Canad. J. Math. 2026, pp. 1–27
http://dx.doi.org/10.4153/S0008414X26102272
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society. This is an Open Access article, distributed under the terms of the Creative Commons Attribution-NonCommercial licence (https://creativecommons.org/licenses/by-nc/4.0), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original article is properly cited. The written permission of Cambridge University Press or the rights holder(s) must be obtained prior to any commercial use.

On the basic sequence structure of variable exponent Lebesgue spaces

José L. Ansorena and Glenier Bello

Abstract. We study the subsymmetric basic sequence structure of variable exponent Lebesgue spaces L_P built from index functions $P\colon \Omega \to (0,\infty]$ on σ-finite measure spaces (Ω, Σ, μ). Specifically, we prove that if P is bounded away from infinity, then any complemented subsymmetric basic sequence of L_P is equivalent to the canonical basis of ℓ_r for some $r \geq 1$ in the essential range of P.

1 Introduction

The theory of variable exponent Lebesgue spaces has matured from a topic of abstract interest into an indispensable tool in partial differential equations, fluid dynamics, and harmonic analysis. Unlike their classical counterparts, these spaces allow the exponent to vary with the spatial variable, providing a natural and powerful framework for modeling phenomena with localized behavior, such as electrorheological fluids or materials with inhomogeneous properties. The rich structure of these spaces is deeply intertwined with the properties of the exponent function. In this article, we regard these spaces from a functional analysis perspective.

Within the study of the geometry of Banach spaces, the structure of basic sequences is a fundamental topic already present in Banach's book [5]. In these early years, specialists were confident of achieving structural results valid for all Banach spaces. In particular, they sought to prove that every Banach space contains an isomorphic copy of c_0 or ℓ_p for some $1 \leq p < \infty$. Since the canonical bases of these spaces are symmetric, the study naturally evolved to determining the structure of symmetric basic sequences of Banach spaces. Any symmetric basis is subsymmetric, that is, unconditional and equivalent to all its subsequences. As a matter of fact, the only feature of symmetric sequences that one needs in many situations is their subsymmetry. Partly because of

Received by the editors December 22, 2025 UTC; revised April 6, 2026 UTC; accepted May 4, 2026 UTC.

Published online on Cambridge Core May 21, 2026.

Both authors acknowledge the support of the Spanish Ministry for Science and Innovation under Grant PID2022-138342NB-I00 for *Functional Analysis Techniques in Approximation Theory and Applications (TAFPAA)*. G.B. has also been partially supported by PID2022-137294NB-I00, DGI-FEDER and by Project E48_23R, D.G. Aragón

AMS subject classification: **46B20**, 46A45, 46B15, 46B25, 46B42.

Keywords: Variable exponent Lebesgue spaces, basic sequences, Orlicz sequence spaces, complemented spaces.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

this, and partly because spreading models of null sequences are subsymmetric, the researchers went on to seek subsymmetric basic sequences.

The study of the basic sequence structure of Banach spaces is part of the study of their subspace structure or, following Banach [5], that of the "linear dimension" of Banach spaces. We point out that the mere existence of an isomorphic copy of a Banach space $\mathbb{X}$ within another Banach space $\mathbb{Y}$ does not entitle us to claim that $\mathbb{X}$ is in a block from which $\mathbb{Y}$ is constructed. To fairly assert this, we must assume that $\mathbb{X}$ can be placed into $\mathbb{Y}$ in a complementary manner, so that $\mathbb{Y}$ is isomorphic to the direct sum of $\mathbb{X}$ and a third Banach space. So, the study of the complemented basic sequence structure of Banach spaces is of special interest.

We refer the reader to [2] for a summary of advances in understanding the structure of subsymmetric basic sequences of Banach spaces. Regarding Lebesgue spaces L_p, the first milestone within this topic is [23], where Paley solved some problems posed in [5]. Namely, he proved that given $p, q \in [1, \infty)$, ℓ_q does not embed in L_p unless $q = 2$ or $p \leq q \leq 2$. The fact that ℓ_2 embeds into L_p is a ready consequence of Khintchine's inequalities [16]. Kadets [12] completed the study by showing that if $p < q < 2$, then ℓ_q do embeds into L_p.

By duality, ℓ_q is not a complemented subspace of L_p unless $p = 1$ or $q \in \{2, p\}$. Clearly, ℓ_p is a complemented subspace of L_p for any $1 \leq p < \infty$. Pełczyński [25] proved that ℓ_2 is a complemented subspace of L_p for any $1 < p < \infty$. Lindenstrauss and Pełczyński [17] completed the picture by proving that any unconditional basic sequence complemented in L_1 is equivalent to the ℓ_1-basis.

Kadets and Pełczyński [13] carried out a systematic study of basic sequences in Lebesgue spaces. Among other results, they proved that, if $2 < p < \infty$, the canonical bases of ℓ_2 and ℓ_p are the only subsymmetric basic sequences of L_p. In contrast, if $1 \leq p < 2$, there exist symmetric basic sequences in L_p other than the canonical ℓ_q-basis for $p \leq q \leq 2$. Indeed, Bretagnolle and Dacunha-Castelle [8] proved the following result (also see [18, 27]).

Theorem 1.1 (See [8]) *Let $1 \leq p \leq 2$. A Banach space $\mathbb{S}$ with a symmetric basis isomorphically embeds into L_p if and only if $\mathbb{S} = \ell_F$ for some Orlicz function F with*

$$p \leq \frac{tF'(t)}{F(t)} \leq 2, \quad 0 < t \leq 1. \tag{1.1}$$

The article that initiated the study of variable exponent Lebesgue spaces by basic sequence techniques was [11] (see also [10]). In it, the authors characterized in terms of the essential range of the variable exponent $\boldsymbol{P}$ the indices $q \in [1, \infty)$ such that ℓ_q isomorphically embeds into a variable exponent Lebesgue $L_{\boldsymbol{P}}$ over the unit interval. Hernández and Ruiz also proved that ℓ_q complementably embeds into $L_{\boldsymbol{P}}$ provided that q belongs to the essential range of $\boldsymbol{P}$.

In this article, we complement the study carried out in [11] by proving that, if $q \neq 2$, being q in the essential range of $\boldsymbol{P}$ is a necessary condition for ℓ_q to be a complemented subspace of $L_{\boldsymbol{P}}$. Besides, we widen the scope of the study by considering not only ℓ_q-spaces but also subsymmetric basic sequences. Specifically, we characterize both the subsymmetric basic sequence structure and the complemented subsymmetric basic sequence structure of Lebesgue spaces with variable exponent essentially bounded.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Oddly enough, we find out that, while symmetric basic sequences other than ℓ_q-bases are contained in these spaces, they fail to be complemented unless they are ℓ_q-bases. Also, we go forward with the study of variable exponent atomic spaces, that is, Nakano spaces, initiated in [24].

