Probability in the Engineering and Informational Sciences, **26**, 2012, 221–244.
doi:10.1017/S0269964811000349

ANALYTICALLY EXPLICIT RESULTS FOR THE *GI*/*C-MSP*/1/∞ QUEUEING SYSTEM USING ROOTS

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In this paper, we present (in terms of roots) a simple closed-form analysis for evaluating system-length distribution at prearrival epochs of the *GI*/*C-MSP*/1 queue. The proposed analysis is based on roots of the associated characteristic equation of the vector-generating function of system-length distribution. We also provide the steady-state system-length distribution at an arbitrary epoch by using the classical argument based on Markov renewal theory. The sojourn-time distribution has also been investigated. The prearrival epoch probabilities have been obtained using the method of roots which is an alternative approach to the matrix-geometric method and the spectral method. Numerical aspects have been tested for a variety of arrival- and service-time distributions and a sample of numerical outputs is presented. The proposed method not only gives an alternative solution to the existing methods, but it is also analytically simple, easy to implement, and computationally efficient. It is hoped that the results obtained will prove beneficial to both theoreticians and practitioners.

1. INTRODUCTION

Queueing models with non-renewal arrivals and service processes are often used to model complex computer and communication systems. Several connections (data,

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https://doi.org/10.1017/S0269964811000349 Published online by Cambridge University Press

voice, video, etc.) generate traffic streams with very different characteristics (required bandwidth, burstiness, correlation, etc.). Traditional teletraffic analysis based on Poisson arrival/service process is not powerful enough to capture this correlative and bursty feature of traffic streams in high-speed packet/cell-based networks. These correlated and bursty arrival and/or service processes in queueing systems have been shown empirically and theoretically to have a significant impact on the queueing behavior. The continuous-time Markovian arrival process (C-MAP) has been introduced due to limitations of the Poisson process in modelling correlated arrivals. The concepts of C-MAP have been explained in the past at several places by many researchers and are available in the literature, see, for example, Pacheco, Tang, and Prabhu [20]. Like this arrival process, the continuous-time Markovian service process (C-MSP) is a versatile service process and can capture the correlation that exists among successive service times. This service process is similar to the C-MAP, where arrivals are replaced with service completions when the server is busy. This service process forms a rich class of point processes that include several known processes such as the Markov-modulated Poisson process (MMPP), the PH-renewal process, and the superposition of such processes. It may be appropriate to point out at this stage that the basic idea of correlated services was first introduced by Chaudhry [5].

The interest in analyzing the $GI/C\text{-}MSP/1$ queue has been mainly due to the fact that $C\text{-}MSP$ is an excellent representation of the bursty and correlated traffic arising in modelling of telecommunications systems. In the past few decades, several methods of inversion of transforms have been developed. As many researchers believed that transform methods were numerically difficult to implement, they started using the matrix-geometric method (MGM) for analyzing various queueing models such as $GI/M/1$ and other related models. However, in this paper, we demonstrate, using roots, that a transform method can be exploited to get the unknown constants k_{ij}'s, $1 \leq i,j \leq m$ (see Section 3.1) and the state probabilities. In this connection, see Tijms [23, p. 471], Janssen and Leeuwaarden [16], and Chaudhry, Harris and Marchal [7], who have used the roots method. The roots (including repeated roots) can be easily found using one of the several commercially available packages such as Maple and Mathematica. At an early stage, a software package called QROOT developed by Chaudhry [6] was used by him and his collaborators to find a large number of roots and use them in solving several queueing problems. The algorithm for finding such roots is available in some of their papers. In particular, see Chaudhry et al. [7].

Our purpose here is to give a unified analytic approach that uses roots of the so-called characteristic equation (c.e.) (defined in Section 3.1) and improves on the available computational aspects. It first accurately calculates the unknown constants k_{ij}'s, $1 \leq i,j \leq m$, and then uses these constants to get probability distributions of the number in system at prearrival, arbitrary, and post-departure epochs as well as the sojourn-time distribution and other performance measures. Further, contrary to the statement made by Daigle [11, p. 142] in his book that the probability generating function method is cumbersome, we find that the procedure proposed here is analytically quite simple to understand and easy to implement. Also, the proposed approach

https://doi.org/10.1017/S0269964811000349 Published online by Cambridge University Press

is simpler than the conventional MGM. For analytic and numerical purposes, we start our work with the vector-generating function (VGF) of system-length distribution at prearrival epochs, evaluate the unknown constants k_{ij}'s, $1 \le i, j \le m$, using roots inside the unit circle of the associated c.e. and then use these unknown constants to evaluate other probabilities. The inversion of associated VGF has also been carried out using the method of roots.

Many researchers have analyzed several queueing models with various types of service as well as arrival processes and such results are available in the literature. However, very little seems to have been done on correlated service processes. Adan and Zhao [1] analyze a single-server $GI/E_r/1$ queueing system and it is shown that the waiting-time distribution can be expressed as a finite sum of exponentials, the exponents of which are the roots of an equation. Ramaswami and Latouche [21] discuss a $GI/PH/1$ queueing model using the MGM. The authors mentioned below use the same method (MGM). A detailed discussion of the stationary distributions of the $MAP/SM/1$ queue is characterized by Lucantoni and Neuts [17]. Alfa, Xue, and Ye [3] discuss the asymptotic behavior of the $GI/MSP/1$ queue using perturbation theory. The analysis of finite- and infinite-buffer $G/MSP/1$ queue has been carried out by Bocharov et al. [4], wherein they derived stationary characteristics of system performance. Shioda [22] analyzed the departure process of the $MAP/SM/1$ queue, and derived various expressions that characterize burstiness and correlation of the departure process. Ozawa [19] carried out the analysis of $MAP/MSP/1$ queue and derived the stationary sojourn-time distribution as well as its asymptotic properties. Gupta and Banik [15] have analyzed $GI/MSP/1$ queue with finite as well as infinite buffer using a combination of the MGM, the imbedded Markov chain, and the supplementary variable techniques. They considered a C-MSP with a new service restart phase distribution such that if the server becomes idle at a certain point the service phase still keeps changing during the subsequent idle period of the system. Then, when a new customer arrives, the service process starts with a new phase distribution independent of the path followed before by the service process.

This paper is organized as follows. In Section 2, we give the description of the model and introduce the notations to describe the model parameters. The steady-state system-length distributions at various epochs and sojourn-time distribution of an arriving customer are analyzed in Section 3. In Section 4, we consider a model with idle–restart service phase distribution. Numerical results are presented in Section 5. Section 6 concludes the paper.

2. MODEL DESCRIPTION AND NOTATIONS

We consider a single-server queue wherein customers are served according to an m-state continuous-time Markovian service process $\{N(t), J(t)\}_{t\ge 0}$ with matrix representation $(\mathbf{L}_0, \mathbf{L}_1)$, called service process. That is, $\{N(t), J(t)\}$ is a two-dimensional Markov process with state space $\{(\ell, i) : \ell \ge 0, 1 \le i \le m\}$ and

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infinitesimal generator

$$\mathbf{Q}^{\text{C-MSP}} = \begin{pmatrix} \mathbf{L}_0 & \mathbf{L}_1 & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{L}_0 & \mathbf{L}_1 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_0 & \mathbf{L}_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here, $N(t)$ denotes the number of customers served in the first t units when the server is busy and $J(t)$ the state of the underlying Markov chain (called service phase) after the same amount of time. Now, if at some instant a customer is under service and the service is in phase i ($1 \le i \le m$), then in a "small" time Δt the phase i changes with probability $[L_0]_{i,j}\Delta t + o(\Delta t)$ to phase j, $j = 1, \ldots, m$, $j \ne i$, and the service of the customer continues, or the phase changes with probability $[L_1]_{i,j}\Delta t + o(\Delta t)$ to phase j, $j = 1, \ldots, m$, and the service is completed so that the customer quits the system. Note that the matrix $\mathbf{L}_0 = [[L_0]_{i,j}]$ has nonnegative off-diagonal and negative diagonal elements (when $i = j$), and the matrix $\mathbf{L}_1 = [[L_1]_{i,j}]$ has nonnegative elements. Let us denote $\mathbf{L}(z) = \mathbf{L}_0 + \mathbf{L}_1 z$ with $\mathbf{L} \equiv \mathbf{L}(1) = \mathbf{L}_0 + \mathbf{L}_1$ being the irreducible infinitesimal generator of the underlying Markov chain $\{J(t)\}$. Since $\mathbf{Q}^{\text{C-MSP}}$ is the infinitesimal generator of the C-MSP, we have $\mathbf{L}\mathbf{e}_m = \mathbf{0}$, where $\mathbf{e}_m$ is an $m \times 1$ column vector with all its elements equal to 1. Throughout the paper we do not use the subscript m in $\mathbf{e}_m$ and write it as $\mathbf{e}$. However, when $\mathbf{e}$'s dimension is other than m we write its subscript as its dimension. The average service rate $\mu^\star$ (the so-called fundamental service rate) of the stationary C-MSP is given by $\mu^\star = \overline{\boldsymbol{\pi}}\mathbf{L}_1\mathbf{e}$, where $\overline{\boldsymbol{\pi}} = [\overline{\pi}_1, \overline{\pi}_2, \ldots, \overline{\pi}_m]$ with $\overline{\pi}_j$ denoting the steady-state probability of service process being in phase j ($1 \le j \le m$). The stationary probability row-vector $\overline{\boldsymbol{\pi}}$ can be calculated from $\overline{\boldsymbol{\pi}}\mathbf{L} = \mathbf{0}$ with $\overline{\boldsymbol{\pi}}\mathbf{e} = 1$. Note that the service process is interrupted during idle periods of the system, implying, in particular, that the service phase does not change during idle periods of the system.

