interact leading to nonlinear transverse diffraction of rays, amplitude modulation and formation of curves of discontinuities (i.e. kinks) on the front $\Omega_t$. A formulation of the propagation of fronts on this line for the MHD equations will be very valuable but the expressions would be very big.

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For a system of equations (3.1) in two space variables, $m = 2$ and there is only one independent tangential derive on the curve $\Omega_t$ in $(x, y)$-plane. The unit normal of $\Omega_t$ is $(n_1, n_2)$. We choose $D_1$ to be the derivative $\partial/\partial\lambda$ (used in Prasad, 2001) defined below

$$D_1 \equiv \frac{\partial}{\partial\lambda} := n_1 \frac{\partial}{\partial y} - n_2 \frac{\partial}{\partial x}. \tag{3.21}$$

Then the two components $L_1$ and $L_2$ of $\mathbf{L}$ in (3.16) are

$$L_1 = n_2 \frac{\partial}{\partial \eta_2^1} = -n_2 \frac{\partial}{\partial\lambda},\ L_2 = n_1 \frac{\partial}{\partial\lambda}. \tag{3.22}$$

In this two-dimensional case, the only operator which appear in (3.20) are the bicharacteristic derivative $d/dt$ (tangential to characteristic surface $\Omega$) and $\partial/\partial\lambda$ (tangential to the wavefront $\Omega_t$).

The ray equations and the compatibility conditions along rays have played important role in numerical solutions of hyperbolic systems. In one-space dimension, the characteristics in $(x, t)$-plane were used not only in development of numerical methods in gas dynamics (Shapiro 1954) but also in proving existence and uniqueness of the solution of a Cauchy problem (Courant and Lax 1949). In two- and three-space dimensions, they have been used to solve numerically MHD equations. The most relevant references are the work of Nakagawa (1981) and Hu and Wu (1984). A systematic procedure for a general hyperbolic system was developed first by Butler (1960) and a recent paper on application of bicharacteristic method is by Arun et al. (2009), where many older references are available. This series of papers was initiated by the work of Reddy et al. (1982) and Lukacova et al. (2000). They are based on the nature of the derivatives in (3.20) which we have just discussed. The aim is to derive an evolution operator for the hyperbolic system (3.1) using all $n$ compatibility conditions. Fix a point $P(\mathbf{x}, t_n + \Delta t)$ in space-time and consider a characteristic conoid (of one of the $n$ families say $j$th family) with $P$ as the apex. Let $Q_j = Q_j(\mathbf{x}(t_n), t_n)$ and $\tilde{Q}_j = \tilde{Q}_j(\mathbf{x}(\tau), \tau)$ be respectively the foot prints of a bicharacteristic of the $j_{th}$ family on the planes $t = t_n$ and $t = \tau \in (t_n, t_{n+1})$. Then we integrate the transport equation (3.20) along the respective bicharacteristics and take an integral average over the wavefronts $\Omega_t^{(j)} (j = 1, 2, \ldots, n)$ to derive the evolution operator. In the integral average the tangential derivatives $D_\alpha$ play important role. The procedure becomes quite complicated compared to the approach of Nakagawa (1981), where one uses bicharacteristics only in coordinate planes $(t, x_M)$, $M = 1, 2, \ldots, m_m$ in space-time (i.e. the projected characteristics).

In the derivation of the ray equations and the compatibility condition, we have just followed one mode of propagation by assuming that the system (3.1) has a real and simple eigenvalue, without saying any thing about the hyperbolic (or otherwise) nature of the system. Had we followed a mode with uniform multiplicity (as we did for the convective waves of multiplicity two) we would have got a number of compatibility conditions equal to the multiplicity under consideration. In MHD, except for the convective waves discussed in detail in the Sec. 2, all other modes have uniform multiplicity *one* when $B_T = n_1 B_2 - n_2 B_1 \neq 0$. For the MHD equations, we shall follow now only the forward-facing fast wave mode. Corresponding expressions for other modes can also be found.