To conclude this introductory section, we outline the structure of the article. In Section 2, we establish the notation about variable Lebesgue spaces that will be used throughout the article. We see these spaces from the perspective of Musielak–Orlicz spaces, including nonconvex spaces. Without intending to tackle an exhaustive study of these spaces, we sketch the more relevant ideas to our purposes. We revisit some well-known results from our perspective and also prove new ones. In Section 3, we study certain Orlicz functions (those that arise as convex combinations of power functions) that naturally appear when studying the geometry of variable Lebesgue spaces. The study of subsymmetric basic sequences in these spaces is addressed in Sections 4 and 5, which contain the main results of the article. While in Section 5, we study complemented basic sequences, in Section 4, we give the results achieved without imposing complementability.

2 Musielak–Orlicz function spaces

Given a measure space (Ω, Σ, μ), we denote by $\Sigma(\mu)$ the set of all measurable sets with finite measure. Let $L(\mu)$ be the vector space of all measurable functions with values in the real or complex field $\mathbb{F}$. As it is customary, we identify functions and sets that differ on a null set. We say that μ is separable if the metric space $(\Sigma(\mu), d_\mu)$ is separable, where d_μ is the distance given by

$$d_\mu(A, B) = \mu(A \triangle B), \quad A, B \in \Sigma(\mu).$$

We endow $L(\mu)$ with the topology associated with the convergence in measure on finite-measure sets, so $L(\mu)$ becomes a complete topological vector space. We denote by $\mathcal{S}(\mu)$ the vector space of all simple integrable functions, that is, the linear span of

$$\{\chi_E : E \in \Sigma(\mu)\}.$$

A function space over (Ω, Σ, μ) will be a quasi-Banach lattice $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ such that

- $\mathbb{X} \subseteq L(\mu)$ continuously,
- $\mathcal{S}(\mu) \subseteq \mathbb{X}$, and
- there is a constant $\boldsymbol{C} \in [0, \infty)$ such that if a sequence $(f_n)_{n=1}^{\infty}$ in $\mathbb{X}$ converges to f μ-a.e. and $S := \limsup_n \|f_n\|_{\mathbb{X}} < \infty$, then $f \in \mathbb{X}$ and $\|f\|_{\mathbb{X}} \le \boldsymbol{C}S$.

Let $L^+(\mu)$ be the cone of all measurable functions with values in $[0, \infty]$. Following [1], a function quasi-norm will be a homogeneous map

$$\rho : L^+(\mu) \to [0, \infty]$$

such that

(F.1) If $f \le g$ μ-a.e., then $\rho(f) \le \rho(g)$.
(F.2) There is $\boldsymbol{k} \in [1, \infty)$, such that, for all $f, g \in L^+(\mu)$,

$$\rho(f + g) \le \boldsymbol{k}\,(\rho(f) + \rho(g)).$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

(F.3) $\rho(\chi_E) < \infty$ for all $E \in \Sigma(\mu)$.
(F.4) For every $E \in \Sigma(\mu)$ and $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(A) \leq \varepsilon$ whenever $A \in \Sigma$ satisfies $A \subseteq E$ and $\rho(\chi_A) \leq \delta$.
(F.5) There is $\boldsymbol{C} \in [1, \infty)$ such that $\rho(\lim_n f_n) \leq \boldsymbol{C} \lim_n \rho(f_n)$ for all non-decreasing sequences $(f_n)_{n=1}^{\infty}$ in $L^+(\mu)$.

Let $0 < r \leq 1$. If the function quasi-norm ρ satisfies

(F.6) $\rho^r(f+g) \leq \rho^r(f) + \rho^r(g)$ for all $f, g \in L^+(\mu)$

we call it a function r-nom (function norm if $r = 1$). Note that (F.6) implies (F.2) with $\boldsymbol{k} = 2^{1/r-1}$.

Given a function quasi-norm ρ the set

$$\boldsymbol{L}_\rho := \left\{ f \in L(\mu) \colon \|f\|_\rho := \rho(|f|) < \infty \right\}$$

endowed with the quasi-norm $\|\cdot\|_\rho$ is a function space. The other way around, if $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a function space, then $\mathbb{X} = \boldsymbol{L}_\rho$, where ρ is defined by $\rho(f) = \|f\|_{\mathbb{X}}$ if $f \in \mathbb{X}$ and $\rho(f) = \infty$ otherwise.

Given $0 < r < \infty$, a function quasi-norm ρ, and a family $\mathcal{F} = (f_j)_{j \in J}$ in $L^+(\mu)$, we set

$$N_\rho(\mathcal{F}; r) = \rho\left(\left(\sum_{j \in J} f_j^r \right)^{1/r} \right), \quad M_\rho(\mathcal{F}; r) = \left(\sum_{j \in J} \rho^r(f_j) \right)^{1/r}.$$

We say that $\mathbb{X} = \boldsymbol{L}_\rho$ is lattice r-convex (resp. r-concave) if there is a constant C such that $N_\rho(\mathcal{F}; r) \leq C M_\rho(\mathcal{F}; r)$ (resp. $M_\rho(\mathcal{F}; r) \leq C N_\rho(\mathcal{F}; r)$) for all families $\mathcal{F}$ in $L^+(\mu)$. We say that $\mathbb{X}$ is L-convex if there is $\varepsilon > 0$ such that $\sup_j \rho(f_j) \geq \varepsilon$ provided that $(f_j)_{j \in J}$ is a finite family in $L^+(\mu)$ for which there is $f \in L^+(\mu)$ such that $\rho(f) = 1$, $f_j \leq f$ for all $j \in J$, and

$$\frac{1}{|J|} \sum_{j \in J} f_j \geq (1 - \varepsilon) f.$$

Since any quasi-Banach is locally p-convex for some $p \in (0, 1]$ by the Aoki–Rolewicz theorem (see [4, 26]), the function space $\mathbb{X}$ is L-convex if and only if it is r-convex for some $r \in (0, \infty)$ (see [14, Theorem 2.2]). If $\mathbb{X}$ is r-concave for some $r \in (0, \infty)$, we say that $\mathbb{X}$ is L-concave. We say that $\mathbb{X}$ is absolutely continuous if

(F.7) $\lim_n \rho(f_n) = 0$ for all non-increasing sequences $(f_n)_{n=1}^{\infty}$ in $L^+(\mu)$ with $\rho(f_1) < \infty$ and $\lim_n f_n = 0$.

Let $\mathbb{X}_0$ be the closure of $\mathcal{S}(\mu)$ in $\mathbb{X}$. We say that $\mathbb{X}$ is minimal if $\mathbb{X}_0 = \mathbb{X}$. If $\mathbb{X}$ is L-concave, then $\mathbb{X}$ is absolutely continuous. In turn, if $\mathbb{X}$ is absolutely continuous, then $\mathbb{X}$ is minimal (see, e.g., [2]).

Given a function quasi-norm ρ on $L^+(\mu)$, we consider the gauge ρ^* defined by

$$\rho^*(g) = \sup \left\{ \int_\Omega f g \, d\mu \colon f \in L^+(\mu), \, \rho(f) \leq 1 \right\}, \quad g \in L^+(\mu).$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

If

(F.8) for every $E \in \Sigma(\mu)$, there is $C_E \in (0, \infty)$ such that

$$\int_E f \, d\mu \leq C_E \rho(f), \quad f \in L^+(\mu),$$

then ρ^* is a function norm. Note that (F.8) implies (F.4) and that ρ^* always satisfies (F.8) as well. We say that ρ^* is the *dual function norm* of ρ. The function space built from ρ^* is called the conjugate space of $\mathbb{X} = \boldsymbol{L}_\rho$ and denoted by $\mathbb{X}'$. If $\mathbb{X}$ is locally convex, then $\mathbb{X}'' = \mathbb{X}$ (see [6]). We can regard $\mathbb{X}'$ as a closed subspace of the dual space $\mathbb{X}^*$. If $\mathbb{X}$ is absolutely continuous, then $\mathbb{X}' = \mathbb{X}^*$ (see [6]).

