Math. Proc. Camb. Phil. Soc. (2001), **130**, 541
Printed in the United Kingdom © 2001 *Cambridge Philosophical Society*

Colombeau generalized ultradistributions

By S. PILIPOVIĆ

Institute of Mathematics, University of Novi Sad, Trg D. Obrado-vića 4, 21000 *Novi Sad, Yugoslavia.*
e-mail: pilipovic@unsim.ns.ac.yu

and D. SCARPALEZOS

U.F.R. de Mathématiques, Université Paris 7, 2 *place Jussieu, Paris* 5^e, 75005, *France.*
e-mail: scarpa@math.jussieu.fr

(*Received* 22 *November* 1999; *revised* 17 *April* 2000)

Abstract

An algebra of Colombeau generalized ultradistributions in which the corresponding space of ultradistributions is embedded via regularizations and in which the multiplication of ultradifferentiable functions of an appropriate class is the ordinary multiplication is constructed. This enables the development of the local and microlocal analysis in the framework of a new scale of Colombeau algebras. Sufficient conditions for a generalized ultradistribution being an ultradifferentiable function are given.

0. *Introduction*

Colombeau algebra of generalized functions (called here generalized distributions) $\mathscr{G}$, containing Schwartz distributions, is a suitable framework for the analysis of linear problems with singular coefficients and data as well as of non-linear problems with multiplications of singular elements. We refer to monographs [**1**, **4**, **5**, **18**] and the references therein. Also, we refer to papers [**2**, **3**, **6**, **7**, **9**–**14**, **17**, **21**] for the local properties of Colombeau generalized distributions.

The aim of this paper is to establish a theory of Colombeau generalized ultradistribution algebras $\mathscr{G}^{\{M_p,N_p\}}(\Omega)$ and $\mathscr{G}^{(M_p,N_p)}(\Omega)$ which correspond to sequences M_p and N_p satisfying conditions (M·1)*, (M·2) and (M·3)$'$, well known in ultradistribution theory (cf. [**11**, **12**, **19**]). In Theorem 1 we give a positive answer to an important question: Is it possible to construct an algebra of Colombeau ultradistributions, containing ultradistributions, in which the multiplication of ultradifferentiable functions of an appropriate subclass is the usual one?

Colombeau generalized distributions are represented by nets of smooth functions f_ε with polynomial growth with respect to ε, while Colombeau generalized ultradistributions are represented by nets with exponential growth, which depend on associated and growth functions characterizing the space of ultradistributions. In

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

several papers ([**6**, **9**, **15**, **16**, **21**]) various scales of algebras and ideals therein with exponential growth are constructed and used in the analysis of various non-linear problems. Gramchev [**9**] was the first who considered ultradistribution spaces of class n^{sn}, $s > 1$, in the framework of algebras of generalized ultradistributions.

In this paper we give an extension of ultradistribution spaces of class (M_p), resp., $\{M_p\}$ in non-quasianalytic case, into corresponding Colombeau type algebras of generalized ultradistributions by determining algebras $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$ and their ideals $\mathscr{N}^{\{M_p,N_p\}}(\Omega)$, $\mathscr{N}^{(M_p,N_p)}(\Omega)$, respectively. If M_p is given and $N_p = M_p/p!^\rho$, $\rho > 0$, then ultradistributions of (M_p), resp., $\{M_p\}$, class are embedded into $\mathscr{G}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{G}^{(M_p,N_p)}(\Omega)$. In these algebras multiplication of ultradifferentiable functions of class $\{M_p/p!^{1+\rho}\}$, resp., $(M_p/p!^{1+\rho})$, is the usual pointwise multiplication.

Scales of ultradistribution Colombeau algebras can be used for the development of local and microlocal analysis. This is the subject of further investigations. In this paper, we introduce subalgebras which correspond to ultradifferentiable function spaces and in Theorem 2 give conditions for a generalized ultradistribution to be an ultradifferentiable function of an appropriate class.

1. *Notation and notions*

We denote by M_p, $p \in \mathbb{N}_0$ a sequence of positive numbers with $M_0 = 1$ satisfying the assumptions

(M·1) $M_p^2 \leqslant M_{p-1}M_{p+1}$, $p \in \mathbb{N}$,

(M·2) $M_p \leqslant AH^pM_qM_{p-q}$, $p \in \mathbb{N}, q \leqslant p$, for some $A > 0$ and $H > 0$,

(M·3)′ $\sum_{p=1}^{\infty} M_{p-1}/M_p \leqslant \infty$.

We refer to [**11**] and [**12**] for the analysis of these conditions. Also, we assume (cf. [**19**])

(M·1)* $(M_p^*)^2 \leqslant M_{p-1}^*M_{p+1}^*$, $p \in \mathbb{N}$.

Recall,

$$M_0^* = 1, \quad M_p^* = M_p/p!, \; m_p = M_p/M_{p-1}, \; m_p^* = M_p^*/M_{p-1}^*, \quad p \in \mathbb{N}.$$

Clearly (M·1)* implies (M·1).

The associated function $M(t)$ and the growth function $M^*(t)$, $t \in \mathbb{R}$, related to M_p are defined by

$$M(t) = \sup_{p\in\mathbf{N}_0} \ln\frac{t^p}{M_p}, \quad M^*(t) = \sup_{p\in\mathbf{N}_0} \ln\frac{t^p}{M_p^*}, \quad t > 0.$$

We will use the usual convention $M(x) = M(|x|)$, $M^*(x) = M^*(|x|)$, $x \in \mathbb{R}$.

Remark. Condition (M·2) enables a transfer of the theory which is to follow from the one-dimensional to the many-dimensional case. For the sake of simplicity, we will consider the case $n = 1$.

Let Ω be an open set in $\mathbb{R}$. Then $K \subset\subset \Omega$ means that K (or its closure) is a compact subset of Ω. Recall, for $\varphi \in C^\infty(\Omega)$,

$$\|\varphi\|_{K,h,M_p} = \sup_{x\in K,\alpha\in\mathbb{N}_0} \frac{h^\alpha \mid \varphi^{(\alpha)}(x) \mid}{M_\alpha}, \quad h > 0, \quad K \subset\subset \Omega.$$

Denote by $\mathscr{E}^{M_p,h}(K)$ the space of smooth functions on Ω for which the above semi-

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

norm is finite and by $\mathscr{D}^{M_p,h}(K)$ its subspace consisting of smooth functions supported by K. Define

$$\mathscr{E}^{(M_p)}(\Omega) = \operatorname{proj}\lim_{K\to\Omega} \operatorname{proj}\lim_{h\to\infty} \mathscr{E}^{M_p,h}(K) = \operatorname{proj}\lim_{K\to\Omega} \mathscr{E}^{(M_p)}(K),$$
$$\mathscr{E}^{\{M_p\}}(\Omega) = \operatorname{proj}\lim_{K\to\Omega} \operatorname{ind}\lim_{h\to 0} \mathscr{E}^{M_p,h}(K) = \operatorname{proj}\lim_{K\to\Omega} \mathscr{E}^{\{M_p\}}(K),$$
$$\mathscr{D}^{(M_p)}(\Omega) = \operatorname{ind}\lim_{K\to\Omega} \operatorname{proj}\lim_{h\to\infty} \mathscr{D}^{M_p,h}(K) = \operatorname{ind}\lim_{K\to\Omega} \mathscr{D}^{(M_p)}(K),$$
$$\mathscr{D}^{\{M_p\}}(\Omega) = \operatorname{ind}\lim_{K\to\Omega} \operatorname{ind}\lim_{h\to 0} \mathscr{D}^{M_p,h}(K) = \operatorname{ind}\lim_{K\to\Omega} \mathscr{D}^{\{M_p\}}(K).$$

($K \to \Omega$ means that K runs over all compact sets exhausting Ω.)