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## 4. Ray or phase velocity components for the fast magneto-acoustic wave

In this section, we shall calculate the two ray velocity components $\chi_1$ and $\chi_2$ by using the result (3.4) and using the expressions of $B^{(1)}$, $B^{(2)}$, $\mathbf{l}$ and $\mathbf{r}$ given in (2.6), (2.38) and (2.41). Some calculations and rearrangements lead to

$$\mathbf{l}B^{(1)}\mathbf{r} = \frac{q_1 a^2}{a_{fn}^2} + \frac{2a^2}{a_{fn}B_T}\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right) + \frac{q_1}{B_T^2}\left\{\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right)^2 + \left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right)^2\right\} + \frac{1}{B_T^2}\left[\frac{n_1}{\rho a_{fn}}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)\left(B^2 - \frac{n_2 a^2}{a_{fn}^2}B_N^2\right) + \frac{n_2^2 q_1}{\rho a_{fn}^2}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)^2 + \frac{n_1}{\rho a_{fn}}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)\left\{\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right) + \frac{n_1 q_1}{a_{fn}}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)\right\}\right], \tag{4.1}$$

where we have used $B_1^2 + B_2^2 = B^2$ and $n_1^2 + n_2^2 = 1$. We find in the above that the coefficient of $q_1$ become $\mathbf{l}A\mathbf{r}$ given in (2.41) and other terms also combine nicely to give

$$\mathbf{l}B^{(1)}\mathbf{r} = (\mathbf{l}A\mathbf{r})q_1 + \frac{2a^2}{a_{fn}B_T}\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right) + \frac{2n_1}{\rho a_{fn}B_T^2}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)^2.$$

Adding $-2n_1 a^2/a_{fn}$ in the second term and subtracting it from the third term on the right-hand side and using

$$-n_1 + \frac{B_2}{B_T} = \frac{B_2 - n_1(n_1 B_2 - n_2 B_1)}{B_T} = \frac{n_2^2 B_2 + n_1 n_2 B_1}{B_T} = \frac{n_2 B_N}{B_T},$$

we get

$$\mathbf{l}B^{(1)}\mathbf{r} = (\mathbf{l}A\mathbf{r})q_1 + \frac{2n_2 a^2 B_N}{a_{fn}B_T}\left(1 - \frac{a^2}{a_{fn}^2}\right) + n_1 a_{fn}\left[\frac{2a^2}{a_{fn}^2} + \frac{2}{a_{fn}^2 B_T^2}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)\right]. \tag{4.2}$$

The right hand side of (4.2) is a very interesting expression. The coefficient of $q_1$, namely $\mathbf{l}A\mathbf{r}$ is given by (2.41). The last term in the square bracket is the same as the expression in (2.42) and both are equal according to the theorem 2.2. The expression for $\mathbf{l}B^{(1)}\mathbf{r}$ finally takes an elegant and simple form:

$$\mathbf{l}B^{(1)}\mathbf{r} = K(q_1 + n_1 a_{fn}) + \frac{2n_2 a^2 B_N}{a_{fn}B_T}\left(1 - \frac{a^2}{a_{fn}^2}\right). \tag{4.3}$$

The expression for the first component of the ray velocity, namely $\chi_1$, given by (3.6) is

$$\chi_1 = q_1 + n_1 a_{fn} + n_2 P, \tag{4.4}$$

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where

$$P = \frac{2a^2 B_N}{K a_{fn} B_T}\left(1 - \frac{a^2}{a_{fn}^2}\right), \tag{4.5}$$

which may take a positive or negative value depending on the values of $B_N$ and $B_T$. Exactly in the same way, we may derive an expression for $\chi_2$

$$\chi_2 = q_2 + n_2 a_{fn} - n_1 P. \tag{4.6}$$

Equation (4.6) is the final expression for $\chi_2 = \mathbf{l}A\mathbf{r}/(\mathbf{l}A\mathbf{r})$ but the intermediate steps are similar to those in the derivation of $\chi_1$. In order to help a reader to derive (4.6), we give here two intermediate steps:

$$\begin{aligned}\mathbf{l}B^{(2)}\mathbf{r} = {} & \frac{a^2}{a_{fn}^2} q_2 - \frac{2a^2}{a_{fn} B_T}\left(B_1 - n_1 \frac{a^2}{a_{fn}^2} B_N\right) + \frac{1}{B_T^2}\left[q_2\left(B_2 - \frac{n_2 a^2}{a_{fn}^2} B_N\right)^2\right. \\ & + q_2\left(B_1 - \frac{n_1 a^2}{a_{fn}^2} B_N\right)^2 + \frac{1}{\rho a_{fn}}\left(B^2 - \frac{a^2}{a_{fn}^2} B_N^2\right)\left\{n_2 B_2\left(B_2 - \frac{n_2 a^2}{a_{fn}^2} B_N\right)\right. \\ & \left.\left. + n_2 B_1\left(B_1 - \frac{n_1 a^2}{a_{fn}^2} B_N\right)\right\} + \frac{n_2}{\rho a_{fn}}\left(B^2 - \frac{a^2}{a_{fn}^2} B_N\right)^2 \right. \\ & \left. + \frac{q_2(n_1^2 + n_2^2 = 1)}{\rho a_{fn}^2}\left(B^2 - \frac{a^2}{a_{fn}^2} B_N^2\right)^2\right] \end{aligned} \tag{4.7}$$

$$= q_2(\mathbf{l}A\mathbf{r}) - \frac{2a^2}{B_T a_{fn}}\left(B_1 - n_1 \frac{a^2}{a_{fn}^2} B_N\right) + \frac{2n_2}{\rho a_{fn} B_T^2}\left(B^2 - \frac{a^2}{a_{fn}^2} B_N^2\right)^2. \tag{4.8}$$

In the step (4.7) for $\mathbf{l}B^{(2)}\mathbf{r}$, the term are less simplified than those in (4.1) for $\mathbf{l}B^{(1)}\mathbf{r}$. This may be helpful as to show how (4.1) has been derived.

Let us look carefully the ray velocity components $\chi_1$ and $\chi_2$. The quantities $q_1 + n_1 a_{fn}$ and $q_2 + n_2 a_{fn}$ are components of a part of the ray velocity $\chi$. This part is recognizable as the usual fast magneto-acoustic mode added to the fluid velocity $\mathbf{q}$. There is an additional part of the ray velocity $(n_2 P, -n_1 P)$, which is tangential to the wavefront. The normal component of the full ray velocity $\chi$ is

$$c_1 = \langle \mathbf{n}, \boldsymbol{\chi} \rangle = n_1 q_1 + n_2 q_2 + a_{fn}, \tag{4.9}$$

and the addition part $(n_2 P, -n_1 P)$ makes no contribution to it.

**Note 4.1.** The expression for this additional part $(n_2 P, -n_1 P)$ of the ray velocity appears to be new, which probably has not been derived so for.

**Note 4.2.** The algebraic calculations to arrive at (4.4)–(4.6) content long expressions. But the final results are remarkably simple and elegant. The result (4.9)

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verifies that our calculations, which have been done with great care, appear to be correct.

## 5. The ray diffraction equation for the fast magneto-acoustic wave

In this section we shall use the general expression (3.7) to deduce the ray diffraction equation for the fast magneto-acoustic wave $c_1 = q_n + a_{fn}$. The calculations are a little more involved compared to those in the last section since differentiation of the state variables are required. Differentiation will also have to be done if we start with the characteristic equation (2.33) and calculations will not be too simple since two square roots appear in the expression for $a_{fn}$. Finally, the results have to be expressed in terms of the derivatives of the original state variables in (2.5).

In two-space dimensions for a planar flow all operators $\partial/\partial\eta_\beta^\alpha$ are zero except the following two, both being expressible in terms of only one tangential derivative $\partial/\partial\lambda$ defined in (3.21):

$$\frac{\partial}{\partial\eta_2^1} = n_2\frac{\partial}{\partial x} - n_1\frac{\partial}{\partial y} = -\frac{\partial}{\partial\lambda},\ \frac{\partial}{\partial\eta_1^2} = -\frac{\partial}{\partial\eta_2^1} = \frac{\partial}{\partial\lambda}. \tag{5.1}$$

Equation (3.7) reduces to two equations, one for $n_1$ and another for $n_2$. However, they are equivalent and in terms of an angle $\theta$ defined by

$$n_1 = \cos\theta,\ n_2 = \sin\theta \tag{5.2}$$

and both give the same ray diffraction equation

$$\frac{d\theta}{dt} = -\frac{1}{\mathbf{lAr}}\mathbf{l}\left(n_1\frac{\partial B^{(1)}}{\partial\lambda} + n_2\frac{\partial B^{(2)}}{\partial\lambda}\right)\mathbf{r}. \tag{5.3}$$