For $j = 1, 2$, let $\mathbb{X}_j$ be a function space over a σ-finite measure space $(\Omega_j, \Sigma_j, \mu_j)$. Let

$$K: \Omega_1 \times \Omega_2 \to [0, \infty)$$

be a measurable function. Suppose there is a bounded linear operator $T: \mathbb{X}_1 \to \mathbb{X}_2$ given by

$$T(f)(\omega_2) = \int_{\Omega_1} K(\omega_1, \omega_2) f(\omega_1) \, d\mu_1(\omega_1), \quad f \in \mathbb{X}_1, \ \omega_2 \in \Omega_2.$$

Then, there is a bounded linear operator $T': \mathbb{X}_2' \to \mathbb{X}_1'$ given by

$$T'(g)(\omega_1) = \int_{\Omega_2} K(\omega_1, \omega_2) g(\omega_2) \, d\mu_2(\omega_2), \quad g \in \mathbb{X}_2, \ \omega_1 \in \Omega_1.$$

We say that T is a kernel operator and that T' is the adjoint operator of T.

Given a measurable function $f: \Omega \to G$, where G is a topological commutative monoid (for instance, $G = [0, \infty]$ or G a vector space), we set

$$\operatorname{supp}(f) = \Omega \backslash f^{-1}(0).$$

Given a function space $\mathbb{X}$ over a σ-finite measure space (Ω, Σ, μ), and a pairwise disjointly supported sequence $\mathcal{X} = (x_n)_{n=1}^\infty$ in $\mathbb{X} \backslash \{0\}$, we set

$$\mathbb{S}[\mathbb{X}, \mathcal{X}] = \left\{ (a_n)_{n=1}^\infty : \sum_{n=1}^\infty a_n x_n \in \mathbb{X} \right\},$$

and we endow $\mathbb{S}[\mathbb{X}, \mathcal{X}]$ with the topology it inherits from $\mathbb{X}$.

An *Orlicz function* will be a non-decreasing left-continuous function

$$F: [0, \infty) \to [0, \infty]$$

such that $\lim_{t \to 0^+} F(t) = 0$ and $F(\infty) := \lim_{t \to \infty} F(t) > 0$. Note that we allow Orlicz functions to take the value ∞. If it is the case for the Orlicz function F, then there is $0 < c < \infty$ such that $F^{-1}(\infty) = (c, \infty)$.

Given a σ-finite measure space (Ω, Σ, μ), a function

$$M: \Omega \times [0, \infty) \to [0, \infty],$$

and $t \in [0, \infty)$, we denote by $\nu_M(\cdot, t)$ the measure on (Ω, Σ) given by

$$\nu_M(A, t) = \int_A M(\omega, t) \, d\mu(\omega), \quad A \in \Sigma.$$

We say that M is a Musielak–Orlicz function (cf. [21, Chapter 7]) if

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

- $M(\omega,\cdot)$ is an Orlicz function for all $\omega \in \Omega$, and
- for every $E \in \Sigma(\mu)$, there exists $t_E \in (0,\infty)$ such that $v_M(E,t_E) < \infty$ and $M(\omega,t_E) > 0$ for all $\omega \in E$.

If $f \in L^+(\mu)$ and M is a Musielak–Orlicz function, then $M(\cdot,f(\cdot))$ is a measurable function. Hence, associated with M, there is a gauge, usually called a modular,

$$\rho_M\colon L^+(\mu) \to [0,\infty], \quad f \mapsto \int_\Omega M(\omega,f(\omega))\,d\mu(\omega).$$

Consider the sets

$$\begin{aligned}
L_M^+ &= \{f \in L^+(\mu)\colon \rho_M(tf) < \infty \text{ for some } t \in (0,\infty)\},\\
L_M &= \{f \in L(\mu)\colon |f| \in L_M^+\},\\
H_M^+ &= \{f \in L^+(\mu)\colon \rho_M(tf) < \infty \text{ for all } t \in (0,\infty)\},\\
H_M &= \{f \in L(\mu)\colon |f| \in H_M^+\},
\end{aligned}$$

and, given $\varepsilon > 0$,

$$B_M(\varepsilon) = \{f \in L(\mu)\colon \rho_M(|f|) < \varepsilon\}.$$

The modular $\rho = \rho_M$ satisfies properties (F.1), (F.4), (F.5), and (F.7), property (F.9) below instead of (F.2), and property (F.10) below instead of (F.3).

(F.9) There are $\boldsymbol{k}$ and $\boldsymbol{r}$ in $(0,\infty)$ such that, for all $f, g \in L^+(\mu)$,

$$\rho(f+g) \le \boldsymbol{k}\,(\rho(\boldsymbol{r}f) + \rho(\boldsymbol{r}g)).$$

(F.10) For every $E \in \Sigma(\mu)$, there is $u_E \in (0,\infty)$ such that $\rho(u_E\chi_E) < \infty$.

In fact, we can choose $\boldsymbol{C} = 1$ in (F.5), $\boldsymbol{k} = 1$ and $\boldsymbol{r} = 2$ in (F.9), and $u_E = t_E$ in (F.10). Consequently,

$$uB_M(\varepsilon), \quad u,\, \varepsilon \in (0,\infty),$$

is a neighborhood basis at the origin for a complete vector topology on L_M. Besides, the F-space L_M continuously embeds into $L(\mu)$. We say that L_M is the Musielak–Orlicz space over (Ω,Σ,μ) built from M. We denote by L_M^0 the closed subspace of L_M generated by simple integrable functions.

By property (F.7), the dominated convergence theorem holds in L_M for sequences dominated by a function in H_M^+. We infer that H_M is the closed linear span of

$$\{\chi_E\colon E \in \Sigma(\mu), \chi_E \in H_M^+\}.$$

Hence, $H_M \subseteq L_M^0 \subseteq L_M$, and $L_M^0 = H_M$ if and only if $\chi_E \in H_M^+$ for all $E \in \Sigma(\mu)$. Consequently, $H_M = L_M^0$ provided that

(F.11) there are $C, D \in (1,\infty)$ such that

$$M(\omega,Dt) \le CM(\omega,t), \quad (\omega,t) \in \Omega \times [0,\infty).$$

If

(F.12) there are $c, d \in (0,1)$ such that

$$M(\omega,dt) \le cM(\omega,t), \quad (\omega,t) \in \Omega \times [0,\infty),$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

then L_M is a function space. Indeed, the Luxemburg map

$$f \mapsto \rho_M^L(f) := \inf\{t \in (0,\infty) : \rho_M(f/t) \le 1\}, \quad f \in L^+(\mu),$$

besides being homogeneous, inherits from ρ_M the properties (F.1), (F.4), (F.5), (F.9), and (F.10). Hence, ρ_M^L is a function quasi-norm such that $L_M = \boldsymbol{L}_{\rho_M^L}$.