Let $\{\mathbf{P}(n,t) : n \ge 0, t \ge 0\}$ denote the matrix of order $m \times m$ whose (i,j)th element $(P_{i,j}(n,t))$ represents the probability that n customers are served in $(0,t]$ with the service process being in phase j at time t, provided initially there were at least $(n+1)$ customers (including the one in service) in the system and the service process was in phase i. Then, the probabilities

$$P_{i,j}(n,t) = \Pr\{N(t) = n, J(t) = j | N(0) = 0, J(0) = i\}, \qquad 1 \le i,j \le m$$

lead to the following equations (in matrix notation):

$$\frac{d}{dt}\mathbf{P}(0,t) = \mathbf{P}(0,t)\mathbf{L}_0 \tag{1}$$

$$\frac{d}{dt}\mathbf{P}(n,t) = \mathbf{P}(n,t)\mathbf{L}_0 + \mathbf{P}(n-1,t)\mathbf{L}_1, \qquad n \ge 1, \tag{2}$$

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with $\mathbf{P}(0,0) = \mathbf{I}_m$ and $\mathbf{P}(n,0) = \mathbf{0}$, $n \geq 1$, where $\mathbf{I}_m$ is the identity matrix of order $m \times m$. Let us define the matrix-generating function $\mathbf{P}^*(z,t)$ as

$$\mathbf{P}^*(z,t) = \sum_{n=0}^{\infty} \mathbf{P}(n,t) z^n, \qquad |z| \leq 1. \tag{3}$$

Multiplying Eqs. (1) and (2) by z^n and summing over $n = 0$ to ∞, after using (3), we get

$$\frac{d}{dt}\mathbf{P}^*(z,t) = \mathbf{P}^*(z,t)\mathbf{L}(z),$$

with $\mathbf{P}^*(z,0) = \mathbf{I}_m$. Solving the above matrix-differential equations, we get

$$\mathbf{P}^*(z,t) = e^{\mathbf{L}(z)t}, \qquad |z| \leq 1,\ t \geq 0. \tag{4}$$

The interarrival times are assumed to be independent and identically distributed (i.i.d.) random variables (r.v.'s) and they are independent of the service process. Further, let $A(x)$ $\{a(x)\}$ $[a^\bullet(s)]$ be the distribution function (DF) [probability density function (pdf)] [Laplace–Stieltjes transform (LST)] of the interarrival time of a typical customer. The mean interarrival time is $1/\lambda = -(d/ds)a^\bullet(s)|_{s=0}$. The traffic intensity is given by $\rho = \lambda/\mu^\star < 1$.

Let $\mathbf{S}_n$ $(n \geq 0)$ denote the matrix of order $m \times m$ whose (i,j)th element represents the probability that n customers are served during an interarrival period of the arrival process with the service process being in phase j at the end of the period, given that there were at least $(n+1)$ customers in the system and the service process was in phase i at the beginning of the interarrival period. Then

$$\mathbf{S}_n = \int_0^\infty \mathbf{P}(n,t)\, dA(t), \qquad n \geq 0. \tag{5}$$

If $\mathbf{S}(z)$ is the matrix-generating function of $\mathbf{S}_n$, then, using (5), (3), and (4) successively, we get

$$\begin{aligned}\mathbf{S}(z) &= \sum_{n=0}^{\infty} \mathbf{S}_n z^n = \int_0^\infty \sum_{n=0}^{\infty} \mathbf{P}(n,t) z^n\, dA(t) \\ &= \int_0^\infty \mathbf{P}^*(z,t)\, dA(t) = \int_0^\infty e^{\mathbf{L}(z)t}\, dA(t).\end{aligned} \tag{6}$$

Note that the derivations of $\mathbf{S}_n$, $n \geq 0$, and $\mathbf{S}(z)$ are given in Appendix B, when $A(t)$ follows a phase type (PH) distribution with irreducible representation $(\boldsymbol{\alpha}, \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension ν.

We now introduce the matrix $\boldsymbol{\Omega}_n$ of order $m \times m$ whose (i,j)th element represents the limiting probability that n customers are served during an elapsed interarrival time of the arrival process with the service process being in phase j, given that there were

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at least $(n+1)$ customers in the system and the service process was in phase i at the beginning of the interarrival period. Then, from Markov renewal theory, we have

$$\mathbf{\Omega}_n = \lambda \int_0^\infty \mathbf{P}(n,t)[1-A(t)]\,dt, \qquad n \geq 0. \tag{7}$$

In order to get $\mathbf{\Omega}_n,\ n \geq 0$, we can write (5) as

$$\begin{aligned}\mathbf{S}_n &= -\int_0^\infty \mathbf{P}(n,t)\,d[1-A(t)] \\ &= -\Big[\mathbf{P}(n,t)[1-A(t)]\Big]_0^\infty + \int_0^\infty \frac{d}{dt}\mathbf{P}(n,t)[1-A(t)]\,dt \\ &= \mathbf{P}(n,0) + \int_0^\infty \frac{d}{dt}\mathbf{P}(n,t)[1-A(t)]\,dt. \end{aligned} \tag{8}$$

Using (1) in (8), for $n=0$, we have

$$\begin{aligned}\mathbf{S}_0 &= \mathbf{I}_m + \int_0^\infty \mathbf{P}(0,t)\mathbf{L}_0[1-A(t)]\,dt \\ &= \mathbf{I}_m + \frac{1}{\lambda}\mathbf{\Omega}_0\mathbf{L}_0 \quad \text{using (7).} \end{aligned} \tag{9}$$

Hence, we have

$$\mathbf{\Omega}_0 = \lambda(\mathbf{I}_m - \mathbf{S}_0)(-\mathbf{L}_0)^{-1}.$$

Using (2) in (8), for $n \geq 1$, we have

$$\begin{aligned}\mathbf{S}_n &= \int_0^\infty [\mathbf{P}(n,t)\mathbf{L}_0 + \mathbf{P}(n-1,t)\mathbf{L}_1][1-A(t)]\,dt \\ &= \frac{1}{\lambda}(\mathbf{\Omega}_n\mathbf{L}_0 + \mathbf{\Omega}_{n-1}\mathbf{L}_1) \quad \text{using (7).} \end{aligned} \tag{10}$$

Thus, we have the following recursive relation for $\mathbf{\Omega}_n$:

$$\mathbf{\Omega}_n = [\mathbf{\Omega}_{n-1}\mathbf{L}_1 - \lambda\mathbf{S}_n](-\mathbf{L}_0)^{-1}, \qquad n \geq 1.$$

Let $\widetilde{P}_{i,j}(n,t)$ be the conditional probability that at least n customers are served in t units of time and the service process is in phase j at the end of the nth service completion, provided initially there were n customers in the system and the service process was in phase i. The probabilities $\widetilde{P}_{i,j}(n,t)$, $n \geq 1, t \geq 0$, then satisfy the Chapman–Kolmogorov equations

$$\widetilde{P}_{i,j}(n,t+\Delta t) = \widetilde{P}_{i,j}(n,t) + \sum_{k=1}^{m} P_{i,k}(n-1,t)[L_1]_{k,j}\Delta t + o(\Delta t), \qquad 1 \leq i,j \leq m,$$

with the initial conditions $\widetilde{P}_{i,j}(n,0) = 0,\ n \geq 1$.

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Rearranging the terms and taking the limit as $\Delta t \to 0$, it reduces to

$$\frac{d}{dt}\widetilde{P}_{i,j}(n,t) = \sum_{k=1}^{m} P_{i,k}(n-1,t)[L_1]_{k,j}, \qquad n \geq 1,$$

for $t \geq 0,\ 1 \leq i,j \leq m$. This system may be written in matrix notation as

$$\frac{d}{dt}\widetilde{\mathbf{P}}(n,t) = \mathbf{P}(n-1,t)\mathbf{L}_1, \qquad n \geq 1, \tag{11}$$

with $\widetilde{\mathbf{P}}(n,0) = \mathbf{0},\ n \geq 1$.