Note that since $a^2\rho = \gamma p$, the right-hand side of (5.3) contains only derivatives of the state variables in (2.5). Using the differential relation $d(a^2\rho) = \gamma dp$, from (2.6) and (2.39), we get

$$(dB^{(1)})\mathbf{r} = \begin{bmatrix} (dq_1)\dfrac{\rho}{a_{fn}} + (d\rho)\dfrac{1}{B_T}\left(B_2 - \dfrac{n_2a^2}{a_{fn}^2}B_N\right) \\ (dq_1)\dfrac{1}{B_T}\left(B_2 - \dfrac{n_2a^2}{a_{fn}^2}B_N\right) + \left(\dfrac{dB_2}{\rho} - \dfrac{B_2}{\rho^2}(d\rho)\right)\dfrac{n_1}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N\right) - \dfrac{(d\rho)a^2}{\rho a_{fn}} \\ -(dq_1)\dfrac{1}{B_T}\left(B_1 - \dfrac{n_1a^2}{a_{fn}^2}B_N\right) + \left(-\dfrac{dB_1}{\rho} + \dfrac{B_1(d\rho)}{\rho^2}\right)\dfrac{n_1}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) \\ -(dq_1)\dfrac{n_2}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) \\ \dfrac{n_1dq_1}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) + \dfrac{dB_2}{B_T}\left(B_2 - \dfrac{n_2a^2}{a_{fn}^2}B_N\right) + \dfrac{dB_1}{B_T}\left(B_1 - \dfrac{n_1a^2}{a_{fn}^2}B_N\right) \\ \dfrac{a^2\rho dq_1}{a_{fn}} + \dfrac{\gamma dp}{B_T}\left(B_2 - \dfrac{n_2a^2}{a_{fn}^2}B_N\right) \end{bmatrix}. \tag{5.4}$$

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Pre-multiplying above by $\mathbf{l}$, we get a big expression. We collect the terms containing the differentials $dq_1, d\rho, dp, dB_1$ and $dB_2$ and write the result below after combining suitable terms

$$\begin{aligned}\mathbf{l}(dB^{(1)})\mathbf{r} = (dq_1)&\left[\frac{1}{B_T^2}\left\{(B_1^2+B_2^2)-2\frac{a^2}{a_{fn}^2}B_N^2+\frac{a^4}{a_{fn}^4}B_N^2\right\}\right.\\&\left.+\frac{(n_1^2+n_2^2)}{\rho a_{fn}^2 B_T^2}(B^2-\frac{a^2}{a_{fn}^2}B_N^2)^2+\frac{a^2}{a_{fn}^2}\right]-(d\rho)\left[\frac{n_1}{\rho^2 a_{fn}B_T^2}\left\{B_2\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)\right.\right.\\&\left.\left.+B_1\left(B_1-\frac{n_1a^2}{a_{fn}^2}B_N\right)\right\}\left(B^2-\frac{a^2}{a_{fn}^2}B_N^2\right)+\frac{a^2}{\rho a_{fn}B_T}\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)\right]\\&+(\gamma dp)\frac{1}{\rho a_{fn}B_T}\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)+\left\{(dB_1)\left(B_1-\frac{n_1a^2}{a_{fn}^2}B_N\right)\right.\\&\left.+(dB_2)\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)\right\}\frac{2n_1}{\rho a_{fn}B_T^2}\left(B^2-\frac{a^2}{a_{fn}^2}B_N^2\right).\end{aligned}\tag{5.5}$$

Note that the coefficient of $dq_1$ is the first expression for $\mathbf{l}A\mathbf{r}$ i.e. $K$ in (2.41). The final expression, which we write, is

$$\begin{aligned}&\mathbf{l}(dB^{(1)})\mathbf{r}\\&=Kdq_1-\frac{n_1}{\rho^2 a_{fn}B_T^2}\left(B^2-\frac{a^2}{a_{fn}^2}B_N^2\right)^2 d\rho+\frac{1}{\rho a_{fn}B_T}\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)(\gamma dp-a^2d\rho)\\&+\frac{2n_1}{\rho a_{fn}B_T^2}\left(B^2-\frac{a^2}{a_{fn}^2}B_N^2\right)\left\{\left(B_1-\frac{n_1a^2}{a_{fn}^2}B_N\right)dB_1+\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)dB_2\right\}.\end{aligned}\tag{5.6}$$