Let $0 < r < \infty$ and M be a Musielak–Orlicz function. If M is r-convex, that is, the mapping

$$t \mapsto M\left(\omega, t^{1/r}\right), \quad t \in [0,\infty),$$

defines a convex function for all $\omega \in \Omega$, then (F.12) holds, and the function space L_M is lattice r-convex. Let $0 < s < \infty$. If M is s-concave, that is, the mapping

$$t \mapsto M\left(\omega, t^{1/s}\right), \quad t \in [0,\infty),$$

defines a concave function for all $\omega \in \Omega$, then it satisfies (F.11). Therefore, if M is s-concave and satisfies (F.12), then L_M is a lattice s-concave absolutely continuous function space, and $H_M = L_M$.

We will use the following two results, whose straightforward proofs we omit.

Lemma 2.1 *Let M be a Musielak–Orlicz function satisfying (F.12). Let $f \in L_M \backslash \{0\}$ and $a = 1/\|f\|_M$. Then, either*

(a) $\rho_M(a|f|) = 1$ *or*
(b) $\rho_M(a|f|) < 1$ *and* $\rho_M(t|f|) = \infty$ *for all* $t \in (a,\infty)$.

In particular, (a) holds for all $f \in H_M$.

Theorem 2.2 (See [3]) *Let M and N be Musielak–Orlicz functions on a σ-finite measure space (Ω, Σ, μ). Suppose that there is $c \in (0,\infty)$ such that*

- $\max\{v_M(\Omega, c), v_N(\Omega, c)\} < \infty$,
- $\min\{M(\omega, c), N(\omega, c)\} > 0$ *for all* $\omega \in \Omega$, *and*
- $M(\omega, t) \le N(\omega, t)$ *for all* $\omega \in \Omega$ *and* $t \in [c,\infty)$.

Then $L_N(\mu) \subseteq L_M(\mu)$ continuously.

Next, we detail three types of Musielak–Orlicz spaces we will deal with.

2.1 Musielak–Orlicz sequence spaces

We regard Musielak–Orlicz functions over $\mathbb{N}$, which we call Musielak–Orlicz sequences, as sequences of Orlicz functions. Given a Musielak–Orlicz sequence $\boldsymbol{F} = (F_n)_{n=1}^\infty$, we denote by ℓ_F the Musielak–Orlicz sequence space built from $\boldsymbol{F}$. These Musielak–Orlicz atomic spaces naturally appear when studying the unconditional basic sequence structure of Musielak–Orlicz function spaces. Indeed, if M is a Musielak–Orlicz function over a σ-finite measure space (Ω, Σ, μ), and $\mathcal{X} = (x_n)_{n=1}^\infty$ is a pairwise disjointly supported sequence in $L_M \backslash \{0\}$, then $\mathbb{S}[L_M, \mathcal{X}] = \ell_F$, where $\boldsymbol{F} = (F_n)_{n=1}^\infty$ is given by

$$F_n(t) = \int_\Omega M\left(\omega, t|x_n(\omega)|\right) d\mu(\omega), \quad n \in \mathbb{N}.$$

We bring up a result in Musielak's memoir [21] that we will use.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Theorem 2.3 [21, Theorem 8.11] *Consider two Musielak–Orlicz sequences* $\mathbf{F} = (F_n)_{n=1}^{\infty}$ *and* $\mathbf{G} = (G_n)_{n=1}^{\infty}$. *Then,* $\ell_F \subseteq \ell_G$ *continuously if and only if there exist* $\mathbf{a} = (a_n)_{n=1}^{\infty} \in \ell_1$ *and* δ, b, *and* C *in* $(0, \infty)$ *such that*

$$G_n(t) \leq a_n + CF_n(bt)$$

for all $n \in \mathbb{N}$ *and all* $t \in (0, \infty)$ *with* $F_n(t) < \delta$.

2.2 Orlicz sequence spaces

Given an Orlicz function F and a measure space (Ω, Σ, μ), the Orlicz space $L_F(\mu)$ is the Musielak–Orlicz space built from the Musielak–Orlicz function M over (Ω, Σ, μ) given by $M(\omega, t) = F(t)$ for all $\omega \in \Omega$ and $t \in [0, \infty)$. If μ is the counting measure on $\mathbb{N}$, we set $\ell_F = L_F(\mu)$, and we call ℓ_F the Orlicz sequence space built from F. Orlicz sequence spaces are Musielak–Orlicz sequence spaces built from constant Musielak–Orlicz sequences. Applying Theorem 2.3 in this particular situation yields the following well-known result.

Corollary 2.4 *Given two Orlicz functions* F *and* G, *then* $\ell_F \subseteq \ell_G$ *if and only if there are* b, c, *and* C *in* $(0, \infty)$ *such that*

$$G(t) \leq CF(bt), \quad 0 \leq t \leq c.$$

Given a convex Orlicz function F, we define its conjugate function $F^*\colon [0, \infty) \to [0, \infty]$ as the optimal function G such that

$$uv \leq F(u) + G(v), \quad u, v \in [0, \infty].$$

F^* is a convex Orlicz function, and the conjugate space of $L_F(\mu)$ is $L_{F^*}(\mu)$. We record some properties of the mapping $F \mapsto F^*$ that we will need.

Given $p \in (0, \infty]$, we consider the power Orlicz functions

$$\mathbf{\Phi}_p, \mathbf{\Psi}_p\colon [0, \infty) \to [0, \infty), \quad \mathbf{\Phi}_p(t) = t^p, \mathbf{\Psi}_p(t) = \frac{t^p}{p},$$

with the convention that $1^{\infty} = 0$. If $p \geq 1$, so that $\mathbf{\Phi}_p$ and $\mathbf{\Psi}_p$ are convex, we denote by p' its conjugate exponent defined by $1/p + 1/p' = 1$. Note that $\mathbf{\Phi}_{\infty} = \mathbf{\Psi}_{\infty} = \infty\chi_{(1,\infty)}$.

We record a couple of lemmas about Orlicz functions. The first of them is clear and well-known.

Lemma 2.5 $(\mathbf{\Psi}_p)^* = \mathbf{\Psi}_{p'}$ *for all* $p \in [1, \infty]$.