Further, let $\widehat{\mathbf{S}}_n$ denote the matrix of order $m \times m$ whose (i,j)th element represents the probability that n customers are served during an interarrival period of the arrival process and the service process is in phase j at the end of the nth service completion, given that there were exactly n customers in the system and the service process was in phase i at the beginning of the interarrival period. Then

$$\begin{aligned}
\widehat{\mathbf{S}}_n &= \int_0^\infty \widetilde{\mathbf{P}}(n,t)\, dA(t) \\
&= -\int_0^\infty \widetilde{\mathbf{P}}(n,t)\, d[1-A(t)] \\
&= -\Big[\widetilde{\mathbf{P}}(n,t)[1-A(t)]\Big]_0^\infty + \int_0^\infty \frac{d}{dt}\widetilde{\mathbf{P}}(n,t)[1-A(t)]\, dt \\
&= \int_0^\infty \mathbf{P}(n-1,t)\mathbf{L}_1[1-A(t)]\, dt \quad \text{using (11)} \\
&= \frac{1}{\lambda}\mathbf{\Omega}_{n-1}\mathbf{L}_1,\ n \geq 1 \quad \text{using (7).}
\end{aligned} \tag{12}$$

3. ANALYSIS OF THE MODEL

In this section, we carry out the analysis of the distributions of system-length at prearrival, arbitrary and post-departure epochs as well as of the sojourn time for an arriving customer.

3.1. System-length Distribution at Prearrival Epoch

We now consider the system-length distribution just before an arrival epoch. Let the kth customer arrive at time instant $\tau_k,\ k = 0,1,\ldots$, with $\tau_0 = 0$, and let τ_k^- denote the prearrival epoch, that is, the time epoch just before the arrival instant τ_k. Then the state of the system at τ_k^- defined as $\{Y_{\tau_k^-}, J_{\tau_k^-}\}$ is a Markov chain, where $Y_{\tau_k^-}$ represents the number of customers in the system and $J_{\tau_k^-}$ the phase of the service process at the embedded point τ_k^-. In the limiting case, let us define

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$\pi_j^-(n) = \lim_{k\to\infty} P(Y_{\tau_k^-} = n, J_{\tau_k^-} = j)$, $n \geq 0, 1 \leq j \leq m$, where $\pi_j^-(n)$ represents the prearrival epoch probability that there are n customers in the system and the server is busy in phase j, when $n \geq 1$, and idle in phase j, when $n = 0$. Let $\boldsymbol{\pi}^-(n)$ be the $1 \times m$ vector whose jth component is $\pi_j^-(n)$. Now observing the state of the system at two consecutive embedded points, we have a Markov chain whose state space is $\{(n,j),\ n \geq 0;\ j = 1,2,\ldots,m\}$. The one-step transition probability matrix $\boldsymbol{\Theta}$ of the above Markov chain is given by

$$\boldsymbol{\Theta} = \begin{pmatrix} \widehat{\mathbf{S}}_1 & \mathbf{S}_0 & \mathbf{0} & \mathbf{0} & \cdots \\ \widehat{\mathbf{S}}_2 & \mathbf{S}_1 & \mathbf{S}_0 & \mathbf{0} & \cdots \\ \widehat{\mathbf{S}}_3 & \mathbf{S}_2 & \mathbf{S}_1 & \mathbf{S}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now we obtain the VGF of the distribution of number of customers in the system at prearrival epoch. Writing $\boldsymbol{\pi}^- = \boldsymbol{\pi}^-\boldsymbol{\Theta}$, where $\boldsymbol{\pi}^- = [\boldsymbol{\pi}^-(0), \boldsymbol{\pi}^-(1), \boldsymbol{\pi}^-(2), \ldots,]$, we have

$$\boldsymbol{\pi}^-(0) = \sum_{r=0}^{\infty} \boldsymbol{\pi}^-(r)\widehat{\mathbf{S}}_{r+1}, \tag{13}$$

$$\boldsymbol{\pi}^-(n) = \sum_{r=n-1}^{\infty} \boldsymbol{\pi}^-(r)\mathbf{S}_{r-n+1}, \qquad n \geq 1. \tag{14}$$

Multiplying (13) by z^0 and (14) by z^n, summing them, and using the VGF $\boldsymbol{\pi}^{-*}(z) = \sum_{n=0}^{\infty} \boldsymbol{\pi}^-(n)z^n$, we get

$$\boldsymbol{\pi}^{-*}(z) = \frac{\sum_{n=0}^{\infty} \boldsymbol{\pi}^-(n)\left[\widehat{\mathbf{S}}_{n+1} - \sum_{r=n+1}^{\infty} \mathbf{S}_r z^{n-r+1}\right] \text{adj}\left[\mathbf{I}_m - z\mathbf{S}(z^{-1})\right]}{\det\left[\mathbf{I}_m - z\mathbf{S}(z^{-1})\right]}, \tag{15}$$

where adj[$\mathbf{M}$] is the adjoint matrix of a square matrix $\mathbf{M}$ and det[$\mathbf{M}$] is the determinant of $\mathbf{M}$.

To evaluate the vector $\boldsymbol{\pi}^-(n)$, $n \geq 0$ in (15), we first determine an analytic expression for each component of $\boldsymbol{\pi}^{-*}(z)$. Each component $\pi_j^{-*}(z)$ defined as $\pi_j^{-*}(z) = \sum_{n=1}^{\infty} \pi_j^-(n)z^n$ of the VGF $\boldsymbol{\pi}^{-*}(z)$ given in (15) being convergent in $|z| \leq 1$ implies that $\boldsymbol{\pi}^{-*}(z)$ is convergent in $|z| \leq 1$. As each component $\pi_j^{-*}(z)$ is convergent in $|z| \leq 1$, the zeros of det[$\mathbf{I}_m - z\mathbf{S}(z^{-1})$] whose absolute value is less than or equal to one must be the zeros of the numerator of each component of (15). Therefore in making partial fractions of (15), these zeros do not play any role. So, we need to have the knowledge of the zeros of det[$\mathbf{I}_m - z\mathbf{S}(z^{-1})$] whose absolute value is more than one. Thus, we first show that the equation det[$\mathbf{I}_m z - \mathbf{S}(z)$] $= 0$ has exactly m roots (including multiplicity) inside the unit circle $|z| = 1$, see Appendix A. Let these roots be γ_i $(1 \leq i \leq m)$. Since the equation det[$\mathbf{I}_m z - \mathbf{S}(z)$] $= 0$ has m roots γ_i inside the unit circle, the function det[$\mathbf{I}_m - z\mathbf{S}(z^{-1})$] has m zeros $1/\gamma_i$ outside the unit circle. We

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assume that all γ_i $(1 \le i \le m)$ are distinct. Hence, using the analyticity of $\pi_j^{-*}(z)$, the procedure used by Chaudhry and Templeton [9, pp. 119–120] for the $GI/M/1$ model, and the partial-fraction method, we have

$$\pi_j^{-*}(z) = \sum_{i=1}^{m} \frac{k_{ij}}{1 - \gamma_i z}, \qquad 1 \le j \le m, \tag{16}$$

where k_{ij} are constants to be determined.

Setting $z = 1$ in (16) and using $\sum_{j=1}^{m} \pi_j^{-*}(1) = 1$, we have

$$\sum_{j=1}^{m} \sum_{i=1}^{m} \frac{k_{ij}}{1 - \gamma_i} = 1. \tag{17}$$

Now, collecting the coefficient of z^n from both sides of (16), we have

$$\pi_j^{-}(n) = \sum_{i=1}^{m} k_{ij} \gamma_i^n, \qquad 1 \le j \le m, \qquad n \ge 0,$$

and hence

$$\boldsymbol{\pi}^{-}(n) = \left[\sum_{i=1}^{m} k_{i1} \gamma_i^n, \sum_{i=1}^{m} k_{i2} \gamma_i^n, \ldots, \sum_{i=1}^{m} k_{im} \gamma_i^n \right], \qquad n \ge 0. \tag{18}$$

Clearly, it is shown that $\pi_j^{-}(n)$ is a sum of geometric terms.
Using (18) in (13), we have

$$\left[\sum_{i=1}^{m} k_{i1}, \sum_{i=1}^{m} k_{i2}, \ldots, \sum_{i=1}^{m} k_{im} \right] = \sum_{r=0}^{\infty} \left[\sum_{i=1}^{m} k_{i1} \gamma_i^r, \sum_{i=1}^{m} k_{i2} \gamma_i^r, \ldots, \sum_{i=1}^{m} k_{im} \gamma_i^r \right] \widehat{\mathbf{S}}_{r+1}. \tag{19}$$

Using (18) in (14) for $n = 1, 2, \ldots, m-1$, we have

$$\left[\sum_{i=1}^{m} k_{i1} \gamma_i^n, \sum_{i=1}^{m} k_{i2} \gamma_i^n, \ldots, \sum_{i=1}^{m} k_{im} \gamma_i^n \right] = \sum_{r=n-1}^{\infty} \left[\sum_{i=1}^{m} k_{i1} \gamma_i^r, \sum_{i=1}^{m} k_{i2} \gamma_i^r, \ldots, \sum_{i=1}^{m} k_{im} \gamma_i^r \right] \mathbf{S}_{r-n+1}. \tag{20}$$

Equations (19) and (20) give m^2 simultaneous equations. Now, replacing any one equation among m^2 equations by (17), we get another m^2 simultaneous equations in m^2 unknowns, k_{ij}'s $(1 \le i, j \le m)$. Solving these m^2 equations, we get the m^2 unknowns. Note that one equation is redundant.