Similarly, we can show that

$$\begin{aligned}&\mathbf{l}(dB^{(2)})\mathbf{r}\\&=Kdq_2-\frac{n_2}{\rho^2 a_{fn}B_T^2}\left(B^2-\frac{a^2}{a_{fn}^2}B_N^2\right)^2 d\rho-\frac{1}{\rho a_{fn}B_T}\left(B_1-\frac{n_1a^2}{a_{fn}^2}B_N\right)(\gamma dp-a^2d\rho)\\&+\frac{2n_2}{\rho a_{fn}B_T^2}\left(B^2-\frac{a^2}{a_{fn}^2}B_N^2\right)\left\{\left(B_1-\frac{n_1a^2}{a_{fn}^2}B_N\right)(dB_1)+\left(B_2-\frac{n_2a^2}{a_{fn}^2}B_N\right)(dB_2)\right\}.\end{aligned}\tag{5.7}$$

We have given the expressions for $\mathbf{l}\left(dB^{(1)}\right)\mathbf{r}$ and $\mathbf{l}\left(dB^{(2)}\right)\mathbf{r}$ side by side. However, we produce here just two intermediate steps to help a reader to derive it, if he

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desires:

$$(dB^{(2)})\mathbf{r}$$

$$= \begin{bmatrix} (dq_2)\dfrac{\rho}{a_{fn}} - (d\rho)\dfrac{1}{B_T}\left(B_1 - \dfrac{n_1 a^2}{a_{fn}^2}B_N\right) \\ (dq_2)\dfrac{1}{B_T}\left(B_2 - \dfrac{n_2 a^2}{a_{fn}^2}B_N\right) + \left(\dfrac{dB_2}{\rho} - \dfrac{B_2}{\rho^2}d\rho\right)\dfrac{n_2}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) \\ -(dq_2)\dfrac{1}{B_T}\left(B_1 - \dfrac{n_1 a^2}{a_{fn}^2}B_N\right) - \left(\dfrac{dB_1}{\rho} - \dfrac{B_1}{\rho^2}d\rho\right)\dfrac{n_2}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) - \dfrac{(d\rho)a^2}{\rho a_{fn}} \\ -\dfrac{n_2 dq_2}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) - \dfrac{dB_2}{B_T}\left(B_2 - \dfrac{n_2 a^2}{a_{fn}^2}B_N\right) - \dfrac{dB_1}{B_T}\left(B_1 - \dfrac{n_1 a^2}{a_{fn}^2}B_N\right) \\ (dq_2)\dfrac{n_1}{a_{fn}B_T}\left(B^2 - \dfrac{a^2}{a_{fn}^2}B_N^2\right) \\ \dfrac{a^2\rho dq_2}{a_{fn}} - \dfrac{\gamma dp}{B_T}\left(B_1 - \dfrac{n_1 a^2}{a_{fn}^2}B_N\right) \end{bmatrix} \tag{5.8}$$

and

$$\begin{aligned} \mathbf{l}(dB^{(2)})\mathbf{r} = (dq_2)\Bigg[&\frac{1}{B_T^2}\left\{B_1^2 + B_2^2 - 2\frac{a^2}{a_{fn}^2}(B_1 n_1 + B_2 n_2)B_N + \frac{a^4}{a_{fn}^4}B_N^2\right\} \\ &+ \frac{(n_1^2 + n_2^2)}{\rho a_{fn}^2 B_T^2}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)^2 + \frac{a^2}{a_{fn}^2}\Bigg] - (d\rho)\Bigg[\frac{n_2}{\rho^2 a_{fn}B_T^2}\Bigg\{B_2\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right) \\ &+ B_1\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right)\Bigg\}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right) - \frac{a^2}{\rho a_{fn}B_T}\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}\right)\Bigg] \\ &- (\gamma dp)\frac{1}{\rho a_{fn}B_T}\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}\right) + \Bigg\{(dB_1)\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right) \\ &+ (dB_2)\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right)\Bigg\}\frac{2n_2}{\rho a_{fn}B_T^2}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right). \end{aligned} \tag{5.9}$$