The strong duals of above spaces are spaces of compactly supported and general Beurling and Roumieu ultradistributions, respectively (cf. [**11**]).

We define spaces of exponentially decreasing ultradifferentiable functions as follows (cf. [**13**, **20**]). Let $b > 0$. The space of smooth functions φ on $\mathbb{R}$ satisfying

$$\sigma_{b,M_p}(\varphi) = \sup_{\alpha,\beta\in\mathbb{N}_0} \int_{\mathbb{R}} \frac{|\,x\,|^\beta}{b^{\alpha+\beta} M_\alpha M_\beta} \mid \varphi^{(\alpha)}(x) \mid dx < \infty,$$

equipped with the topology induced by the norm σ_{b,M_p}, is denoted by $\mathscr{S}^{M_p,b}$. The strong dual of

$$\mathscr{S}^{(M_p)} = \operatorname{proj}\lim_{b\to 0} \mathscr{S}^{M_p,b} \quad \text{resp.,} \quad \mathscr{S}^{\{M_p\}} = \operatorname{ind}\lim_{b\to\infty} \mathscr{S}^{M_p,b}$$

is called the space of tempered ultradistributions of Beurling, resp., Roumieu type.

In particular, we are interested in spaces $\mathscr{S}^{(p^{!1+\rho})}$ and $\mathscr{S}^{\{p^{!1+\rho}\}}$, where $\rho > 0$. In the theorems which are to follow mollifiers are elements of this space.

Definition 1. Let $\phi \in \mathscr{S}^{\{p^{!1+\rho}\}}$ or $\phi \in \mathscr{S}^{(p^{!1+\rho})}$ be such that

$$\int_{\mathbb{R}} \phi(t)dt = 1, \quad \int_{\mathbb{R}} t^n \phi(t)dt = 0, \quad n = 1, 2, \ldots.$$

Then ϕ is called the mollifier; $\phi_\varepsilon = \varepsilon^{-1}\phi(\cdot/\varepsilon)$ is a net of mollifiers.

Example. Take a function $\theta \in \mathscr{D}^{\{p^{!1+\rho}\}}(\mathbb{R})$, resp., $\theta \in \mathscr{D}^{(p^{!1+\rho})}(\mathbb{R})$ equals 1 in a neighbourhood of 0. Then, its Fourier transformation is a mollifier $\phi \in \mathscr{S}^{\{p^{!1+\rho}\}}$, resp., $\phi \in \mathscr{S}^{(p^{!1+\rho})}$.

In the sequel, we will use also a positive sequence N_p satisfying conditions (M·1)*, (M·2) and (M·3)′. The corresponding associated and growth functions are denoted by N and N^*.

Definition 2. The sets of exponentially growth generalized ultradistribution nets and null nets which correspond to the given sequences M_p and N_p are defined, respectively, by

$$\mathscr{E}_{exp}^{\{M_p,N_p\}}(\Omega) = \Bigg\{(f_\varepsilon)_\varepsilon; \quad (\forall K \subset\subset \Omega) \quad (\exists h > 0) \quad (\exists k > 0)$$
$$\sup\left\{\frac{h^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|; \quad x \in K, \quad \alpha \in \mathbb{N}\right\} = \mathscr{O}(e^{N^*(k/\varepsilon)})\Bigg\},$$

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

$$\mathscr{N}^{\{M_p,N_p\}}(\Omega)=\Big\{(f_\varepsilon)_\varepsilon;(\forall K\subset\subset\Omega)(\exists h>0)(\forall k>0)$$
$$\sup\Big\{\frac{h^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|;\quad x\in K,\quad \alpha\in\mathbb{N}\Big\}=\mathscr{O}(e^{-N^*(k/\varepsilon)})\Big\}$$

(Roumieu type spaces).

$$\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)=\Big\{(f_\varepsilon)_\varepsilon;\quad(\forall K\subset\subset\Omega)\quad(\forall h>0)\quad(\exists k>0)$$
$$\sup\Big\{\frac{h^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|;\quad x\in K,\quad \alpha\in\mathbb{N}\Big\}=\mathscr{O}(e^{M^*(k/\varepsilon)})\Big\},$$
$$\mathscr{N}^{(M_p,N_p)}(\Omega)=\Big\{(f_\varepsilon)_\varepsilon;\quad(\forall K\subset\subset\Omega)\quad(\forall h>0)\quad(\forall k>0)$$
$$\sup\Big\{\frac{h^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|;\quad x\in K,\quad \alpha\in\mathbb{N}\Big\}=\mathscr{O}(e^{-M^*(k/\varepsilon)})\Big\}$$

(Beurling type spaces).

Recall that an operator of the form $P(D)=\sum_{k\in\mathbb{N}_0}a_kD^k$ $(D=d/dx)$ is called an ultradifferential operator of class (M_p), respectively of class $\{M_p\}$, if there exist $h>0, B>0$, respectively, such that for every $h>0$ there exists $B>0$ such that

$$|a_k|\leqslant Bh^{|k|}/M_k,\ k\in\mathbb{N}_0. \tag{1}$$

PROPOSITION 1. (i) $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$ *is an algebra under pointwise multiplication and* $\mathscr{N}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{N}^{(M_p,N_p)}(\Omega)$, *is an ideal of* $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$.

(ii) $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$ *is closed under the action of ultradifferential operators of class* $\{M_p\}$, resp., *class* (M_p).

Proof. (i) Let us prove that $\mathscr{N}^{\{M_p,N_p\}}(\Omega)$ is an ideal in $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$. The other assertions of (i) can be proved in a similar way.

Let $(f_\varepsilon)_\varepsilon\in\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, $(r_\varepsilon)_\varepsilon\in\mathscr{N}_{exp}^{\{M_p,N_p\}}(\Omega)$ and $K_0\subset\subset\Omega$ be given. We know

$$(\exists h_0>0)\quad(\exists k_0>0)$$
$$\sup\Big\{\frac{h_0^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|;\quad x\in K_0,\quad \alpha\in\mathbb{N}\Big\}=\mathscr{O}(e^{N^*(k_0/\varepsilon)}) \tag{2}$$
$$(\exists h_1>0)\quad(\forall k_1>0)$$
$$\sup\Big\{\frac{h_1^\alpha}{M_\alpha}|r_\varepsilon^{(\alpha)}(x)|;\quad x\in K_0,\quad \alpha\in\mathbb{N}\Big\}=\mathscr{O}(e^{N^*(k_1/\varepsilon)}).$$

Let $0<t<\frac{1}{2}\min\{h_0,h_1\}$. We will use inequality $M_pM_q\leqslant M_{p+q},\ p,q\in\mathbb{N}$, which follows from (M·1).