Lemma 2.6 *Let* F *be a convex Orlicz function. Let* c, C, A, *and* v *in* $(0, \infty)$ *be such that*

$$uv \leq CF(u) + A, \quad 0 \leq u \leq c.$$

Then, $F^*(w) \leq A/C$, *where*

$$w = \min\left\{\frac{v}{C}, \frac{F(c)}{c}\right\}.$$

Proof On the one hand, if $u \in [0, c]$,

$$uw \leq \frac{uv}{C} \leq F(u) + \frac{A}{C}.$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

On the other hand, if $u \in [c, \infty)$, by convexity,

$$uw \leq u\frac{F(c)}{c} \leq F(u) \leq F(u) + \frac{A}{C}. \qquad \blacksquare$$

2.3 Variable exponent Lebesgue spaces

A variable exponent on a measure space (Ω, Σ, μ) will be a measurable function

$$\boldsymbol{P}\colon \Omega \to (0, \infty].$$

The variable exponent Lebesgue space $L_{\boldsymbol{P}}$ associated with $\boldsymbol{P}$ will be the F-space over Ω built from the Musielak–Orlicz function

$$(\omega, t) \mapsto M_{\boldsymbol{P}}(\omega, t) = \boldsymbol{\Phi}_{\boldsymbol{P}(\omega)}(t), \quad (\omega, t) \in \Omega \times [0, \infty),$$

that is, $L_{\boldsymbol{P}} = L_{M_{\boldsymbol{P}}}$. We denote by $\rho_{\boldsymbol{P}} := \rho_{M_{\boldsymbol{P}}}$ the modular constructed from $M_{\boldsymbol{P}}$. If the implicit measure μ over (Ω, Σ) we are considering could be in doubt, we put $L_{\boldsymbol{P}} = L_{\boldsymbol{P}}(\mu)$. Set also $H_{\boldsymbol{P}} = H_{M_{\boldsymbol{P}}}$.

Recall that the support of a Borel measure ν on a Lindelöf space X, denoted by $\operatorname{supp}(\nu)$, is the smallest closed set K such that $\nu(X \backslash K) = 0$. Let $\mu \circ \boldsymbol{P}^{-1}$ be the pushforward of μ by $\boldsymbol{P}$. Let

$$R(\boldsymbol{P}) := \operatorname{supp}\left(\mu \circ \boldsymbol{P}^{-1}\right) \subseteq [0, \infty]$$

be the essential range of $\boldsymbol{P}$, that is, the set of all $p \in [0, \infty]$ such that

$$\mu\left(\boldsymbol{P}^{-1}(V)\right) > 0$$

for all measurable neighborhoods V of p. Set

$$\boldsymbol{P}^- := \min\left(R(\boldsymbol{P})\right), \quad \boldsymbol{P}^+ = \max\left(R(\boldsymbol{P})\right).$$

Suppose that $\boldsymbol{P}^- > 0$. Then, $M_{\boldsymbol{P}}$ is $\boldsymbol{P}^-$-convex, whence $L_{\boldsymbol{P}}$ is a lattice $\boldsymbol{P}^-$-convex function space over (Ω, Σ, μ). We denote by $\rho_{\boldsymbol{P}}^L$ the function quasi-norm constructed from $\rho_{\boldsymbol{P}}$.

We will also consider the Musielak–Orlicz function over (Ω, Σ, μ) given by

$$(\omega, t) \mapsto M_{\boldsymbol{P}}^a(\omega, t) := \boldsymbol{\Psi}_{\boldsymbol{P}(\omega)}(t).$$

Let $\rho_{\boldsymbol{P}}^a$ be the modular associated with $M_{\boldsymbol{P}}^a$. In turn, let $\rho_{\boldsymbol{P}}^{L,a}$ be the function quasi-norm constructed from $\rho_{\boldsymbol{P}}^a$. Since $p^{1/p}$ is bounded away from zero at infinity as long as p is bounded away from zero, the function quasi-norms $\rho_{\boldsymbol{P}}^L$ and $\rho_{\boldsymbol{P}}^{L,a}$ are equivalent. In fact,

$$\boldsymbol{P}^- \rho_{\boldsymbol{P}}^{L,a} \leq \rho_{\boldsymbol{P}}^L \leq \sup\left\{p^{1/p} \colon \boldsymbol{P}^- \leq p < \infty\right\} \rho_{\boldsymbol{P}}^{L,a}.$$

Thus, there are two natural quasi-norms $L_{\boldsymbol{P}}$ can be equipped with, namely, those constructed from $\rho_{\boldsymbol{P}}^L$ and $\rho_{\boldsymbol{P}}^{L,a}$. We denote by $\|\cdot\|_{\boldsymbol{P}}$ the one constructed from the former function quasi-norm, that is, $\|f\|_{\boldsymbol{P}} = \rho_{\boldsymbol{P}}^L(|f|)$ for all $f \in L_{\boldsymbol{P}}$.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Both ρ_P^L and $\rho_P^{L,a}$ satisfy (F.6) with

$$r = \boldsymbol{P}^c := \min\{1, \boldsymbol{P}^-\}.$$

In other words, ρ_P^L and $\rho_P^{L,a}$ are function $\boldsymbol{P}^c$-norms.

When dealing with variable exponent Lebesgue spaces, we must be aware that the identity

$$\boldsymbol{\Psi}_p(st) = \boldsymbol{\Phi}_p(s)\boldsymbol{\Psi}_p(t)$$

breaks down when $p = \infty$ and $0 \le s < 1 < t < \infty$. So, we must pay attention to the set

$$\Omega_\infty^{\boldsymbol{P}} := \boldsymbol{P}^{-1}(\infty).$$

In fact, for any $f \in L^+(\mu)$, if $\|\cdot\|_\infty$ denotes the norm of $L_\infty(\mu|_{\Omega_\infty^{\boldsymbol{P}}})$,

$$\rho_P^L(f) = \max\left\{\left\|f|_{\Omega_\infty^{\boldsymbol{P}}}\right\|_\infty, \inf\left\{t > 0 \colon \int_{\Omega\setminus\Omega_\infty^{\boldsymbol{P}}} \frac{f(\omega)^{\boldsymbol{P}(\omega)}}{t^{\boldsymbol{P}(\omega)}} d\mu(\omega) \le 1\right\}\right\}.$$

If $\boldsymbol{P}^+ < \infty$, then $L_{\boldsymbol{P}}$ is $\boldsymbol{P}^+$-concave. Consequently, if $\boldsymbol{P}^- > 0$ and $\boldsymbol{P}^+ < \infty$, then $L_{\boldsymbol{P}}$ is lattice $\boldsymbol{P}^+$-concave, whence $H_{\boldsymbol{P}} = L_{\boldsymbol{P}}$ and $L_{\boldsymbol{P}}$ is absolutely continuous.

If μ is a finite measure and $\boldsymbol{P}$ and $\boldsymbol{Q}$ are variable exponents over (Ω, Σ, μ) with $\boldsymbol{P} \le \boldsymbol{Q}$, then $L_{\boldsymbol{Q}} \subseteq L_{\boldsymbol{P}}$ continuously by Theorem 2.2.

Given a measurable function $h\colon \Omega \to (0, \infty)$, the pointwise multiplier

$$f \mapsto fh$$

defines an isomorphism from $L_{\boldsymbol{P}}(\nu)$ onto $L_{\boldsymbol{P}}(\mu)$, where

$$d\nu = h_{\boldsymbol{P}}\, d\mu, \quad h_{\boldsymbol{P}} = h^{\boldsymbol{P}} \chi_{\Omega\setminus\Omega_\infty^{\boldsymbol{P}}} + h\chi_{\Omega_\infty^{\boldsymbol{P}}}.$$

Since we can choose h so that $\int_\Omega h_{\boldsymbol{P}}\, d\mu = 1$, any variable exponent Lebesgue space is isomorphic to a variable exponent Lebesgue space over a probability space. If this probability space is nonatomic and separable, then, by [9], the corresponding variable exponent Lebesgue space is isomorphic to a variable exponent Lebesgue space over $[0, 1]$.