When we use (5.6) and (5.7) to evaluate $\mathbf{l}(n_1 dB^{(1)} + n_2 dB^{(2)})\mathbf{r}$, terms simplify and there appears a term $(a^2/\rho a_{fn}B_T)(\gamma dp - a^2 d\rho)$ whose coefficient simply becomes $n_1 B_2 - n_2 B_1 = B_T$. Using this in (5.3), we finally get

$$\begin{aligned} \frac{d\theta}{dt} = \frac{1}{\rho K a_{fn}B_T^2}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)\Bigg[&\frac{1}{\rho}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right)\frac{\partial\rho}{\partial\lambda} - 2\Bigg\{\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right)\frac{\partial B_1}{\partial\lambda} \\ &+ \left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right)\frac{\partial B_2}{\partial\lambda}\Bigg\}\Bigg] - \left(n_1\frac{\partial q_1}{\partial\lambda} + n_2\frac{\partial q_2}{\partial\lambda}\right) - \frac{1}{\rho K a_{fn}}\left(\gamma\frac{\partial p}{\partial\lambda} - a^2\frac{\partial\rho}{\partial\lambda}\right), \end{aligned} \tag{5.10}$$

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which is a relatively a simple expression after such long calculations. This gives the time rate of change of direction of a ray due to gradients of $\rho, \mathbf{q}, p$ and $\mathbf{B}$ in a direction tangential to the wavefront propagating with the fast magneto-acoustic speed.

## 6. The characteristic compatibility condition for the fast magneto-acoustic wave

The form of the characteristic compatibility condition (3.9), for a general system (3.1), finally reduces to (3.20). In two-space dimension, there is only one independent tangential derivative $D_1$ and we choose $D_1$ to be $\partial/\partial\lambda$ introduced in (3.22). In this case, we can write (3.20) with $A = I$ in a simpler form. In terms of the tangential derivative $\partial/\partial\lambda$ and normal derivative $\partial/\partial n = n_1(\partial/\partial x) + n_2(\partial/\partial y)$ the derivative $\partial/\partial x$ and $\partial/\partial y$ are given by

$$\frac{\partial}{\partial x} = -n_2\frac{\partial}{\partial\lambda} + n_1\frac{\partial}{\partial n},\ \frac{\partial}{\partial y} = n_1\frac{\partial}{\partial\lambda} + n_2\frac{\partial}{\partial n}. \tag{6.1}$$

Substituting in (3.9) with $\alpha = 2$ and $\mathbf{C} = 0$ and using

$$\mathbf{l}\left\{n_1\left(B^{(1)} - \chi_1 A\right) + n_2\left(B^{(2)} - \chi_2 A\right)\right\} = 0 \tag{6.2}$$

and (3.5), we get (note $A = I$)

$$\mathbf{l}\frac{d\mathbf{u}}{dt} + \left\{-n_2\mathbf{l}\left(B^{(1)} - \chi_1 A\right) + n_1\mathbf{l}\left(B^{(2)} - \chi_2 A\right)\right\}\frac{\partial\mathbf{u}}{\partial\lambda} = 0. \tag{6.3}$$

We shall use this simple expression to the MHD equations (2.4).

After some calculations and using (4.5), we find following expressions for $s_j^{(\alpha)} = l_i(B_{ij}^{(\alpha)} - \chi_\alpha A_{ij})$, defined in (3.11)

$$s_1^{(1)} = l_i\left(B_{i1}^{(1)} - \chi_1 A_{i1}\right) = 0, \tag{6.4}$$

$$s_2^{(1)} = \frac{a^2}{a_{fn}^2} + \frac{n_1 B_2}{\rho a_{fn} B_T}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right) - \frac{n_1 a_{fn} + n_2 P}{B_T}\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right), \tag{6.5}$$

$$s_3^{(1)} = -\frac{n_1 B_1}{\rho a_{fn} B_T}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right) + \frac{n_1 a_{fn} + n_2 P}{B_T}\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right), \tag{6.6}$$

$$s_4^{(1)} = \left(\frac{n_1 n_2}{\rho B_T} + \frac{n_2^2 P}{\rho a_{fn} B_T}\right)\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right), \tag{6.7}$$

$$s_5^{(1)} = \left(\frac{n_2^2}{\rho B_T} - \frac{n_1 n_2 P}{\rho a_{fn} B_T}\right)\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right), \tag{6.8}$$

$$s_6^{(1)} = \frac{1}{\rho B_T}\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right) - \frac{n_1}{\rho} - \frac{n_2 P}{\rho a_{fn}}, \tag{6.9}$$