For every $s>0$, $k\in\mathbb{N}$ and $x\in K_0$,

$$e^{N^*(s/\varepsilon)}\frac{t^k}{M_k}|(r_\varepsilon f_\varepsilon)^{(k)}(x)|\leqslant e^{N^*(s/\varepsilon)}\sum_{i=0}^{k}\binom{k}{i}t^{k-i}\frac{|r_\varepsilon^{(k-i)}(x)|}{M_{k-i}}t^i\frac{|f_\varepsilon^{(i)}(x)|}{M_i}$$
$$\leqslant e^{N^*(k_1/\varepsilon)}||r_\varepsilon||_{K_0,2h_1,M_p}e^{-N^*(k_0/\varepsilon)}||f_\varepsilon||_{K_0,2h_0,M_p}<\infty,$$

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

where we choose $k_1 > 0$ such that

$$N^*(k_0/\varepsilon) + N^*(s/\varepsilon) \leqslant N^*(k_1/\varepsilon),\ \varepsilon < 1.$$

This implies the assertion.

(ii) Let us prove that $P(D)f_\varepsilon \in \mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, where $P(D)$ is of class $\{M_p\}$ (cf. (1)) and f_ε satisfies (2). Let $t, h > 0$ such that $2H(t+h) < h_0$, where H is a constant from condition (M·2). Let $s > 0$, $k \in \mathbb{N}$ and $x \in K_0$. With the constant B corresponding to h in (1), we have

$$\begin{aligned} e^{N^*(s/\varepsilon)}|P(D)f_\varepsilon(x)| &\leqslant e^{N^*(s/\varepsilon)} \sum_{i=0}^{\infty} |a_i| \frac{t^k}{M_k} |f_\varepsilon^{(k+i)}(x)| \\ &\leqslant B e^{N^*(s/\varepsilon)} \sum_{i=0}^{\infty} \frac{h^i t^k}{M_i M_k} |f_\varepsilon^{(k+i)}(x)| \\ &\leqslant AB e^{N^*(s/\varepsilon)} \sum_{i=0}^{\infty} 2^{-i} \frac{(2H(h+t))^{k+i}}{M_{k+i}} |f_\varepsilon^{(k+i)}(x)| \\ &\leqslant AB e^{N^*(s/\varepsilon)} ||f_\varepsilon||_{K_0,h_0,M_p}. \end{aligned}$$

Colombeau generalized ultradistribution algebras $\mathscr{G}^{\{M_p,N_p\}}(\Omega)$ and $\mathscr{G}^{(M_p,N_p)}(\Omega)$ are constructed in the same way as the standard Colombeau algebra:

$$\begin{aligned} \mathscr{G}^{\{M_p,N_p\}}(\Omega) &= \mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)/\mathscr{N}^{\{M_p,N_p\}}(\Omega), \\ \mathscr{G}^{(M_p,N_p)}(\Omega) &= \mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)/\mathscr{N}^{(M_p,N_p)}(\Omega). \end{aligned}$$

Their elements are denoted by $[f_\varepsilon], \ldots$ or just by $f, \ldots$.

These algebras have similar properties to $\mathscr{G}(\Omega)$. As a direct consequence of Proposition 1 we have that $\mathscr{G}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{G}^{(M_p,N_p)}(\Omega)$, are closed under the action of ultradifferential operators of $\{M_p\}$, resp., (M_p), class.

Our construction involves a new construction of generalized complex numbers, elements of $\bar{\mathbb{C}}^{N_p}$:

$$\begin{aligned} \mathbb{C}_{\exp}^{N_p} &= \{(f_\varepsilon)_\varepsilon;\quad (\exists k > 0)\quad (|f_\varepsilon| = \mathcal{O}(e^{N^*(k/\varepsilon)}))\} \\ \mathcal{N}_0^{N_p} &= \{(f_\varepsilon)_\varepsilon;\quad (\forall k > 0)\quad (|f_\varepsilon| = \mathcal{O}(e^{-N^*(k/\varepsilon)}))\}. \\ \bar{\mathbb{C}}^{N_p} &= \mathbb{C}_{\exp}^{N_p}/\mathcal{N}_0^{N_p}. \end{aligned}$$

Similarly, with real nets, one defines $\bar{\mathbb{R}}^{N_p}$. Note, $\bar{\mathbb{C}}^{N_p}$ and $\bar{\mathbb{R}}^{N_p}$ are rings not fields. They are subalgebras of $\mathscr{G}^{\{M_p,N_p\}}(\Omega)$ and $\mathscr{G}^{(M_p,N_p)}(\Omega)$, for every sequence M_p.

Denote by $\mathscr{E}_{\mathrm{mod}}(\Omega)$ and $\mathscr{N}(\Omega)$ Colombeau algebra of moderate nets and Colombeau algebra of null nets (cf. [**4**]). We have

$$\begin{aligned} \mathscr{E}_{\mathrm{mod}}(\Omega) \subset \mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega),\quad \mathscr{N}^{\{M_p,N_p\}}(\Omega) \subset \mathscr{N}(\Omega), \\ \mathscr{E}_{\mathrm{mod}}(\Omega) \subset \mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega),\quad \mathscr{N}^{(M_p,N_p)}(\Omega) \subset \mathscr{N}(\Omega), \\ \mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega) \subset \mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega),\quad \mathscr{N}^{(M_p,N_p)}(\Omega) \subset \mathscr{N}^{\{M_p,N_p\}}(\Omega). \end{aligned}$$

Remark. One can show that there exist elements of $\mathscr{N}(\Omega)$ which determine different elements of $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$ and $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$, respectively. We postpone the discussion of relations between various Colombeau distribution and ultradistributuion spaces to another paper.

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

As in the case of conventional Colombeau generalized functions, one can prove that $\Omega \to \mathscr{G}^{\{M_p,N_p\}}(\Omega)$, resp., $\Omega \to \mathscr{G}^{(M_p,N_p)}(\Omega)$, constitutes a sheaf which is fine but not flabby (cf. [**8**, **12**] for the definitions and the proofs of these properties in ultradistribution spaces). Thus, we have the notion of support in the new sense.

Definition 3. Let $[f_\varepsilon]$ be an element of $\mathscr{G}^{\{M_p,N_p\}}(\Omega)$, resp., $\mathscr{G}^{(M_p,N_p)}(\Omega)$. Then $\operatorname{supp}_{\{M_p,N_p\}}[f_\varepsilon]$ is the complement of the union of all open sets where it is represented by an element of $\mathscr{N}^{\{M_p,N_p\}}(\Omega)$, resp., $\operatorname{supp}_{(M_p,N_p)}[f_\varepsilon]$ is the complement of the union of all open sets where it is represented by an element of $\mathscr{N}^{(M_p,N_p)}(\Omega)$.