Remark 3.1: If some roots of $\det[\mathbf{I}_m z - \mathbf{S}(z)] = 0$ inside the unit circle $(|z| < 1)$ are repeated, the above procedure needs to be modified slightly to get the unknown k_{ij}'s $(1 \le i, j \le m)$.

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3.2. Sojourn-time Distribution

In this section, we obtain certain characteristics related to the sojourn time of an arriving customer in the system. By the sojourn time we mean the total time spent by a customer in the system (from its arrival until departure). Here, let $\mathbf{\Psi}_n(x)$ denote the matrix whose (i,j)th element represents the probability that n customers will be served within a time x and the service phase is j at time x, provided at the initial instant of arrival epoch there were n customers in the system and the service process was in phase i. Let $\mathbf{\Psi}_n^{\bullet}(s)$ denote the Laplace–Stieltjes transform of the matrix $\mathbf{\Psi}_n(x)$. Since the probability that the service of a customer is completed in the time interval $(x, x+dx]$ is given by the matrix $e^{\mathbf{L}_0 x}\mathbf{L}_1 dx + o(dx)$ and the total service time of n customers is the sum of their service times, we have

$$\mathbf{\Psi}_n^{\bullet}(s) = [\mathbf{\Psi}_1^{\bullet}(s)]^n, \qquad n \geq 1,$$

where

$$\begin{aligned}\mathbf{\Psi}_1^{\bullet}(s) &= \int_0^{\infty} e^{-sx} e^{\mathbf{L}_0 x}\mathbf{L}_1\, dx \\ &= (s\mathbf{I}_m - \mathbf{L}_0)^{-1}\mathbf{L}_1.\end{aligned}$$

Hence, the Laplace–Stieltjes transform $\mathbf{W}^{\bullet}(s)$ of the stationary distribution of the sojourn time of an arriving customer is given by

$$\mathbf{W}^{\bullet}(s) = \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n)[\mathbf{\Psi}_1^{\bullet}(s)]^{n+1}.$$

Differentiating the above expression at the point $s = 0$, we obtain the mean sojourn time W of an arriving customer and it is given by

$$W = \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n) \sum_{k=1}^{n+1} [(-\mathbf{L}_0)^{-1}\mathbf{L}_1]^{n+1-k}(-\mathbf{L}_0)^{-1}[(-\mathbf{L}_0)^{-1}\mathbf{L}_1]^k \mathbf{e}.$$

Using $(\mathbf{L}_0 + \mathbf{L}_1)\mathbf{e} = \mathbf{0}$ in the above equation, we obtain

$$W = \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n) \sum_{k=0}^{n} [(-\mathbf{L}_0)^{-1}\mathbf{L}_1]^k (-\mathbf{L}_0)^{-1}\mathbf{e}.$$

3.3. System-length Distribution at an Arbitrary Epoch

We now derive explicit expressions for the steady-state system-length distribution $\boldsymbol{\pi}(n) = [\pi_1(n), \pi_2(n), \ldots, \pi_m(n)]$, $n \geq 0$, at an arbitrary epoch using the classical argument based on Markov renewal theory as given in Chaudhry and Templeton [9, pp. 74–77], and Ferreira and Pacheco [13], which relates the steady-state system-length distribution at an arbitrary epoch to that at the corresponding prearrival epoch.

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Let $\mathbf{\Phi}_n$ denote the matrix of order $m \times m$ whose (i,j)th element represents the limiting probability that n customers are served during an elapsed interarrival time of the arrival process and the service process is in phase j at the end of the nth service completion, given that there were exactly n customers in the system and the service process was in phase i at the beginning of the interarrival period. Then, from Markov renewal theory, we have

$$\mathbf{\Phi}_n = \lambda \int_0^\infty \widetilde{\mathbf{P}}(n,t)[1 - A(t)]\, dt, \qquad n \geq 1.$$

Using (11) in the above, we have

$$\mathbf{\Phi}_n = \lambda \int_0^\infty \left(\int_0^t \mathbf{P}(n-1,x)\mathbf{L}_1\, dx \right) [1 - A(t)]\, dt, \qquad n \geq 1. \tag{21}$$

Using (1) in (21), for $n = 1$, we obtain

$$\begin{aligned} \mathbf{\Phi}_1 &= \lambda \int_0^\infty \left(\int_0^t e^{\mathbf{L}_0 x}\mathbf{L}_1\, dx \right) [1 - A(t)]\, dt \\ &= (\mathbf{I}_m - \mathbf{\Omega}_0)\,(-\mathbf{L}_0)^{-1}\,\mathbf{L}_1, \end{aligned} \tag{22}$$

taking into account (7) and the fact that $\mathbf{P}(0,t) = e^{\mathbf{L}_0 t}$, $t \geq 0$.
Using (2) in (21), we have

$$\mathbf{\Phi}_n = \lambda \int_0^\infty \int_0^t \left(\frac{d}{dx}\mathbf{P}(n,x) - \mathbf{P}(n,x)\mathbf{L}_0 \right) dx[1 - A(t)]\, dt$$

which, in view of (7), leads to

$$\mathbf{\Phi}_{n+1} = (\mathbf{\Phi}_n - \mathbf{\Omega}_n)\,(-\mathbf{L}_0)^{-1}\,\mathbf{L}_1, \qquad n \geq 1. \tag{23}$$

Using similar results of Markov renewal theory and semi-Markov processes, see, for example, Çinlar [10] or Lucantoni and Neuts [17], we obtain

$$\boldsymbol{\pi}(0) = \sum_{r=0}^{\infty} \boldsymbol{\pi}^-(r)\mathbf{\Phi}_{r+1}, \tag{24}$$

$$\boldsymbol{\pi}(n) = \sum_{k=n}^{\infty} \boldsymbol{\pi}^-(k-1)\mathbf{\Omega}_{k-n} \qquad n \geq 1. \tag{25}$$

The mean system length can be obtained as

$$L = \sum_{n=1}^{\infty} n\boldsymbol{\pi}(n)\mathbf{e}.$$

From Little's law, we also have mean sojourn time W as

$$W = \frac{L}{\lambda}.$$

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Remark 3.2: It may be pointed out that $\sum_{n=0}^{\infty} \boldsymbol{\pi}(n) \neq \overline{\boldsymbol{\pi}}$ as the service process is interrupted during idle periods of the system.

System-length Distribution at Post-departure Epoch It is also possible to derive the probabilities for the states of the system immediately after a service completion takes place. Let $\boldsymbol{\pi}^{+}(n) = [\pi_1^{+}(n), \pi_2^{+}(n), \ldots, \pi_m^{+}(n)]$, $n \geq 0$ be the $1 \times m$ vector whose ith component $\pi_i^{+}(n)$ represents the post-departure epoch probability that there are n customers in the system immediately after a service completion and the server is busy in phase i when $n \geq 1$, and idle in phase i when $n = 0$. The post-departure epoch thus occurs immediately after the server has either reduced the queue or has become idle. Hence, using level-crossing arguments given in Chaudhry and Templeton [9, p. 299], we have

$$\boldsymbol{\pi}^{+}(n) = \frac{1}{\lambda}\boldsymbol{\pi}(n+1)\mathbf{L}_1, \qquad n \geq 0. \tag{26}$$

Using Theorem A.4 given in Appendix A, it may be noted that $\sum_{n=1}^{\infty} \boldsymbol{\pi}(n)\mathbf{L}_1\mathbf{e} = \lambda$. Further, one may note, see Theorem A.5 given in Appendix A, that the relations

$$\boldsymbol{\pi}^{+}(0) = \boldsymbol{\pi}^{-}(0),$$
$$\boldsymbol{\pi}^{+}(n)\mathbf{e} = \boldsymbol{\pi}^{-}(n)\mathbf{e}, \qquad n \geq 1$$

also hold, as they should.

4. THE MODEL WITH IDLE–RESTART SERVICE PHASE DISTRIBUTION

We consider a C-MSP with a new service restart phase distribution such that if the server becomes idle at a certain point the service phase still keeps changing during the subsequent idle period of the system. Then, when a new customer arrives, the service process starts with a new phase distribution f_j, $j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} f_j = 1$, independent of the path followed before by the service process. Therefore, C-MSP is characterized by the matrices $\mathbf{L}_0$, $\mathbf{L}_1$ along with the vector $\mathbf{f} = [f_1, f_2, \ldots, f_m]$. This kind of idle–restart service phase distribution was suggested in [18] and subsequently it was used in [2,15].