$$s_1^{(2)} = l_i\left(B_{i1}^{(2)} - \chi_2 A_{i1}\right) = 0, \tag{6.10}$$

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$$s_2^{(2)} = \frac{n_2 B_2}{\rho a_{fn} B_T}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right) - \frac{n_2 a_{fn} - n_1 P}{B_T}\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right), \tag{6.11}$$

$$s_3^{(2)} = \frac{a^2}{a_{fn}^2} - \frac{n_2 B_1}{\rho a_{fn} B_T}\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right) + \frac{n_2 a_{fn} - n_1 P}{B_T}\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right), \tag{6.12}$$

$$s_4^{(2)} = -\left(\frac{n_1^2}{\rho B_T} + \frac{n_1 n_2 P}{a_{fn} B_T}\right)\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right), \tag{6.13}$$

$$s_5^{(2)} = \left(-\frac{n_2 n_1}{\rho B_T} + \frac{n_1^2 P}{\rho a_{fn} B_T}\right)\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right), \tag{6.14}$$

$$s_6^{(2)} = -\frac{1}{\rho B_T}\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right) - \frac{n_2}{\rho} + \frac{n_1 P}{\rho a_{fn}}. \tag{6.15}$$

It is interesting to check that the of above expressions satisfy correct identities $n_1 s_j^{(1)} + n_2 s_j^{(2)} = 0$ i.e. (3.12) for each $j$. This result is obviously true for $j = 1$. Using the above expressions, some calculations lead to

$$n_1 s_2^{(1)} + n_2 s_2^{(2)} = -\frac{B_2}{a_{fn}^3 B_T}\left\{a_{fn}^4 - \left(a^2 + a_A^2\right) a_{fn}^2 + a^2 a_{An}^2\right\}. \tag{6.16}$$

The right-hand side of (6.16) vanishes since $a_{fn}$ is a zero of the polynomial in $a_{fn}$ in the curly bracket. The expression $a_{fn}^4 - \left(a^2 + a_A^2\right) a_{fn}^2 + a^2 a_{An}^2$ appears as a factor for the expression for $n_1 s_3^{(1)} + n_2 s_3^{(2)}$ also. Hence $n_1 s_3^{(1)} + n_2 s_3^{(2)}$ vanishes. $n_1 s_j^{(1)} + n_2 s_j^{(2)}$ for $j = 4$ and 5 vanish due to cancellation of terms. For the last relation we find

$$n_1 s_6^{(1)} + n_2 s_6^{(2)} = -\frac{1}{\rho} + \frac{1}{\rho B_T}(n_1 B_2 - n_2 B_T) = -\frac{1}{\rho} + \frac{B_T}{\rho B_T} = 0. \tag{6.17}$$

We are now ready to calculate the coefficient

$$-n_2\left\{l_i\left(B_{ij}^{(1)} - \chi_1 A_{ij}\right)\right\} + n_1\left\{l_i\left(B_{ij}^{(2)} - \chi_2 A_{ij}\right)\right\} = -n_2 s_j^{(1)} + n_1 s_j^{(2)} = h_j, \text{say} \tag{6.18}$$

of $\partial u_j/\partial\lambda$ in (6.3). We get

$$h_1 = -n_2 s_1^{(1)} + n_2 s_1^{(2)} = 0, \tag{6.19}$$

$$h_2 = -\frac{n_2 a^2}{a_{fn}} + \frac{P}{B_T}\left(B_2 - \frac{n_2 a^2}{a_{fn}^2}B_N\right), \tag{6.20}$$

$$h_3 = \frac{n_1 a^2}{a_{fn}} + \frac{P}{B_T}\left(B_1 - \frac{n_1 a^2}{a_{fn}^2}B_N\right), \tag{6.21}$$

$$h_4 = -\frac{1}{\rho B_T}\left(n_1 + \frac{n_2}{a_{fn}}P\right)\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right), \tag{6.22}$$

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$$h_5 = \frac{1}{\rho B_T}\left(-n_2 + \frac{n_1}{a_{fn}}P\right)\left(B^2 - \frac{a^2}{a_{fn}^2}B_N^2\right), \tag{6.23}$$

$$h_6 = \frac{P}{\rho a_{fn}} - \frac{B_N}{\rho B_T}\left(1 - \frac{a^2}{a_{fn}^2}\right). \tag{6.24}$$