To understand the structure of a variable exponent Lebesgue space we must take into account that if $(\Omega_j)_{j=1}^2$ is a partition of the measure space (Ω, Σ, μ) then, for any variable exponent $\boldsymbol{P}\colon \Omega \to (0, \infty]$, setting $\boldsymbol{P}_j = \boldsymbol{P}|_{\Omega_j}$ and $\mu_j = \mu|_{\Omega_j}$ for $j = 1, 2$,

$$L_{\boldsymbol{P}} \simeq L_{\boldsymbol{P}_1} \oplus L_{\boldsymbol{P}_2}. \tag{2.1}$$

If J is a countable set and $\boldsymbol{Q}\colon J \to (0, \infty]$ is a function, we denote by $\ell_{\boldsymbol{Q}}$ the variable exponent Lebesgue space over the counting measure on J associated with the variable exponent $\boldsymbol{Q}$. These variable exponent Lebesgue atomic spaces were introduced by Nakano [22] in the case when $\boldsymbol{Q}^- \ge 1$, and by Bourgin [7] in the case when $\boldsymbol{Q}^+ \le 1$. In general, as is customary, we will call them Nakano sequence spaces.

Splitting σ-finite measures into their purely atomic part and their nonatomic part, we infer that any variable exponent Lebesgue space is isomorphic to

$$L_{\boldsymbol{P}} \oplus \ell_{\boldsymbol{Q}}$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

for some nonatomic probability measure space (Ω, Σ, μ), some countable set J and some variable exponents $\boldsymbol{P}\colon \Omega \to (0,\infty]$ and $\boldsymbol{Q}\colon J \to (0,\infty]$. Summing up, any variable exponent Lebesgue space over any separable σ-finite measure space is isomorphic to $L_{\boldsymbol{P}} \oplus \ell_{\boldsymbol{Q}}$ for some countable set J and some variable exponents $\boldsymbol{P}\colon [0,1] \to (0,\infty]$ and $\boldsymbol{Q}\colon J \to (0,\infty]$.

The dual space of $L_{\boldsymbol{P}}$ only depends on the behavior of $\boldsymbol{P}$ on

$$\Omega_c^{\boldsymbol{P}} = \boldsymbol{P}^{-1}([1,\infty]).$$

Indeed, we have the following.

Proposition 2.7 *Let $\boldsymbol{P}\colon \Omega \to (0,1)$ be a variable exponent on a nonatomic σ-finite measure space (Ω, Σ, μ). Then, the dual space of $L_{\boldsymbol{P}}$ is null.*

Proof Set $\Omega_{p,A} = \boldsymbol{P}^{-1}((0,p]) \cap A$ for each $p \in (0,1)$ and $A \in \Sigma(\mu)$. Since

$$\bigcup_{\substack{0<p<1 \\ A\in\Sigma(\mu)}} \left\{ f \in L_{\boldsymbol{P}} \colon \operatorname{supp}(f) \subseteq \Omega_{p,A} \right\}$$

is dense in $L_{\boldsymbol{P}}$, it suffices to prove the result in the case when μ is finite and $\boldsymbol{P}(\Omega) \subseteq (0,p]$ for some $0 < p < 1$. In this case, since $L_p(\mu) \subseteq L_{\boldsymbol{P}}$ continuously and $\mathcal{S}(\mu)$ is dense in $L_{\boldsymbol{P}}$, the result follows from the fact that $(L_p(\mu))^* = \{0\}$. ∎

We say that a variable exponent $\boldsymbol{P}$ over a σ-finite measure space (Ω, Σ, μ) is convex if $\boldsymbol{P}(\Omega) \subseteq [1,\infty]$. The conjugate variable exponent of $\boldsymbol{P}$ is the convex variable exponent $\boldsymbol{Q}$ over (Ω, Σ, μ) defined by

$$\boldsymbol{Q}(\omega) = (\boldsymbol{P}(\omega))' = \frac{\boldsymbol{P}(\omega)}{\boldsymbol{P}(\omega) - 1}.$$

The well-known duality relation between $L_{\boldsymbol{P}}$ and $L_{\boldsymbol{Q}}$ relies on the following lemma.

Lemma 2.8 *Let $\boldsymbol{Q}$ be the conjugate variable exponent of a variable exponent $\boldsymbol{P}$ over a σ-finite measure space (Ω, Σ, μ). Let $g \in L^+(\mu)$ with $\rho_{\boldsymbol{Q}}^L(g) = 1$ and $\operatorname{supp}(g) \subseteq \boldsymbol{P}^{-1}([p,\infty])$ for some $p > 1$. Set $f = g^{\boldsymbol{Q}-1}$, with the convention that $0^0 = 0$. Then,*

$$\int_\Omega fg \, d\mu = \rho_{\boldsymbol{P}}(f) = \rho_{\boldsymbol{Q}}(g) = 1.$$

Proof Our conditions on g yields $g \in H_{\boldsymbol{Q}}^+$, whence $\rho_{\boldsymbol{Q}}(g) = 1$ by Lemma 2.1. Since $fg = f^{\boldsymbol{P}} = g^{\boldsymbol{Q}}$, we are done. ∎

With the information we have gathered, proving that the conjugate space of $L_{\boldsymbol{P}}$ is $L_{\boldsymbol{Q}}$ takes little effort. On the one hand, by Lemma 2.5,

$$\int_\Omega fg \, d\mu \le 2, \quad f, g \in L^+(\mu), \ \max\left\{ \rho_{\boldsymbol{P}}^{L,a}(f), \rho_{\boldsymbol{Q}}^{L,a}(g) \right\} \le 1.$$

Therefore, the dual function norm ρ^* of $\rho_{\boldsymbol{Q}}^L$ satisfies $\rho^* \le 2\rho_{\boldsymbol{Q}}^{L,a}$. On the other hand, by Lemma 2.8 and homogeneity, $\rho^*(g) \ge \rho_{\boldsymbol{Q}}^L(g)$ for all $g \in L^+(\mu)$ with $\operatorname{supp}(g) \subseteq \boldsymbol{P}^{-1}([p,\infty])$ for some $p > 1$. By the Fatou property, this inequality extends to any $g \in L^+(\mu)$ with $\operatorname{supp}(g) \subseteq \boldsymbol{P}^{-1}((1,\infty])$. Since the conjugate space of $L_1(\Omega_\infty^{\boldsymbol{Q}})$ is $L_\infty(\Omega_\infty^{\boldsymbol{Q}})$, the inequality extends to any $g \in L^+(\mu)$.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

We close this section with a result that connects variable exponent Lebesgue spaces to classical Lebesgue spaces. Given $0 < p \leq 2$ and $t \in (0, \infty)$, we say that a random variable is *p-stablewith parameter t* if its characteristic function is

$$\nu \mapsto e^{-t|\nu|^p}, \quad \nu \in \mathbb{R}.$$

Note that if $s \in (0, \infty)$ and X is a p-stable random variable with parameter t, then sX is a p-stable random variable with parameter $s^p t$. We say that a continuous-time stochastic process $(X_t)_{t\in[0,1]}$ is p-stable if $X_0 = 0$ a.e., it has stationary and independent increments, and, for all $0 \leq s < t \leq 1$, $X_t - X_s$ is a p-stable random variable with parameter $t - s$.