Let us give an asymptotic interpretation of the given construction which enables a simple introduction of topologies in algebras defined above.

Let

$$a_k(\varepsilon) = e^{N^*(k/\varepsilon)}, \quad a_k^-(\varepsilon) = e^{-N^*(k/\varepsilon)}, \quad \varepsilon \in (0,1), \quad k \in \mathbb{N}.$$

These sequences of functions constitute asymptotic scales in the sense that

$$a_k(\varepsilon) = \mathcal{O}(a_{k+1}(\varepsilon)), \quad a_{k+1}^-(\varepsilon) = \mathcal{O}(a_k^-(\varepsilon)), \quad k \in \mathbb{N},$$
$$(\forall n \in \mathbb{N}) \quad (\forall m \in \mathbb{N}) \quad (\exists N \in \mathbb{N})$$
$$a_n(\varepsilon)a_m(\varepsilon) = \mathcal{O}(a_N(\varepsilon)), \quad a_N^-(\varepsilon) = \mathcal{O}(a_n^-(\varepsilon)a_m^-(\varepsilon)).$$

This follows from the inequality equivalent to (M·2):

$$2N^*(\rho) \leqslant N^*(h\rho) + \log A, \quad \rho > 0, \quad \text{for some } A > 0,\ h > 1, \tag{3}$$

which implies (with $m \leqslant n$)

$$N^*(m/\varepsilon) + N^*(n/\varepsilon) \leqslant 2N^*(n/\varepsilon) \leqslant N^*(N/\varepsilon) + \text{const.},$$

where $N = [Hn] + 1$ and $[Hh]$ is the greatest integer less or equal to Hh.

Recall that N_p satisfies (M·2) if and only if N_p^* satisfies (M·2) ([**19**]).

Thus, Definition 2 can be phrased by using asymptotic scales.

Let $\Omega_n, n \in \mathbb{N}$ be an exhaustive sequence ($\Omega_n \subset\subset \Omega_{n+1}$) of relatively compact open sets with the union Ω. Put, for a net of smooth functions f_ε on Ω,

$$v_{k,n,M_p}(f_\varepsilon) = \inf\{s;\ \text{such that } ||f_\varepsilon||_{\Omega_k,n,M_p} = \mathcal{O}(a_s(\varepsilon)) \text{ or } ||f_\varepsilon||_{\Omega_k,n,M_p} = \mathcal{O}(a_s^-(\varepsilon))\}.$$

This is called the k,n,M_p-valuation. The space of nets for which the k,n,M_p-valuation is finite is denoted by $\mathscr{E}_{\exp}^{N_p}(\Omega_k,n,M_p)$.

It defines an ultra-semi-metric topology on $\mathscr{E}_{\exp}^{N_p}(\Omega_k,n,M_p)$

$$e_{k,n,M_p}(f_\varepsilon,g_\varepsilon) \coloneqq e^{v_{k,n,M_p}(f_\varepsilon - g_\varepsilon)},\ (f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathscr{E}_{\exp}^{N_p}(\Omega_k,n,M_p).$$

We have

$$\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega) = \operatorname{proj}\lim_{k\to\infty} \operatorname{ind}\lim_{n\to\infty} \mathscr{E}_{\exp}^{N_p}(\Omega_k,n,M_p),$$
$$\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega) = \operatorname{proj}\lim_{k\to\infty} \operatorname{proj}\lim_{n\to\infty} \mathscr{E}_{\exp}^{N_p}(\Omega_k,n,M_p),$$
$$\mathscr{N}^{\{M_p,N_p\}}(\Omega) = \{(f_\varepsilon)_\varepsilon; \quad (\forall k) \quad (\exists n) \quad (\forall s)\big(||f_\varepsilon||_{\Omega_k,n} = \mathcal{O}(a_s^-(\varepsilon))\big)\}$$
$$\mathscr{N}^{(M_p,N_p)}(\Omega) = \{(f_\varepsilon)_\varepsilon; \quad (\forall k) \quad (\forall n) \quad (\forall s)\big(||f_\varepsilon||_{\Omega_k,n} = \mathcal{O}(a_s^-(\varepsilon))\big)\}.$$

The corresponding topologies in $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$, $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$, $\mathscr{G}_{\exp}^{\{M_p,N_p\}}(\Omega)$ and $\mathscr{G}_{\exp}^{(M_p,N_p)}(\Omega)$ are called sharp topologies.

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

Remark. One can verify that ultradifferential operators of $\{M_p\}$, resp., (M_p) class act continuously on $\mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega)$ and $\mathscr{G}_{\exp}^{\{M_p,N_p\}}(\Omega)$, resp., on $\mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega)$ and $\mathscr{G}_{\exp}^{(M_p,N_p)}(\Omega)$.

2. *Embeddings*

Let $\phi \in \mathscr{E}^{\{M_p\}}(\Omega)$, resp., $\phi \in \mathscr{E}^{(M_p)}(\Omega)$ and $(\phi)_\varepsilon$ be a constant net, $\phi_\varepsilon = \phi$, $\varepsilon < 1$. Then, one can easily prove that for every sequence N_p (satisfying conditions (M·1)*, (M·2) and (M·3)′)

$$(\phi)_\varepsilon \in \mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega), \text{ resp., } (\phi)_\varepsilon \in \mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega).$$

In this section we will consider the case where M_p is a sequence which satisfies (M·1)*, (M·2) and, additionally,

$$N_p := \frac{M_p}{p!^{1+\rho}} \text{ satisfies } (\mathrm{M}\cdot 3)' \quad (\rho > 0).$$

Note that N_p satisfies (M·1)*, (M·2) and M_p satisfies (M·3)′. Also we have

$$\mathscr{E}_{\exp}^{\{M_p,M_p\}}(\Omega) \subset \mathscr{E}_{\exp}^{\{M_p,N_p\}}(\Omega), \quad \mathscr{E}_{\exp}^{(M_p,M_p)}(\Omega) \subset \mathscr{E}_{\exp}^{(M_p,N_p)}(\Omega).$$
$$\mathscr{N}_{\exp}^{\{N_p,N_p\}}(\Omega) \subset \mathscr{N}_{\exp}^{\{M_p,N_p\}}(\Omega), \quad \mathscr{N}_{\exp}^{(N_p,N_p)}(\Omega) \subset \mathscr{N}_{\exp}^{(M_p,N_p)}(\Omega).$$

Our basic results are given in the next theorem. The analogue of the first part is known in the theory of Colombeau generalized functions as the fundamental lemma. The second part gives the natural embeddings of Beurling, resp., Roumieu, ultradistributions into the algebra of Beurling, resp., Roumieu, Colombeau ultradistributions.