The steady-state system-length distributions at various epochs and sojourn-time distribution of an arriving customer can be obtained along the same lines as discussed in Section 3. For this, we need to replace $\widehat{\mathbf{S}}_n$ and $\boldsymbol{\Phi}_n$ in (13) and (24), respectively, with the results given in this section.

Let $\widehat{\mathbf{S}}_n$ denote the matrix of order $m \times m$ whose (i,j)th element represents the probability that n customers are served during an interarrival period of the arrival process and the service process of next customer starts in phase j, given that there were n customers in the system and the service process was in phase i at the beginning

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of the interarrival period. Then

$$\widehat{\mathbf{S}}_n = \int_0^\infty \sum_{r=n}^{\infty} \mathbf{P}(n,t)\mathbf{ef}\, dA(t), \qquad n \geq 1. \tag{27}$$

Using (5) in (27), we have

$$\widehat{\mathbf{S}}_n = \left(\mathbf{I}_m - \sum_{k=0}^{n-1} \mathbf{S}_k\right)\mathbf{ef}, \qquad n \geq 1.$$

Let $\mathbf{\Phi}_n$ denote the matrix of order $m \times m$ whose (i,j)th element represents the limiting probability that n customers are served during an elapsed interarrival time of the arrival process and the service process of next customer starts in phase j, given that there were n customers in the system and the service process was in phase i at the beginning of the interarrival period. Then, from Markov renewal theory, we have

$$\mathbf{\Phi}_n = \lambda \int_0^\infty \sum_{r=n}^{\infty} \mathbf{P}(n,t)\mathbf{ef}[1 - A(t)]\, dt, \qquad n \geq 1. \tag{28}$$

Using (7) in (28), we have

$$\mathbf{\Phi}_n = \left(\mathbf{I}_m - \sum_{r=0}^{n-1} \mathbf{\Omega}_r\right)\mathbf{ef}, \qquad n \geq 1.$$

It may be noted here that

$$\boldsymbol{\pi}(0)\mathbf{e} \neq 1 - \rho$$

and

$$\boldsymbol{\pi}^+(0) \neq \boldsymbol{\pi}^-(0).$$

This is due to the fact that the service phase of the underlying Markov chain still keeps changing during idle periods of the system. However, the relations $\boldsymbol{\pi}^+(0)\mathbf{ef} = \boldsymbol{\pi}^-(0)\mathbf{ef}$ and $\boldsymbol{\pi}^+(n)\mathbf{e} = \boldsymbol{\pi}^-(n)\mathbf{e}$, $n \geq 1$, hold, as they should.

Remark 4.1: We may also evaluate $\widehat{\mathbf{S}}_n$, $n \geq 1$, from (12), and $\mathbf{\Phi}_n$, $n \geq 1$, from (22) and (23) by postmultiplying $\mathbf{ef}$.

Remark 4.2: It may be pointed out that $\sum_{n=0}^{\infty} \boldsymbol{\pi}(n) \neq \overline{\boldsymbol{\pi}}$ as the service process is interrupted by introducing a new phase distribution $\mathbf{f}$.

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5. NUMERICAL RESULTS

For numerical work, we take the interarrival-time distribution to be of phase-type having the representation $(\boldsymbol{\alpha}, \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension ν. Then

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \boldsymbol{\alpha})(\mathbf{L}(z) \oplus \mathbf{T})^{-1}(\mathbf{I}_m \otimes \mathbf{T}\mathbf{e}_\nu)$$

with $\mathbf{L}(z) \oplus \mathbf{T} = (\mathbf{L}(z) \otimes \mathbf{I}_\nu) + (\mathbf{I}_m \otimes \mathbf{T})$, where $\otimes$ and $\oplus$ are used for the Kronecker product and the Kronecker sum, respectively. For the derivation of $\mathbf{S}(z)$, see Theorem B.2 in Appendix B. Knowing that each element of $\mathbf{L}(z)$ is a polynomial in z, each element of $\mathbf{L}(z) \oplus \mathbf{T}$ is also a polynomial in z and hence the determinant of $\mathbf{L}(z) \oplus \mathbf{T}$ is a polynomial. This implies that each element of $(\mathbf{L}(z) \oplus \mathbf{T})^{-1}$ is a rational function in z. Hence, from the above expression of $\mathbf{S}(z)$, we can immediately say that each element of $\mathbf{S}(z)$ is a rational function in z with the same denominator.

We have carried out an extensive numerical work based on the procedure discussed in this paper by considering different service matrices *C-MSP*$(\mathbf{L}_0, \mathbf{L}_1)$ and phase-type interarrival-time distributions *PH*$(\boldsymbol{\alpha}, \mathbf{T})$. Further, we tested our procedure for different values of $\mathbf{L}_0$ and $\mathbf{L}_1$ with reasonable positive lag-1 correlation coefficient, that is, correlation coefficient between two consecutive service times. Several outputs were generated and the results were matched against some existing results, for example, Gupta and Banik [15]. In all the cases matching was found to be perfect. All the calculations were performed on a PC having configurations as Intel(R) Core(TM)2 Duo CPU Processor T9300 @ 2.50 GHz with 4.00 GB of RAM, using Maple 12. Further, though all the numerical results were carried out in 190 decimal places, they are reported here in seven decimal places due to lack of space. Some sample outputs are shown in Tables 1–3, where the results are given for the *PH*$(\boldsymbol{\alpha}, \mathbf{T})$/*C-MSP*$(\mathbf{L}_0, \mathbf{L}_1)$/1 queue. It is found that the mean sojourn time W using Little's rule given in Table 2 matches with the result obtained from the sojourn-time distribution. We choose the service matrices $\mathbf{L}_0$ and $\mathbf{L}_1$ with positive lag-1 correlation coefficient of 0.5381025 from Chaudhry, Singh, and Gupta [8] as

$$\mathbf{L}_0 = \begin{bmatrix} -15.449 & 0.116 & 0.052 & 0.000 \\ 0.092 & -5.069 & 0.000 & 0.062 \\ 0.000 & 0.213 & -4.832 & 4.430 \\ 0.059 & 5.033 & 0.000 & -5.321 \end{bmatrix},$$

$$\mathbf{L}_1 = \begin{bmatrix} 15.068 & 0.157 & 0.000 & 0.056 \\ 0.123 & 0.047 & 4.745 & 0.000 \\ 0.084 & 0.000 & 0.000 & 0.105 \\ 0.051 & 0.000 & 0.077 & 0.101 \end{bmatrix}$$

and the interarrival-time representation as

$$\boldsymbol{\alpha} = \begin{bmatrix} 0.22 & 0.33 & 0.45 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} -7.423 & 0.000 & 3.812 \\ 8.542 & -9.942 & 1.000 \\ 5.210 & 0.000 & -8.640 \end{bmatrix}.$$

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TABLE 1. System-length distribution at prearrival epoch

n	$\pi_1^-(n)$	$\pi_2^-(n)$	$\pi_3^-(n)$	$\pi_4^-(n)$	$\boldsymbol{\pi}^-(n)\mathbf{e}$
0	0.4032867	0.0042989	0.0105668	0.0019870	0.4201394
1	0.0755046	0.0071079	0.0091968	0.0060182	0.0978275
2	0.0160503	0.0074954	0.0081615	0.0068075	0.0385146
3	0.0051611	0.0073521	0.0075018	0.0067036	0.0267186
4	0.0030663	0.0070470	0.0070175	0.0063892	0.0235200
5	0.0025692	0.0066933	0.0066078	0.0060435	0.0219138
10	0.0018724	0.0050446	0.0049594	0.0045402	0.0164167
50	0.0001902	0.0005125	0.0005038	0.0004612	0.0016678
100	0.0000109	0.0000294	0.0000289	0.0000265	0.0000957
150	0.0000006	0.0000017	0.0000017	0.0000015	0.0000055
⋮	⋮	⋮	⋮	⋮	⋮
≥221	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
Sum	0.5480103	0.1540982	0.1612395	0.1366519	1.0000000

TABLE 2. System-length distribution at an arbitrary epoch

n	$\pi_1(n)$	$\pi_2(n)$	$\pi_3(n)$	$\pi_4(n)$	$\boldsymbol{\pi}(n)\mathbf{e}$
0	0.3905207	0.0041593	0.0098459	0.0019074	0.4064333
1	0.0854676	0.0070268	0.0092894	0.0058718	0.1076556
2	0.0178619	0.0074895	0.0082187	0.0067896	0.0403597
3	0.0054973	0.0073668	0.0075393	0.0067188	0.0271222
4	0.0031351	0.0070686	0.0070467	0.0064113	0.0236618
5	0.0025892	0.0067167	0.0066334	0.0060660	0.0220053
10	0.0018795	0.0050636	0.0049781	0.0045572	0.0164783
50	0.0001909	0.0005144	0.0005057	0.0004630	0.0016741
100	0.0000110	0.0000295	0.0000290	0.0000266	0.0000960
150	0.0000006	0.0000017	0.0000017	0.0000015	0.0000055
⋮	⋮	⋮	⋮	⋮	⋮
≥221	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
Sum	0.5476054	0.1543590	0.1611821	0.1368536	1.0000000
		$L = 9.0464098$, $W = 2.8286412$			