Proposition 2.9 *Let $\mathbf{P}$ be a variable exponent on a nonatomic σ-finite measure space (Ω, Σ, μ). Suppose that $\mathbf{P}^- < 2$. Then, L_p is almost isometrically isomorphic to a subspace of $L_{\mathbf{P}}$ for all $p \in (\mathbf{P}^-, 2]$.*

Proof For each $0 < s < \infty$, let $C_{p,s} \in (0, \infty]$ be the s-norm of a p-stable random variable with parameter one. Pick $r := \mathbf{P}^- < q < p$. By (2.1), we can assume that $P(\omega) \in [\mathbf{P}^-, q]$ for all $\omega \in \Omega$. We can also assume that μ is a probability measure and that there is a p-stable continuous-time stochastic process on (Ω, Σ, μ) (see, e.g., [15, Section 4]). Then, there is a linear map $T: L_p \to L(\mu)$ such that, for all $f \in L_p$, $T(f)$ is a p-stable random variable with parameter $\|f\|_p^p$ (see, e.g., [15, Section 5]). Therefore, $\|T(f)\|_s = C_{p,s} \|f\|_p$ for all $0 < s < p$ and $f \in L_p$. Since $L_q \subseteq L_{\mathbf{P}} \subseteq L_r$ and $C_{p,s} < \infty$ for all $0 < s < p$ (see, e.g., [15, Section 5]),

$$\frac{1}{C_{p,q}} \|T(f)\|_{\mathbf{P}} \leq \|f\|_p \leq \frac{1}{C_{p,r}} \|T(f)\|_{\mathbf{P}}, \quad f \in L_p.$$

Consequently, L_p is $(C_{p,q}/C_{p,r})$-isomorphic to a subspace of $L_{\mathbf{P}}$. Since

$$\lim_{q \to r^+} C_{p,q} = C_{p,r},$$

we are done. ∎

3 Orlicz functions generated near zero by power functions

Given a signed measure μ on $(0, \infty]$, we define

$$\boldsymbol{\psi}_\mu : [0,1] \to \mathbb{R}, \quad t \mapsto \int_{(0,\infty]} \boldsymbol{\Psi}_p(t)\, d\mu(p).$$

Let δ_p be the Dirac measure on $p \in (0, \infty]$. We have

$$\boldsymbol{\psi}_p := \boldsymbol{\psi}_{\delta_p} = \boldsymbol{\Psi}_p\big|_{[0,1]}.$$

We will use the elementary inequality

$$\frac{v^s - u^s}{s} \leq \frac{v^r - u^r}{r}, \quad 0 < r \leq s \leq \infty,\ 0 \leq u \leq v \leq 1, \tag{3.1}$$

several times.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Lemma 3.1 *The mapping $p \mapsto H(p) := \boldsymbol{\psi}_p$ defines a nonincreasing continuous function from $(0,\infty]$ into the Banach lattice $\mathcal{C}([0,1])$.*

Proof Applying (3.1) with $u = 0$ gives that H is nonincreasing. This inequality also gives

$$\|H(r) - H(s)\|_\infty = \boldsymbol{\psi}_r(1) - \boldsymbol{\psi}_s(1) = \frac{1}{r} - \frac{1}{s}, \quad 0 < r \le s \le \infty. \qquad \blacksquare$$

Definition 3.2 Given $R \subseteq (0,\infty]$ nonempty, we denote by $\mathcal{K}(R)$ the the closed convex hull in $\mathcal{C}([0,1])$ of $\{\boldsymbol{\psi}_p : p \in R\}$. We denote by $\mathcal{O}(R)$ the set of all Orlicz functions F for which there are $0 < c \le 1$ and $\varphi \in \mathcal{K}(R)$ such that

$$F(t) \approx \varphi(t), \quad 0 \le t \le c.$$

Put

$$\mathcal{O}((0,\infty]) = \bigcup_{p>0} \mathcal{O}([p,\infty]).$$

Given $r \in (0,\infty]$, we set

$$\mathcal{O}(R, r^+) = \bigcap_{s>r} \mathcal{O}(R \cap [r,s]).$$

Finally, we put $\mathcal{O}(r) = \mathcal{O}(\{r\})$ and $\mathcal{O}(r^+) = \mathcal{O}([r,\infty], r^+)$.

Note that $\mathcal{O}(\infty) = \mathcal{O}(\infty^+)$ consists of all Orlicz functions that are null in a neighborhood of the origin.

Lemma 3.3 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Then, any function in $\mathcal{K}(R)$ extends to a function in $\mathcal{O}(R)$.*

Proof Just notice that any function $F \in \mathcal{K}(R)$ is nondecreasing, continuous, and satisfies $F(0) = 0$. $\blacksquare$

Given $f : [0,1] \to \mathbb{R}$ and $p \in (0,\infty]$, we consider the function

$$\pi_p(f) : [0,1] \to \mathbb{R}, \quad \pi_p(f)(t) = f\left(t^p\right).$$

Definition 3.4 Let $0 < r \le s \le \infty$. We denote by $\mathcal{K}_{r,s}$ the set consisting of all nondecreasing functions $\varphi : [0,1] \to [0,\infty)$ such that $\pi_r(\varphi)$ is convex, $\pi_s(\varphi)$ is concave,

$$\frac{t^s - u^s}{s} \le \varphi(t) - \varphi(u) \le \frac{t^r - u^r}{r}, \quad 0 \le u < t \le 1,$$

and

$$\varphi(bt) \le b^s \varphi(t), \quad 1 < b < \infty,\ 0 \le t \le \frac{1}{b}.$$

Lemma 3.5 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Set $r = \min(R)$ and $s = \max(R)$. Then $\mathcal{K}(R) \subseteq \mathcal{K}_{r,s}$.*

Proof We infer from inequality (3.1) that $\boldsymbol{\psi}_p \in \mathcal{K}_{r,s}$ for all $p \in R$. Since $\mathcal{K}_{r,s}$ is convex and closed, we are done. $\blacksquare$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Lemma 3.6 *For $j = 1, 2$, let $R_j \subseteq (0, \infty]$ be closed, nonempty, and bounded away from zero. Set $r_1 := \max(R_1)$ and $r_2 := \min(R_2)$.*

(i) *If $r_1 = r_2$, then $\mathcal{O}(R_1) \cap \mathcal{O}(R_2) = \mathcal{O}(r_1)$.*
(ii) *If $r_1 < r_2$, then $\mathcal{O}(R_1) \cap \mathcal{O}(R_2) = \varnothing$.*

Proof Pick $F \in \mathcal{O}(R_1) \cap \mathcal{O}(R_2)$ and $r_1 \le r \le s \le r_2$. There are $0 < c \le 1$, $\varphi_j \in \mathcal{O}(R_j)$, and $C_j \in (0, \infty)$, $j = 1, 2$, such that

$$\frac{\varphi_1(t)}{C_1} \le F(t) \le C_2 \varphi_2(t), \quad 0 \le t \le c.$$

By Lemma 3.5,

$$\frac{\psi_r(t)}{C_1} \le F(t) \le C_2 \psi_s(t), \quad 0 \le t \le c.$$

Consequently, $r \ge s$. ∎

Lemma 3.7 *Let $0 < q < r \le \infty$. Then, $\mathcal{O}(q^+) \cap \mathcal{O}(r^+) = \varnothing$.*