THEOREM 1. (i) (*a*) *Let* $\psi \in \mathscr{D}^{\left(\frac{M_p}{p!^{1+\rho}}\right)}(K)$ *and* ϕ *be a mollifier, in* $\mathscr{S}^{\{p!^{1+\rho}\}}$. *Then*

$$\psi * \phi_\varepsilon - \psi \in \mathscr{N}^{\left(\frac{M_p}{p!^\rho},M_p\right)}(\Omega) \quad (\phi_\varepsilon = \varepsilon^{-1}\phi(\cdot\varepsilon^{-1})).$$

(*b*) *Let* $\psi \in \mathscr{D}^{\left\{\frac{M_p}{p!^{1+\rho}}\right\}}(K)$ *and* ϕ *be a mollifier, in* $\mathscr{S}^{\{p!^{1+\rho}\}}$, resp., *in* $\mathscr{S}^{(p!^{1+\rho})}$. *Then*

$$\psi * \phi_\varepsilon - \psi \in \mathscr{N}^{\left\{\frac{M_p}{p!^\rho},M_p\right\}}(\Omega), \text{ resp., } \psi * \phi_\varepsilon - \psi \in \mathscr{N}^{\left(\frac{M_p}{p!^\rho},M_p\right)}(\Omega).$$

(ii) *Let* $f \in \mathscr{E}'^{(M_p)}(\Omega)$ *and* ϕ *be a mollifier in* $\mathscr{S}^{\{p!^{1+\rho}\}}$. *Then*

(*a*) $f * \phi_\varepsilon \in \mathscr{E}_{\exp}^{\left(M_p,\frac{M_p}{p!^\rho}\right)}(\Omega)$.

If ϕ' *is another mollifier in* $\mathscr{S}^{\{p!^{1+\rho}\}}$ *then*

$$\langle f * \phi_\varepsilon - f * \phi'_\varepsilon, \psi\rangle \in \mathscr{N}_0^{M_p}, \quad \psi \in \mathscr{D}^{\left(\frac{M_p}{p!^{1+\rho}}\right)}(\Omega).$$

(*b*) *If* $f \in \mathscr{E}'^{\{M_p\}}(\Omega)$, *then* $f * \phi_\varepsilon \in \mathscr{E}_{\exp}^{\left\{M_p,\frac{M_p}{p!^\rho}\right\}}(\Omega)$. *If* ϕ' *is another mollifier in* $\mathscr{S}^{\{p!^{1+\rho}\}}$, *then*

$$\langle f * \phi_\varepsilon - f * \phi'_\varepsilon, \psi\rangle \in \mathscr{N}_0^{M_p}, \quad \psi \in \mathscr{D}^{\left\{\frac{M_p}{p!^{1+\rho}}\right\}}(\Omega).$$

Proof. (i) We will prove the assertions in part (*a*). Part (*b*) can be proved similarly.

Let $r > 0$ and $k > 1$ be given. We will prove that for suitable $C > 0, c > 0$, which do not depend on r and k,

$$\sup_{s\in\mathbb{N}_0,x\in K}\left|\frac{r^s}{(M_s/s!^\rho)}(\psi * \phi_\varepsilon - \psi)^{(s)}(x)\right| \leqslant Ce^{-M^*\left(\frac{ck}{\varepsilon}\right)},$$

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

Let $x \in K \subset\subset \Omega$ and $s \in \mathbb{N}$. We have

$$(\psi * \phi_\varepsilon - \psi)^{(s)}(x) = \int_{\mathbb{R}} (\psi^{(s)}(x+\varepsilon t) - \psi^{(s)}(x))\phi^\varepsilon(t)dt$$
$$= \int_{\mathbb{R}} \left(\sum_{p=0}^{N-1} \frac{(\varepsilon t)^p}{p!} \psi^{(p+s)}(x) + \frac{(\varepsilon t)^N}{N!} \psi^{(N+s)}(\xi) - \psi^{(s)}(x) \right) \phi^\varepsilon(t)dt,$$

where $x \leqslant \xi \leqslant x + \varepsilon t$.

We will show that for every $\varepsilon < \bar{\varepsilon}$, where $\bar{\varepsilon}$ satisfies $\bar{\varepsilon} m_1^*/k < 1$, there exists a natural number $N = N_\varepsilon$ such that

$$1 \leqslant \frac{\varepsilon}{k} m_N^* \leqslant T, \tag{4}$$

where T is a constant appearing in the estimate

$$m_{2p}^*/m_p^* \leqslant T, \quad p \in \mathbb{N},$$

for which we know that it exists since this estimate is equivalent to condition (M·2) ([**19**]). First, let us prove

$$m_{p+1}^*/m_p^* \leqslant T, \quad p \in \mathbb{N}.$$

In fact, it is obvious for $p = 2k$ since m_p^* increases to infinity (cf. (M·1)*) and for $p = 2k+1, k \geqslant 1$, it follows from

$$m_{2k+1}^*/m_{2k}^* \leqslant m_{2k+2}^*/m_{2k}^* \leqslant m_{2k+2}^*/m_{k+1}^* \leqslant T, \qquad k \in \mathbb{N}.$$

With this T, we will prove that for every $\varepsilon < \bar{\varepsilon}$ there exists N such that (4) holds.

Assume, on the contrary, that for some ε_0 such an N_{ε_0} does not exist. This implies the existence of an index p_0 such that

$$\varepsilon_0 m_{p_0}^*/k < 1 \quad \text{and} \quad \varepsilon_0 m_{p_0+1}^*/k > T.$$

But

$$\varepsilon_0 m_{p_0+1}^*/k = \varepsilon_0 m_{p_0}^*/k + (m_{p_0+1}^*/m_{p_0}^* - 1)\varepsilon_0 m_{p_0}^*/k$$
$$\leqslant \varepsilon_0 m_{p_0}^*/k + (T-1)\varepsilon_0 m_{p_0}^*/k = \frac{\varepsilon_0 m_{p_0}^*}{k} T < T$$

which gives the contradiction.

With such $N = N_\varepsilon$, we have

$$\frac{r^s}{(M_s/s!^\rho)} (\psi * \phi_\varepsilon - \psi)^{(s)}(x) = \int_{\mathbb{R}} \frac{(\varepsilon t)^N r^s}{N!(M_s/s!^\rho)} \psi^{(N+s)}(\xi)\phi(t)dt.$$

Let $d > 0$ such that $d + r > b$ and $\sigma_{b,p^{!1+\rho}}(\phi) < \infty$. We have

$$\left| \frac{r^s}{(M_s/s!^\rho)} (\psi * \phi_\varepsilon - \psi)^{(s)}(x) \right|$$
$$\leqslant \int_{\mathbb{R}} \frac{(d+r)^{N+s}}{(M_{N+s}/(N+s)!^{1+\rho})} |\psi^{(N+s)}(\xi)| \frac{M_{N+s}\varepsilon^N}{(d+r)^{N+s}(M_s/s!^\rho)N!(N+s)!^{1+\rho}} t^N |\phi(t)|dt.$$

We will use $M_N^* = m_N^* m_{N-1}^* \cdots m_1^* \leqslant (m_N^*)^N$. Since $\varepsilon/k \leqslant T/m_N^*$, using condition