These lead to $\lambda = 3.1981468$, $\mu^\star = 5.3880162$, $\rho = 0.5935667$, and

$$\overline{\boldsymbol{\pi}} = \begin{bmatrix} 0.2646453 & 0.2530460 & 0.2549608 & 0.2273479 \end{bmatrix}.$$

6. CONCLUSION AND FUTURE SCOPE

In this paper, we have presented a simple procedure to evaluate the steady-state system-length distribution at three epochs of time (prearrival, arbitrary, and post-departure) and sojourn-time distribution of the $GI/C\text{-}MSP/1$ queue. The proposed analysis is based on roots of the associated characteristic equation of the VGF of system-length

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TABLE 3. System-length distribution at post-departure epoch

n	$\pi_1^+(n)$	$\pi_2^+(n)$	$\pi_3^+(n)$	$\pi_4^+(n)$	$\boldsymbol{\pi}^+(n)\mathbf{e}$
0	0.4032867	0.0042990	0.0105668	0.0019870	0.4201394
1	0.0847682	0.0009869	0.0112754	0.0007970	0.0978275
2	0.0264888	0.0003781	0.0110917	0.0005560	0.0385146
3	0.0153302	0.0002578	0.0106418	0.0004887	0.0267186
4	0.0127281	0.0002258	0.0101114	0.0004547	0.0235200
5	0.0117050	0.0002102	0.0095705	0.0004281	0.0219138
10	0.0087390	0.0001574	0.0071989	0.0003214	0.0164167
50	0.0008878	0.0000160	0.0007313	0.0000326	0.0016678
100	0.0000509	0.0000009	0.0000419	0.0000019	0.0000957
150	0.0000029	0.0000000	0.0000024	0.0000001	0.0000055
⋮	⋮	⋮	⋮	⋮	⋮
≥221	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
Sum	0.7520043	0.0099188	0.2260960	0.0119809	1.0000000

distribution. Motivation behind the procedure is to acquaint researchers and practitioners with an alternative procedure, which is analytically simple, easy to implement, and computationally efficient. The proposed method can be applied to get computational results of more complex models such as bulk arrival and/or bulk service queue.

Acknowledgments

The first author was supported partially by NSERC. The second and third authors received partial support from Portuguese National Funds through FCT (Fundação para a Ciência e a Tecnologia) under the scope of projects PEst-OE/MAT/UI0822/2011 and PTDC/MAT-STA/3169/2012. The second author was supported by FCT research grant SFRH/BPD/64372/2009.

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APPENDIX A

THEOREM A.1: *Every function $z - S_{i,i}(z)$, $1 \le i \le m$, has exactly one zero inside the unit circle.*

PROOF: Consider absolute values of $f(z) = z$ and $g(z) = -S_{i,i}(z)$ on the circle $|z| = 1 - \delta$, where δ is positive and sufficiently small. First, note that $\mathbf{S} = \mathbf{S}(1)$ is the embedded transition probability matrix of $J(t)$ in a random amount of time with the distribution of an interarrival time. Since the state space of $J(t)$ is finite, then $\mathbf{S}$ is an irreducible and aperiodic discrete-time Markov chain, which is necessarily ergodic. This implies, in particular, that $S_{i,i}(1) < 1$, for $1 \le i \le m$. As Eq. (6) yields $\mathbf{S}'(1) = \sum_{n=1}^{\infty} n\mathbf{S}_n$ which represents the mean number of customers served and the phase changes of the underlying Markov chain, we have $S'_{i,i}(1) \ge 0$. Now $|g(z)| \le S_{i,i}(|z|) \le S_{i,i}(1) - \delta S'_{i,i}(1) + o(\delta)$ and $|f(z)| = 1 - \delta$. Further, $S_{i,i}(1) < 1$ leads to $|g(z)| < |f(z)|$ as δ is positive and sufficiently small. In view of this, using the well-known Rouché's theorem, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the unit circle. It is obvious that $f(z) = z$ has exactly one zero inside the unit circle. Thus, $f(z) + g(z) = z - S_{i,i}(z)$ has exactly one zero inside the unit circle. ■

THEOREM A.2: *The following inequalities hold on the circle $|z| = 1 - \delta$:*

$$|z - S_{i,i}(z)| > \sum_{j=1,\, j\neq i}^{m} |S_{i,j}(z)|, \qquad 1 \le i \le m. \tag{A.1}$$

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PROOF: On the circle $|z| = 1 - \delta$, using the Taylor series expansion, we have

$$\begin{aligned}
|z - S_{i,i}(z)| &\geq |z| - |S_{i,i}(z)| \\
&\geq |z| - S_{i,i}(|z|) \\
&= 1 - \delta - (S_{i,i}(1) - \delta S'_{i,i}(1)) + o(\delta) \\
&= \sum_{j=1,\, j\neq i}^{m} S_{i,j}(1) - \delta(1 - S'_{i,i}(1)) + o(\delta), \qquad 1 \leq i \leq m,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1,\, j\neq i}^{m} |S_{i,j}(z)| &\leq \sum_{j=1,\, j\neq i}^{m} S_{i,j}(|z|) \\
&\leq \sum_{j=1,\, j\neq i}^{m} S_{i,j}(1) - \delta \sum_{j=1,\, j\neq i}^{m} S'_{i,j}(1) + o(\delta), \qquad 1 \leq i \leq m.
\end{aligned}$$

Now assume the following inequality holds for some $i \in \{1, 2, \ldots, m\}$:

$$|z - S_{i,i}(z)| \leq \sum_{j=1,\, j\neq i}^{m} |S_{i,j}(z)|.$$

This implies

$$\sum_{j=1,\, j\neq i}^{m} S_{i,j}(1) - \delta(1 - S'_{i,i}(1)) + o(\delta) \leq \sum_{j=1,\, j\neq i}^{m} S_{i,j}(1) - \delta \sum_{j=1,\, j\neq i}^{m} S'_{i,j}(1) + o(\delta)$$

$$\Rightarrow 1 - S'_{i,i}(1) \geq \sum_{j=1,\, j\neq i}^{m} S'_{i,j}(1)$$

$$\Rightarrow 1 \geq \sum_{j=1}^{m} S'_{i,j}(1) = \mathbf{S}'(1)\mathbf{e}$$

$$\Rightarrow \sum_{i=1}^{m} \overline{\pi}_i \geq \sum_{i=1}^{m} \overline{\pi}_i \sum_{j=1}^{m} S'_{i,j}(1) = \overline{\boldsymbol{\pi}}\mathbf{S}'(1)\mathbf{e}$$

$$\Rightarrow 1 \geq \frac{1}{\rho} \Rightarrow \rho \geq 1.$$

This contradicts the system stability condition $\rho < 1$, and hence Eq. (A.1) is satisfied. ■

THEOREM A.3: *The determinant* $\det[z\mathbf{I}_m - \mathbf{S}(z)]$ *has exactly m zeros (including multiplicity) inside the unit circle.*

PROOF: Mathematical induction is used to prove this theorem. Let us denote

$$D_n(z) = \det[z\mathbf{I}_n - \mathbf{S}(z)] \tag{A.2}$$

represents the determinant of the underlying square matrix of order n.

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1. We show that the statement is true for $n = 1$.
   For $n = 1$, Eq. (A.2) becomes $D_1(z) = z - S(z)$, where $S(z) = S_{1,1}(z)$. This is the characteristic function for the classical $GI/M/1$ queue. It has been proved at several places that $z = S(z)$ has exactly one root inside the unit circle, see, for example, Chaudhry and Templeton [9, p. 67].
2. For $n = 2$, Eq. (A.2) becomes

$$D_2(z) = \begin{vmatrix} z - S_{1,1}(z) & -S_{1,2}(z) \\ -S_{2,1}(z) & z - S_{2,2}(z) \end{vmatrix}$$
$$= -S_{2,1}(z)C_{2,1}(z) + [z - S_{2,2}(z)]D_1(z). \quad \textbf{(A.3)}$$

   where $C_{2,1}(z) = S_{1,2}(z)$ is the cofactor of $-S_{2,1}(z)$, and $D_1(z) = z - S_{1,1}(z)$.
   Again, we can write (A.3) as

$$\left|\frac{D_2(z) - [z - S_{2,2}(z)]D_1(z)}{[z - S_{2,2}(z)]D_1(z)}\right| = \left|\frac{-S_{2,1}(z)C_{2,1}(z)}{[z - S_{2,2}(z)]D_1(z)}\right|$$
$$= \frac{|S_{2,1}(z)|\,|y_{2,1}(z)|}{|z - S_{2,2}(z)|} < 1,$$

   where

$$|y_{2,1}(z)| = \frac{|C_{2,1}(z)|}{|D_1(z)|} = \frac{|S_{1,2}(z)|}{|z - S_{1,1}(z)|} < 1$$

   and

$$\frac{|S_{2,1}(z)|}{|z - S_{2,2}(z)|} < 1$$

   by Theorem A.2 for $m = 2$.