Finally, an explicit form of the compatibility condition (6.3) is

$$\begin{aligned} &l_2\frac{dq_1}{dt} + l_3\frac{dq_2}{dt} + l_4\frac{dB_1}{dt} + l_5\frac{dB_2}{dt} + l_6\frac{dp}{dt} \\ &\quad + h_2\frac{\partial q_1}{\partial\lambda} + h_3\frac{\partial q_2}{\partial\lambda} + h_4\frac{\partial B_1}{\partial\lambda} + h_5\frac{\partial B_2}{\partial\lambda} + h_6\frac{\partial p}{\partial\lambda} = 0, \end{aligned} \tag{6.25}$$

where $l_i$ are given in (2.40) and $h_i$ are given above. The time derivative $d/dt$ along a ray, given by (see (4.4) and (4.6))

$$\frac{dx}{dt} = q_1 + n_1 a_{fn} + n_2 P, \quad \frac{dy}{dt} = q_2 + n_2 a_{fn} - n_1 P \tag{6.26}$$

is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (q_1 + n_1 a_{fn} + n_2 P)\frac{\partial}{\partial x} + (q_2 + n_2 a_{fn} - n_1 P)\frac{\partial}{\partial y}. \tag{6.27}$$

Here $d/dt$ is a tangential derivative on the characteristic surface $\Omega$ in space-time and $\partial/\partial\lambda$ is a tangential derivative on the wavefront $\Omega_t$ in $(x_1, x_2)$-plane and also on the characteristic surface $\Omega$ in space-time.

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## Appendix. List of symbols and frequently used results

$\mathbf{u} = (\rho, \mathbf{q}, \mathbf{B}, P)$, $q = |\mathbf{q}|$, $B = |\mathbf{B}|$,
$\mathbf{n}$ = unit normal to wavefront $\Omega_t$. In two dimensions $\mathbf{n} = (n_1, n_2)$,
$a^2 = \dfrac{\gamma p}{\rho}$, $a_A^2 = \dfrac{B^2}{\rho}$, $a_{An}^2 = \dfrac{B_N^2}{\rho}$, $B_N = \langle \mathbf{n}, \mathbf{B}\rangle$,

$$a_{fn}^2, a_{sn}^2 = \frac{1}{2}\left[a^2 + a_A^2 \pm \left\{\left(a^2 + a_A^2\right)^2 - 4a^2 a_{An}^2\right\}^{1/2}\right],$$

$$a_{fn}^4 - \left(a^2 + a_A^2\right)a_{fn}^2 + a^2 a_{An}^2 = 0.$$

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**The following results are for the characteristic velocity** $c = q_n + a_{fn}$

Left eigenvector $\mathbf{l} = (l_1, l_2, l_3, l_4, l_5, l_6)$,

Right eigenvector $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6)^T$,

Expressions for components of $\mathbf{l}$ and $\mathbf{r}$ are given in (2.38) and (2.40), respectively.

$$K = \mathbf{l}A\mathbf{r} = \frac{2a^2}{a_{fn}^2} + \frac{2}{a_{fn}^2 a_{AT}^2}\left(a_A^2 - \frac{a^2}{a_{fn}^2}a_{An}^2\right),$$

$$P = \frac{2a^2 B_N}{K a_{fn} B_T}\left(1 - \frac{a^2}{a_{fn}^2}\right),$$

$$\chi_1 = q_1 + n_1 a_{fn} + n_2 P, \quad \chi_2 = q_2 + n_2 a_{fn} - n_1 P,$$

$$\mathbf{L} = \nabla - \mathbf{n}\langle \mathbf{n}, \nabla \rangle, \quad \frac{\partial}{\partial \eta_\beta^\alpha} = n_\beta \frac{\partial}{\partial x_\alpha} - n_\alpha \frac{\partial}{\partial x_\beta}.$$

**In two-space dimensions**

$$a_{At}^2 = \frac{B_T^2}{\rho}, \quad B_T = n_1 B_2 - n_2 B_1,$$

$$\frac{\partial}{\partial \eta_1^2} = -\frac{\partial}{\partial \eta_2^1} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1} = \frac{\partial}{\partial \lambda},$$

$$L_1 = -n_2 \frac{\partial}{\partial \lambda}, \quad L_2 = n_1 \frac{\partial}{\partial \lambda}.$$

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