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

(M·2), we obtain

$$\left|\frac{r^s}{(M_s/s!^\rho)}(\psi*\phi_\varepsilon-\psi)^{(s)}(x)\right| \leqslant \int_{\mathbb{R}}\frac{(2kT(r+d))^{N+s}}{(M_{N+s}/(N+s)!^{1+\rho})}|\psi^{(N+s)}(\xi)|2^{-N}AH^{N+s}\frac{M_N^*}{(m_N^*)^N}\frac{|t|^N}{(r+d)^N N!^{1+\rho}}|\phi(t)|dt.$$

Thus, with $r_0 = 2HkT(r+d)$, we have

$$\left|\frac{r^s}{(M_s/s!^\rho)}(\psi*\phi_\varepsilon-\psi)^{(s)}(x)\right| \leqslant A2^{-N}||\psi||_{K,r_0,\frac{M_p}{p!^{1+\rho}}}\sigma_{(r+d),p!^{1+\rho}}(\phi).$$

Now we use an equivalent condition for (M·2), given in [**19**],

$$2^{-N} \leqslant e^{-M^*\left(\frac{m_N^*}{h}\right)},$$

where $h>0$ is a suitable constant which does not depend on N. This and (4) imply, with suitable $C>0$ and $c>0$,

$$\left|\frac{r^s}{(M_s/s!^\rho)}(\psi*\phi_\varepsilon-\psi)^{(s)}(x)\right| \leqslant Ce^{-M^*\left(\frac{cr}{\varepsilon}\right)}.$$

By taking the supremum over all s and x we obtain that $\psi*\phi_\varepsilon-\psi \in \mathcal{N}^{\left(\frac{M_p}{p!^\rho},M_p\right)}(\mathbf{\Omega})$.

(ii) (a) Recall ([**11**]), if $f \in \mathscr{E}'^{(M_p)}(\mathbf{\Omega})$, then there exists an ultradifferential operator of class (M_p), $P(D)=\sum_{k=0}^\infty a_k D^k$, and continuous functions F_k, $\operatorname{supp} F_k \subset K_0 \subset\subset \mathbf{\Omega}$, $k\in\mathbb{N}_0$ with the property $\sup\{|F_k(x)|;\ k\in\mathbb{N}_0,\ x\in\mathbf{\Omega}\} \leqslant M$, such that $f=\sum_{k=0}^\infty a_k D^k F_k$. This implies

$$f*\phi_\varepsilon(x)=\sum_{k=0}^\infty\frac{(-1)^k a_k}{\varepsilon^k}\int_{\mathbb{R}}F_k(x+\varepsilon t)D^k\phi(t)dt, \quad x\in\mathbb{R},$$

where ϕ is a mollifier in $\mathscr{E}^{\{p!^{1+\rho}\}}$ such that $\sigma_{b,p!^{1+\rho}}(\phi)<\infty$ and a_k, $k\in\mathbb{N}_0$ satisfy (1). Let $K\subset\subset\mathbf{\Omega}$, $x\in K$ and $r>0$ be given. We have

$$\begin{aligned}
\frac{r^p}{M_p}|f^{(p)}*\phi_\varepsilon(x)| &= \left|\sum_{k=0}^\infty\frac{(-1)^k a_k}{\varepsilon^{k+p}}\frac{r^p}{M_p}\int_{\mathbb{R}}F_k(x+\varepsilon t)D^{k+p}\phi(t)dt\right| \\
&\leqslant \sum_{k=0}^\infty B\frac{r^p h^k}{\varepsilon^{k+p}M_k M_p}\int_{\mathbb{R}}|F_k(x+\varepsilon t)||D^{k+p}\phi(t)|dt \\
&\leqslant \sum_{k=0}^\infty BA\frac{(H(r+h))^{p+k}}{\varepsilon^{k+p}(M_{k+p}/(k+p)!^{1+\rho})}\int_{\mathbb{R}}|F_k(x+\varepsilon t)|\frac{|D^{k+p}\phi(t)|}{(k+p)!^{1+\rho}}dt \\
&\leqslant \sum_{k=0}^\infty\frac{1}{2^k}BA\frac{(2Hb(r+h))^{p+k}}{\varepsilon^{k+p}N_{k+p}^*}\int_{\mathbb{R}}|F_k(x+\varepsilon t)|\frac{1}{b^{k+p}(k+p)!^{1+\rho}}|D^{k+p}\phi(t)|dt \\
&\leqslant Ce^{N^*\left(\frac{2Hb(r+h)}{\varepsilon}\right)}\sigma_{b,p!^{1+\rho}}(\phi)
\end{aligned}$$

($N^*(t)$ corresponds to $N_p=M_p/p!^\rho$). This proves that $f*\phi_\varepsilon \in \mathscr{E}_{\exp}^{\left(M_p,\frac{M_p}{p!^\rho}\right)}(\mathbf{\Omega})$.

Let us prove the second assertion in (a), i.e. that

$$\langle f,(\check{\phi}_\varepsilon-\check{\phi}'_\varepsilon)*\psi\rangle \in \mathcal{N}_0^{M_p}.$$

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

By the general theory ([**12**]) we know that there exist $K \subset\subset \Omega$, $h > 0$ and $C > 0$ such that

$$|\langle f, (\check{\phi}_\varepsilon - \check{\phi}'_\varepsilon) * \psi\rangle| \leqslant C||(\check{\phi}_\varepsilon - \check{\phi}'_\varepsilon) * \psi||_{K,h,M_p/p!^\rho}$$
$$\leqslant ||\check{\phi}_\varepsilon * \psi - \psi||_{K,h,M_p/p!^\rho} + ||\check{\phi}'_\varepsilon * \psi - \psi||_{K,h,M_p/p!^\rho}. \qquad (5)$$

By the first part of the theorem we have that $\psi * \phi_\varepsilon - \psi$, $\psi * \phi'_\varepsilon - \psi \in \mathcal{N}^{\left(\frac{M_p}{p!^\rho}, M_p\right)}(\Omega)$. This and Definition 2 imply that for every $k > 0$ and $\varepsilon < 1$ both addents in (5) are $\leqslant e^{-M^*(k/\varepsilon)}$.

Example. The following assertion is proved in [**17**].

The equation

$$(\partial_t^2 - \Delta)G = -G^3, \quad G|_{t=0} = A, \quad \partial_t G|_{t=0} = B,$$

$A, B \in \mathcal{G}_{L^2(\mathbb{R})}$, has a unique solution in $\mathcal{G}_{L^2}([0,T) \times \mathbb{R}^3)$, for every $T > 0$ if there exist representatives of initial data such that

$$\|(\nabla^2 A_\varepsilon, \nabla B_\varepsilon)\|_{L^2} = o((\log \varepsilon^{-1})^{\frac{1}{2}}).$$

In the framework of ultradistribution Colombeau spaces, this assertion can be reformulated as follows: if

$$\|(\nabla^2 A_\varepsilon, \nabla B_\varepsilon)\|_{L^2} = o(\varepsilon^{-\nu/2}), \quad \nu < 1,$$

then there exists a unique solution to this equation in the space $\mathcal{G}^{(M_p,p!^\mu)}(\Omega)$, where M_p is an arbitrary sequence and $1 < \mu < 1/\nu$.