   Hence, by Rouché's theorem $D_2(z)$ has two roots inside the unit circle, $|z| = 1$, since $(z - S_{2,2}(z))D_1(z)$ has two roots.
3. We assume that the statement is true for $n = m - 1$. It must then be shown that the statement holds for $n = m$.

The determinant $D_n(z)$ for $n = m$ is given by

$$D_m(z) = \begin{vmatrix} z - S_{1,1}(z) & -S_{1,2}(z) & \cdots & -S_{1,m-1}(z) & -S_{1,m}(z) \\ -S_{2,1}(z) & z - S_{2,2}(z) & \cdots & -S_{2,m-1}(z) & -S_{2,m}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -S_{m-1,1}(z) & -S_{m-1,2}(z) & \cdots & z - S_{m-1,m-1}(z) & -S_{m-1,m}(z) \\ -S_{m,1}(z) & -S_{m,2}(z) & \cdots & -S_{m,m-1}(z) & z - S_{m,m}(z) \end{vmatrix}. \quad \textbf{(A.4)}$$

Now we can rewrite (A.4) in the following way:

$$D_m(z) = -\sum_{j=1}^{m-1} S_{m,j}(z)C_{m,j}(z) + [z - S_{m,m}(z)]D_{m-1}(z), \quad \textbf{(A.5)}$$

where $C_{m,j}(z)$ is the cofactor of $-S_{m,j}(z)$.

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Again, we can write (A.5) as

$$\left|\frac{D_m(z)-[z-S_{m,m}(z)]D_{m-1}(z)}{[z-S_{m,m}(z)]D_{m-1}(z)}\right| = \left|\frac{-\sum_{j=1}^{m-1} S_{m,j}(z)C_{m,j}(z)}{[z-S_{m,m}(z)]D_{m-1}(z)}\right| \leq \frac{\sum_{j=1}^{m-1}\left|S_{m,j}(z)\right|\left|y_{m,j}(z)\right|}{\left|z-S_{m,m}(z)\right|}, \tag{A.6}$$

where $|y_{m,j}(z)| = |C_{m,j}(z)|/|D_{m-1}(z)|$ is the unique solution (by Cramer's rule, provided $D_{m-1}(z) \neq 0$) of the system of equations

$$\begin{pmatrix} z-S_{1,1}(z) & -S_{1,2}(z) & \cdots & -S_{1,m-1}(z) \\ -S_{2,1}(z) & z-S_{2,2}(z) & \cdots & -S_{2,m-1}(z) \\ \vdots & \vdots & \vdots & \vdots \\ -S_{m-1,1}(z) & -S_{m-1,2}(z) & \cdots & z-S_{m-1,m-1}(z) \end{pmatrix} \cdot \begin{pmatrix} y_{m,1}(z) \\ y_{m,2}(z) \\ \vdots \\ y_{m,m-1}(z) \end{pmatrix} = \begin{pmatrix} -S_{1,m}(z) \\ -S_{2,m}(z) \\ \vdots \\ -S_{m-1,m}(z) \end{pmatrix}. \tag{A.7}$$

The kth equation of (A.7) is given by

$$[z-S_{k,k}(z)]y_{m,k}(z) - \sum_{j=1, j\neq k}^{m-1} S_{k,j}(z)y_{m,j}(z) = -S_{k,m}(z), \qquad 1 \leq k \leq m-1. \tag{A.8}$$

Now, we assume the contrary that

$$\mathrm{Max}_j|y_{m,j}(z)| = |y_{m,k}(z)| \geq 1.$$

Because of our assumption $|[y_{m,j}(z)]/[y_{m,k}(z)]| \leq 1$ and $|1/[y_{m,k}(z)]| \leq 1$, we can rewrite (A.8) in the form

$$|z-S_{k,k}(z)| \leq \sum_{j=1, j\neq k}^{m-1} \left|S_{k,j}(z)\right|\left|\frac{y_{m,j}(z)}{y_{m,k}(z)}\right| + \left|S_{k,m}(z)\right|\left|\frac{1}{y_{m,k}(z)}\right| \leq \sum_{j=1, j\neq k}^{m} \left|S_{k,j}(z)\right|.$$

This contradicts Theorem A.2. Thus we have $|y_{m,j}(z)| < 1$.

Hence, using Theorem A.2 and $|y_{m,j}(z)| < 1$, the right-hand side expression of Eq. (A.6) is less than one, and therefore $|g(z)| < |f(z)|$, where $f(z) = [z - S_{m,m}(z)]D_{m-1}(z)$ and $g(z) = D_m(z) - [z - S_{m,m}(z)]D_{m-1}(z)$. By Rouché's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the unit circle, $|z| = 1$. By our assumption $D_{m-1}(z)$ has $(m-1)$ zeros, and $[z - S_{m,m}(z)]$ has one zero by Theorem A.1. This implies $f(z)$ has m zeros inside the unit circle. Hence, it has now been proved by mathematical induction that $f(z) + g(z) = D_m(z)$ has exactly m zeros inside the unit circle, $|z| = 1$. ■

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Remark A.1: For alternative proofs of this theorem, one may see references such as Gail, Hantler, and Taylor [14] and Dudin and Klimenok [12]. However, it may be pointed out that the proof given here is simpler than those given in Gail et al. [14] and Dudin and Klimenok [12].

THEOREM A.4: *The following equality holds:*

$$\boldsymbol{\pi}(0)\mathbf{e} = 1 - \rho.$$

PROOF: Adding (13) and (14) for n from 1 to ∞, and then using (12), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n) &= \frac{1}{\lambda}\sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n)\boldsymbol{\Omega}_n\mathbf{L}_1 + \sum_{n=1}^{\infty}\sum_{r=n-1}^{\infty} \boldsymbol{\pi}^{-}(r)\mathbf{S}_{r-n+1} \\ &= \frac{1}{\lambda}\sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n)\boldsymbol{\Omega}_n\mathbf{L}_1 + \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n)\mathbf{S}_0 + \sum_{n=1}^{\infty}\sum_{r=n}^{\infty} \boldsymbol{\pi}^{-}(r)\mathbf{S}_{r-n+1}.\end{aligned}$$

Using the expressions for $\mathbf{S}_0$ and $\mathbf{S}_n$, $n \geq 1$ given in (9) and (10), respectively, in the above, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n) &= \frac{1}{\lambda}\sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n)\boldsymbol{\Omega}_n\mathbf{L}_1 + \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n)\left(\mathbf{I}_m + \frac{1}{\lambda}\boldsymbol{\Omega}_0\mathbf{L}_0\right) \\ &\quad + \frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{r=n}^{\infty} \boldsymbol{\pi}^{-}(r)\left(\boldsymbol{\Omega}_{r-n+1}\mathbf{L}_0 + \boldsymbol{\Omega}_{r-n}\mathbf{L}_1\right) \\ &= \sum_{n=0}^{\infty} \boldsymbol{\pi}^{-}(n) + \frac{1}{\lambda}\sum_{n=1}^{\infty}\sum_{r=n}^{\infty} \boldsymbol{\pi}^{-}(r-1)\boldsymbol{\Omega}_{r-n}\left(\mathbf{L}_0 + \mathbf{L}_1\right). \qquad \textbf{(A.9)}\end{aligned}$$

Using (25) in (A.9), we have

$$\sum_{n=1}^{\infty} \boldsymbol{\pi}(n)\left(\mathbf{L}_0 + \mathbf{L}_1\right) = \mathbf{0}.$$

This relation implies that

$$\sum_{n=1}^{\infty} \boldsymbol{\pi}(n) = C\overline{\boldsymbol{\pi}}, \qquad \textbf{(A.10)}$$

where C is the normalization constant.