For example, if $A \in H^{2-\mu}$, $B \in H^{1-\mu}$, then we know that their regularizations satisfy the above assumptions and there exists a unique solution to this equation in the space $\mathcal{G}^{(M_p,p!^\mu)}(\Omega)$, where M_p is arbitrary and $1 < \mu < 1/\nu$. The similar analysis in some other cases of semilinear wave equation (depending on the space dimension and the growth order of the non-linear term) shows that the Cauchy problem for this equation can be considered in appropriate ultradistribution Colombeau spaces if initial data posses appropriate types of singularities.

3. *Regularity and localization*

We continue to assume that M_p and N_p satisfy $(\mathrm{M}\cdot 1)^*$, $(\mathrm{M}\cdot 2)$ and $(\mathrm{M}\cdot 3)'$. Moreover, we assume $N_p \leqslant M_p$, which means that for some $h > 0$ there exists $B > 0$ such that $N_p \leqslant Bh^p M_p$, $p \in \mathbb{N}$. This implies that there exist $c > 0$ and $C > 0$ such that $M^*(t) \leqslant N^*(ct) + C$, $t > 0$ (cf. [**11**]).

The regularity properties of Colombeau generalized functions can be considered according to the following algebras.

Definition 4. Roumieu case:

$$\mathcal{E}^{\{M_p\},\infty}(\Omega) = \left\{(f_\varepsilon)_\varepsilon; (\forall K \subset\subset \Omega) \quad (\exists k > 0) \quad (\exists h > 0)\right.$$
$$\left.\sup\left\{\frac{h^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|; \quad x \in K, \quad \alpha \in \mathbb{N}\right\} = \mathcal{O}(e^{M^*(k/\varepsilon)})\right\},$$
$$\mathcal{G}^{\{M_p,N_p\},\infty}(\Omega) = \mathcal{E}_{\exp}^{\{M_p\},\infty}(\Omega)/\mathcal{N}^{\{M_p,N_p\}}(\Omega).$$

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

Beurling case:

$$\mathscr{E}^{(M_p),\infty}(\Omega) = \Big\{(f_\varepsilon)_\varepsilon;\quad (\forall K \subset\subset \Omega)\quad (\exists k>0)\quad (\forall h>0)$$
$$\sup\left\{\frac{h^\alpha}{M_\alpha}|f_\varepsilon^{(\alpha)}(x)|; x\in K, \alpha\in\mathbb{N}\right\} = \mathscr{O}(e^{M^*(k/\varepsilon)})\Big\},$$
$$\mathscr{G}^{(M_p,N_p),\infty}(\Omega) = \mathscr{E}^{(M_p),\infty}_{\exp}(\Omega)/\mathscr{N}^{(M_p,N_p)}(\Omega).$$

One can prove that

$$\Omega \to \mathscr{G}^{\{M_p,N_p\},\infty}(\Omega),\ \text{resp.},\ \Omega\to\mathscr{G}^{(M_p,N_p),\infty}(\Omega)$$

constitutes a sheaf. This implies the definition of singular supports.

Definition 5. Let $[f_\varepsilon] \in \mathscr{G}^{\{M_p,N_p\}}(\Omega)$, resp., $[f_\varepsilon]\in\mathscr{G}^{(M_p,N_p)}(\Omega)$. Then, $\operatorname{singsupp}_{\{M_p,N_p\}}[f_\varepsilon]$ is the complement of the union of all open sets in Ω where there exists a representative $f_\varepsilon\in\mathscr{E}^{\{M_p\},\infty}(\Omega)$, resp., $\operatorname{singsupp}_{(M_p,N_p)}[f_\varepsilon]$ is the complement of the union of all open sets in Ω where there exists a representative $f_\varepsilon\in\mathscr{E}^{(M_p),\infty}(\Omega)$.

Theorem 2. *Let $G\in\mathscr{G}^{\{M_p,N_p\},\infty}(\mathbb{R})$, resp., $G\in\mathscr{G}^{(M_p,N_p),\infty}(\mathbb{R})$, and $T\in\mathscr{D}'^{\{M_p\}}(\mathbb{R})$, resp., $T\in\mathscr{D}'^{(M_p)}(\mathbb{R})$, such that for every compact set P there exists $c_p>0$ such that*

$$\langle T-G_\varepsilon,\phi\rangle = \mathscr{O}(e^{-M^*(c_p/\varepsilon)}),\ \phi\in\mathscr{D}^{\{M_p\}}(\mathbb{R}),\ \textit{resp.},\ \phi\in\mathscr{D}^{(M_p)}(\mathbb{R}),\ \textit{supp}\,\phi\subset P.$$

Then, $T\in\mathscr{E}^{\{M_p\}}(\mathbb{R})$, resp., $T\in\mathscr{E}^{(M_p)}(\mathbb{R})$.

Proof. We will give the proof in the first case. By using the partition of unity, one can suppose without loss of generality that T and G have both compact supports. Let K be a compact set containing supports of T and $G_\varepsilon, \varepsilon<1$, where $G=[G_\varepsilon]$. There exist $r>0$ and $h>0$ such that

$$\sup_{x\in K,\alpha\in\mathbb{N}}\frac{h^\alpha}{M_\alpha}|G_\varepsilon^{(\alpha)}(x)| = \mathscr{O}(e^{M^*(r/\varepsilon)}).$$

Since

$$\left|\xi^m\int_{\mathbb{R}} e^{-i\xi x}G_\varepsilon(x)dx\right| = \left|\int_{\mathbb{R}} e^{-ix\xi}G_\varepsilon^{(m)}(x)dx\right| \leqslant C\sup_{x\in K}\left|G_\varepsilon^{(m)}(x)\right|,\quad \xi\in\mathbb{R},$$

the definition of the associated function M implies that there exists $C_h>0$ such that

$$|\widehat{G_\varepsilon}(\xi)| \leqslant C_h e^{M^*(r/\varepsilon)}e^{-M(h\xi)},\quad \xi\in\mathbb{R},\quad \varepsilon\in(0,1). \tag{6}$$

Let $\varphi\in\mathscr{E}^{\{M_p\}}(\mathbb{R})$. The assumption of the theorem implies that there exists $b>0$ such that

$$|\langle T-G_\varepsilon,\varphi\rangle| = \mathscr{O}(e^{-M^*(b/\varepsilon)}).$$

So, for suitable $a<b$, $S_\varepsilon\to 0$ in $\mathscr{E}'^{\{M_p\}}(\Omega)$ as $\varepsilon\to 0$, where $S_\varepsilon = e^{M^*(a/\varepsilon)}(T-G_\varepsilon)$, $\varepsilon\in(0,1)$. There exists a compact set $K_1\supset K$ such that for every $k_1>0$ and corresponding $C_1>0$ the following holds

$$|\langle S_\varepsilon,\varphi\rangle| \leqslant C_1\|\varphi\|_{K_1,k_1,M_p},\quad \varphi\in\mathscr{E}^{(M_p)}(\mathbb{R}). \tag{7}$$

Therefore, (7) holds for every $\varphi\in\mathscr{E}^{k_1,M_p}(K_1)$.

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

Note,

$$\|e^{i\xi x}\|_{K_1,k_1,M_p} \leqslant e^{M(k_1\xi)}, \quad \xi \in \mathbb{R}. \tag{8}$$

Thus, by (7) and (8), and by letting $\phi(x) = e^{i\xi x}$, $x \in \mathbb{R}$, it follows that there exists $C > 0$ such that

$$|\hat{S}_\varepsilon(\xi)| = (e^{M^*(a/\varepsilon)})|\widehat{T}(\xi) - \widehat{G_\varepsilon}(\xi)| \leqslant Ce^{M(k_1\xi)}, \quad \xi \in \mathbb{R}, \quad \varepsilon \in (0,1).$$

This and (6) imply

$$|\widehat{T}(\xi)| \leqslant C(e^{-M^*(a/\varepsilon)+M(k_1\xi)} + e^{M^*(r/\varepsilon)-M(h\xi)}), \quad \xi \in \mathbb{R}, \quad \varepsilon \in (0,1). \tag{9}$$

Let

$$\varepsilon = \frac{r}{M^{*-1}(M((h-\delta)\xi))}, \quad |\xi| \geqslant |\xi_0|, \tag{10}$$

where we choose $\delta \in (0,h)$, ξ_0 and k_1 in (9) and (10) as follows. By (3), we have that there exist $C > 0$ and $c > 0$ such that

$$e^{-M^*(\frac{a}{r}t)} \leqslant Ce^{-cM^*(t)}, \quad t > 0.$$

This implies that we have to choose k_1 such that there exists $s_1 > 0$ such that

$$-cM((h-\rho)\xi) + M(k_1\xi) \leqslant -M(s_1\xi), \quad \xi \in \mathbb{R}.$$

Again this is possible by (3).

Since $M^{*-1}(M((h-\delta)\xi)) \to \infty$, $|\xi| \to \infty$, there exists $\xi_0 \in \mathbb{R}$ such that in (10) $\varepsilon < 1$ for $|\xi| \geqslant |\xi_0|$. By putting ε of the form (10) in (9) we have

$$e^{-M^*(a/\varepsilon)+M(k_1\xi)} \leqslant e^{-M(s_1\xi)},$$
$$e^{M^*(r/\varepsilon)-M(h\xi)} \leqslant e^{M((h-\delta)\xi)-M(h\xi)}, \quad \xi \in \mathbb{R}.$$

This implies that there exists $s > 0$ and $C_s > 0$ such that

$$|\widehat{T}(\xi)| \leqslant C_s e^{-M(s\xi)}, \quad \xi \in \mathbb{R}.$$

By Lemma 3·3 and [**11**, theorem 9·1], this implies that $T \in \mathscr{E}^{\{M_p\}}(\mathbb{R})$, supp $\mathrm{T} \subset K$.

Remark. As in the case of Colombeau distributions and the definition of generalized wave front ([**7**]), we can define ultradistributional generalized wave front sets for Colombeau ultradistributions. This will be discussed in a separate paper.

Acknowledgements. Grateful thanks are due to the ESI for support.

REFERENCES

[**1**] H. A. BIAGIONI. *A nonlinear theory of generalized functions* (Springer-Verlag, 1990).
[**2**] H. A. BIAGIONI and J. F. COLOMBEAU. New generalized functions and C^∞ functions with values in generalized complex numbers. *J. London Math. Soc.* (2) **33**, 1 (1986), 169–179.
[**3**] H. A. BIAGIONI and J. F. COLOMBEAU. Borel's theorem for generalized functions. *Studia Math.* **81**, 2 (1985), 179–183.
[**4**] J. F. COLOMBEAU. *New generalized functions and multiplication of the distributions* (North Holland, 1983).
[**5**] J. F. COLOMBEAU. Multiplication of distributions. *Lecture Notes in Math.* **1532** (Springer, 1992).
[**6**] A. DELCROIX and D. SCARPALÉZOS. Asymptotic scales-asymptotic algebras. *Integral Transforms and Special Functions* **6** (1998), 181–190.
[**7**] N. DJAPIĆ, S. PILIPOVIĆ and D. SCARPALÉZOS. Intrinsic microlocal characterization of Colombeau's generalized functions. *J. Anal. Math.* **75** (1998), 51–66.

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press

[**8**] A. Eida and S. Pilipović. On the Microlocal Decomposition of some classes of hyperfunctions. *Math. Proc. Camb. Phil. Soc.* **125** (1999), 455–461.
[**9**] T. Gramchev. Nonlinear maps in spaces of distributions. *Math. Zeitschr.* **209** (1992), 101–114.
[**10**] L. Hörmander. *The analysis of linear partial differential operators* I (Springer, 1983).
[**11**] H. Komatsu. Ultradistributions, I – III, *J. Fac. Sci. Univ. Tokyo*, Sect. IA **20** (1973), 25–105; **24** (1977), 607–628; **29** (1982), 653–717.
[**12**] H. Komatsu. Microlocal analysis in Gevrey classes and in convex domains. *Lecture Notes in Math.* **1726** (Springer, 1989), 426–493.
[**13**] D. Kovačević and S. Pilipović. Structural properties of the space of tempered ultradistributions, complex analysis and generalized functions (1993). *Proc. Conf. 'Comp. Analysis and Applications '91 with Symposium on Generalized Functions'*, Varna 1991, 169–184.
[**14**] M. Kunziger and M. Oberguggenberger. Characterization of Colombeau generalized functions by their pointvalues. *Math. Nachr.* (1997), 147–157.
[**15**] J. A. Marti. Analyse locale et microlocale des algébres de fonctions généralisées. *Annales de la Faculté des Sciences de Toulouse* (1998).
[**16**] J. A. Marti. Fundamental structures and asymptotic microlocalization in sheaves of generalized functions. *Integral Transforms and Special Functions* **6** (1998), 223–228.
[**17**] M. Nedeljkov, M. Oberguggenberger and S. Pilipović. Generalized solution to a semilinear wave equation, preprint.
[**18**] M. Oberguggenberger. *Multiplications of distributions and applications to partial differential equations* (Longman, 1992).
[**19**] H. J. Petzsche. Generalized functions and the boundary values of holomorphic functions. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **31** (1984), 391–431.
[**20**] S. Pilipović. Microlocal analysis of ultradistributions. *Proc. AMS* **126** (1998), 105–113.
[**21**] D. Scarpalézos. Some remarks on functoriality of Colombeau's construction: topological and microlocal aspects and applications. *Integral Transforms and Special Functions* **6** (1998), 295–307.

https://doi.org/10.1017/S0305004101005072 Published online by Cambridge University Press