Post-multiplying (A.10) by $\mathbf{L}_1\mathbf{e}$, we have

$$C\mu^{\star} = \sum_{n=1}^{\infty} \boldsymbol{\pi}(n)\mathbf{L}_1\mathbf{e}.$$

Using (25) in the above equation, we have

$$\begin{aligned}C\mu^{\star} &= \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-n}\mathbf{L}_1\mathbf{e} \\ &= \sum_{k=1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-1}\mathbf{L}_1\mathbf{e} + \sum_{n=2}^{\infty}\sum_{k=n}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-n}\mathbf{L}_1\mathbf{e}. \qquad \textbf{(A.11)}\end{aligned}$$

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Now, using (12) and (13) successively, we obtain

$$\sum_{k=1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-1}\mathbf{L}_1\mathbf{e} = \lambda \sum_{k=1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\widehat{\mathbf{S}}_k\mathbf{e} = \lambda\boldsymbol{\pi}^{-}(0)\mathbf{e}. \tag{A.12}$$

Similarly, for $n \geq 2$, using (9), (14), and (10) successively, we obtain

$$\begin{aligned}\sum_{k=n}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-n}\mathbf{L}_1\mathbf{e} &= \lambda\boldsymbol{\pi}^{-}(n-1)\mathbf{e} - \lambda\boldsymbol{\pi}^{-}(n)\mathbf{e} + \sum_{k=n+1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-n-1}\mathbf{L}_1\mathbf{e} \\ &= \lambda\boldsymbol{\pi}^{-}(n-1)\mathbf{e} - \lambda\boldsymbol{\pi}^{-}(n)\mathbf{e} + \lambda\boldsymbol{\pi}^{-}(n)\mathbf{e} - \lambda\boldsymbol{\pi}^{-}(n+1)\mathbf{e} \\ &\quad + \sum_{k=n+2}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-n-2}\mathbf{L}_1\mathbf{e} \\ &= \cdots = \lambda\boldsymbol{\pi}^{-}(n-1)\mathbf{e}. \end{aligned} \tag{A.13}$$

Using (A.12) and (A.13) in (A.11), we have

$$C = \rho. \tag{A.14}$$

Now, using (A.14) with the normalization condition $\sum_{n=0}^{\infty} \boldsymbol{\pi}(n)\mathbf{e} = 1$ in (A.10), we have

$$\boldsymbol{\pi}(0)\mathbf{e} = 1 - \rho.$$

■

THEOREM A.5: *The following equalities hold:*

$$\begin{aligned}\boldsymbol{\pi}^{+}(0) &= \boldsymbol{\pi}^{-}(0), \\ \boldsymbol{\pi}^{+}(n)\mathbf{e} &= \boldsymbol{\pi}^{-}(n)\mathbf{e}, \quad n \geq 1.\end{aligned}$$

PROOF: Using (25) in (26), we have

$$\boldsymbol{\pi}^{+}(n) = \frac{1}{\lambda} \sum_{k=n+1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-n-1}\mathbf{L}_1, \qquad n \geq 0.$$

For $n = 0$, we have

$$\begin{aligned}\boldsymbol{\pi}^{+}(0) &= \frac{1}{\lambda} \sum_{k=1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\boldsymbol{\Omega}_{k-1}\mathbf{L}_1 \\ &= \sum_{k=1}^{\infty} \boldsymbol{\pi}^{-}(k-1)\widehat{\mathbf{S}}_k \qquad \text{using (12)} \\ &= \boldsymbol{\pi}^{-}(0) \qquad \text{using (13).}\end{aligned}$$

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For $n \geq 1$, using (9), (14), and (10) successively, we obtain

$$\begin{aligned}\boldsymbol{\pi}^+(n)\mathbf{e} &= \boldsymbol{\pi}^-(n)\mathbf{e} - \boldsymbol{\pi}^-(n+1)\mathbf{e} + \frac{1}{\lambda}\sum_{k=n+2}^{\infty}\boldsymbol{\pi}^-(k-1)\boldsymbol{\Omega}_{k-n-2}\mathbf{L}_1\mathbf{e}\\ &= \boldsymbol{\pi}^-(n)\mathbf{e} - \boldsymbol{\pi}^-(n+1)\mathbf{e} + \boldsymbol{\pi}^-(n+1)\mathbf{e} - \boldsymbol{\pi}^-(n+2)\mathbf{e}\\ &\quad + \frac{1}{\lambda}\sum_{k=n+3}^{\infty}\boldsymbol{\pi}^-(k-1)\boldsymbol{\Omega}_{k-n-3}\mathbf{L}_1\mathbf{e}\\ &= \cdots = \boldsymbol{\pi}^-(n)\mathbf{e}.\end{aligned}$$

■

APPENDIX B

THEOREM B.1: *If $A(t)$ follows a PH-distribution with irreducible representation $(\boldsymbol{\alpha}, \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension v, then the matrices $\mathbf{S}_n$ are given by*

$$\mathbf{S}_n = \mathbf{U}_n(\mathbf{I}_m \otimes \mathbf{T}^0), \quad n \geq 0,$$

where

$$\begin{aligned}\mathbf{U}_0 &= -(\mathbf{I}_m \otimes \boldsymbol{\alpha})[\mathbf{L}_0 \otimes \mathbf{I}_v + \mathbf{I}_m \otimes \mathbf{T}]^{-1},\\ \mathbf{U}_n &= -\mathbf{U}_{n-1}(\mathbf{L}_1 \otimes \mathbf{I}_v)[\mathbf{L}_0 \otimes \mathbf{I}_v + \mathbf{I}_m \otimes \mathbf{T}]^{-1}, \qquad n \geq 1,\end{aligned}$$

with $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}_v$ and the symbol $\otimes$ denotes the Kronecker product of two matrices.

PROOF: The proof is similar to the one given in Neuts [18] for PH-type service. ■

THEOREM B.2: *If an interarrival-time distribution is of phase type with representation $(\boldsymbol{\alpha}, \mathbf{T})$, then*

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \boldsymbol{\alpha})(\mathbf{L}(z) \oplus \mathbf{T})^{-1}(\mathbf{I}_m \otimes \mathbf{T}\mathbf{e}_v),$$

where $\mathbf{L}(z) \oplus \mathbf{T} = (\mathbf{L}(z) \otimes \mathbf{I}_v) + (\mathbf{I}_m \otimes \mathbf{T})$ with the symbol $\oplus$ denoting the Kronecker sum of two matrices.

PROOF: As interarrival-time distribution is of phase type with representation $(\boldsymbol{\alpha}, \mathbf{T})$, we can write its density function as $a(t) = \boldsymbol{\alpha}e^{\mathbf{T}t}\mathbf{T}^0$ with $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}_v$. We can write (6) as

$$\mathbf{S}(z) = \int_0^{\infty} e^{\mathbf{L}(z)t} \otimes a(t)\, dt, \tag{B.1}$$

where $\otimes$ is a Kronecker product of two square matrices of order m and 1, respectively. We write (B.1) in a form so that each square matrix is the product of three different matrices (not

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necessarily square matrices) as

$$\mathbf{S}(z) = \int_0^\infty \left(\mathbf{I}_m e^{\mathbf{L}(z)t}\mathbf{I}_m\right) \otimes \left(\boldsymbol{\alpha} e^{\mathbf{T}t}\mathbf{T}^0\right)\, dt.$$

Using the well-known rule of Kronecker product $(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3) \otimes (\mathbf{F}_1\mathbf{F}_2\mathbf{F}_3) = (\mathbf{E}_1 \otimes \mathbf{F}_1)(\mathbf{E}_2 \otimes \mathbf{F}_2)(\mathbf{E}_3 \otimes \mathbf{F}_3)$, we write

$$\begin{aligned}\mathbf{S}(z) &= \int_0^\infty (\mathbf{I}_m \otimes \boldsymbol{\alpha}) \left(e^{\mathbf{L}(z)t} \otimes e^{\mathbf{T}t}\right) \left(\mathbf{I}_m \otimes \mathbf{T}^0\right) dt \\ &= (\mathbf{I}_m \otimes \boldsymbol{\alpha}) \left(\int_0^\infty \left(e^{\mathbf{L}(z)t} \otimes e^{\mathbf{T}t}\right) dt\right) \left(\mathbf{I}_m \otimes \mathbf{T}^0\right).\end{aligned}$$

We use another important rule of Kronecker product, that is, $e^{\mathbf{E}_1} \otimes e^{\mathbf{F}_1} = e^{\mathbf{E}_1 \oplus \mathbf{F}_1}$, where $\oplus$ is the Kronecker sum defined as $\mathbf{E}_1 \oplus \mathbf{F}_1 = \mathbf{E}_1 \otimes \mathbf{I}_\nu + \mathbf{I}_m \otimes \mathbf{F}_1$. Here, it is assumed that $\mathbf{E}_1$ and $\mathbf{F}_1$ are square matrices of order m and ν, respectively. We let $\mathbf{E} = (\mathbf{L}(z) \oplus \mathbf{T})$ and assume that $\mathbf{E}^{-1}$ exists and in fact this is the case here. From the fact that $\int_0^\infty e^{\mathbf{E}t}\, dt = -\mathbf{E}^{-1}$, we finally get

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \boldsymbol{\alpha}) \left(-(\mathbf{L}(z) \oplus \mathbf{T})^{-1}\right) (\mathbf{I}_m \otimes \mathbf{T}^0),$$

where $\mathbf{L}(z) \oplus \mathbf{T} = (\mathbf{L}(z) \otimes \mathbf{I}_\nu) + (\mathbf{I}_m \otimes \mathbf{T})$.
Using $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}_\nu$, we can finally write

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \boldsymbol{\alpha})(\mathbf{L}(z) \oplus \mathbf{T})^{-1}(\mathbf{I}_m \otimes \mathbf{T}\mathbf{e}_\nu). \qquad \blacksquare$$

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