Proof Choose $q < s_1 < r < s_2$. Since $\mathcal{O}(q^+) \cap \mathcal{O}(r^+) \subseteq \mathcal{O}([q, s_1]) \cap \mathcal{O}([r, s_2])$, the result follows from Lemma 3.6. ∎

Proposition 3.8 *Let $R \subseteq (0, \infty]$ be closed, nonempty, and bounded away from zero.*

(i) *$\varphi \in \mathcal{K}(R)$ if and only if there is a probability measure μ over R such that $\varphi = \psi_\mu$.*
(ii) *$\mathcal{K}(R)$ is a compact set.*

Proof Let us identify the space $\mathcal{M}(R)$ of all signed measures over R with the dual space of $\mathcal{C}(R)$. The set $\mathcal{P}(R)$ of all probability measures over R is a convex subspace of $\mathcal{M}(R)$. Besides, $\mathcal{P}(R)$ is closed relative to the weak* topology, and

$$\{\delta_p : p \in R\}$$

is the set of extreme points of $\mathcal{P}(R)$.

By (3.1), the map

$$G : [0,1] \to \mathcal{C}(R), \quad G(t)(p) = \psi_p(t)$$

is continuous. Consequently, the map

$$D : [0,1] \times \mathcal{M}(R), \quad (t, \mu) \mapsto \int_R \psi_p(t)\, d\mu(p)$$

is continuous. We infer that the map

$$E : \mathcal{M}(R) \to \mathcal{C}([0,1]), \quad \mu \mapsto D(\cdot, \mu) = \psi_\mu,$$

besides linear, is continuous. Note that $E(\delta_p) = \psi_p$ for all $p \in R$ and that

$$E(\mathcal{P}(R)) = \mathcal{L}(R) := \left\{ \psi_\mu : \mu \in \mathcal{P}(R) \right\}.$$

Therefore, $\mathcal{L}(R)$ is compact by the Banach–Alaoglu theorem, and the convex hull of $\{\psi_p : p \in R\}$ is dense in $\mathcal{L}(R)$ by the Krein–Milman theorem. Consequently, $\mathcal{L}(R) = \mathcal{K}(R)$. ∎

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Example 3.9 Let $0 < r < \infty$ and $a \in \mathbb{R}$. Define $F_{r,a}\colon [0,\infty) \to [0,\infty)$ by

$$F_{r,a}(t) = \begin{cases} 0 & \text{if } t = 0, \\ t^r(-\log(t))^a & \text{if } 0 < t \le 1/e, \\ t^r & \text{otherwise.} \end{cases}$$

Suppose that $a > 0$. Aiming to prove that $F_{r,a} \in \mathcal{O}(r^+)$, we fix $s > r$ and consider the probability measure $\mu_{a,r,s}$ on $(0,\infty)$ given by

$$d\mu_{a,r,s}(p) = \frac{a\,(p-r)^{a-1}}{(s-r)^a}\chi_{[r,s]}(p)\,dp.$$

By definition, $\mu_{a,r,s}$ is supported on $[r,s]$. Set $\varphi_{a,r,s} = \psi_{\mu_{a,r,s}}$. We have

$$\varphi_{a,r,s}(t) = \int_r^s \frac{t^p}{p}\frac{a\,(p-r)^{a-1}}{(s-r)^a}\,dp = C(t)t^r\,(-\log(t))^a$$

for all $0 \le t < 1$, where

$$C(t) = \frac{a}{(s-r)^a}\int_0^{(r-s)/\log(t)} \frac{v^{a-1}e^{-v}}{r - v\log(t)}dv.$$

Fix $0 \le c < 1$. Since $C(t)$ is bounded away from zero and infinity when t runs over $[0,c]$, $\varphi_{a,r,s} \approx F_{r,a}$ on $[0,c]$. Therefore, $F_{r,a} \in \mathcal{O}([r,s])$.

Lemma 3.10 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Let $\mu \in \mathcal{P}(R)$ and F be an Orlicz function with $F \approx \psi_\mu$. If $r = \min(\mathrm{supp}(\mu))$, then $F \in \mathcal{O}(R, r^+)$.*

Proof Fix $s > r$. We have

$$\lambda = \mu([r,s]) > 0.$$

Let ν be the probability measure obtained by restricting μ/λ to $[r,s]$. Let $0 < t \le 1$. By Lemma 3.1,

$$\psi_\mu(t) \le \psi_\nu(t) \le \frac{1}{\lambda}\psi_\mu(t).$$

Since $\nu \in \mathcal{P}(R \cap [r,s])$, we are done. ∎

Corollary 3.11 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Let $F \in \mathcal{O}(R)$ and $b \in (0,\infty)$. Then, there is $c \in (0,\infty)$ such that $F(bt) \approx F(t)$ for $0 \le t \le c$.*

Proof If $F \in \mathcal{O}(\infty)$, the result is clear. Otherwise, there are $0 < r < s < \infty$ such that $F \in \mathcal{O}([r,s])$. Applying Lemma 3.5 puts an end to the proof. ∎

Lemma 3.12 *Suppose that a decreasing sequence $(r_n)_{n=1}^\infty$ in $(0,\infty)$ converges to $r > 0$. Set $R = \{r\} \cup \{r_n : n \in \mathbb{N}\}$. Let $(b_n)_{n=1}^\infty$ be a positive sequence with $\sum_{n=1}^\infty b_n = 1$. Define $F\colon [0,\infty) \to [0,\infty]$ by*

$$F(t) = \sum_{n=1}^\infty t^{r_n} b_n.$$

Then, $F \in \mathcal{O}(R, r^+)\backslash\mathcal{O}(r)$.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Proof By Lemma 3.10, $F \in \mathcal{O}(R, r^+)$. By the dominated convergence theorem,

$$\lim_{t\to 0^+} \frac{F(t)}{t^r} = 0.$$

Hence, $F \notin \mathcal{O}(r)$. ∎

Recall that $r \in [0,\infty]$ is an accumulation point of a set $R \subseteq [0,\infty]$ if $R \cap (V\backslash\{r\}) \neq \varnothing$ for every neighborhood V of r. If $R \cap (r,s) \neq \varnothing$ for all $s > r$, we say that r is a right-sided accumulation point of R.

Proposition 3.13 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Given $r \in R$, $\mathcal{O}(R, r^+)\backslash\mathcal{O}(r)$ is nonempty if and only if r is a right-sided accumulation point of R.*

Proof Let $F \notin \mathcal{O}(r)$ be such that $F \in \mathcal{O}(R \cap [r,s])$ for all $s > r$. Then, $\{r\} \subsetneq R \cap [r,s]$ for all $s > r$.

Suppose that r is a right-sided accumulation point of R. Let $(r_n)_{n=1}^\infty$ in R be decreasing to r. Pick $(b_n)_{n=1}^\infty$ with $\sum_{n=1}^\infty b_n = 1$. If F is as in Lemma 3.12, $F \in \mathcal{O}(R \cap [r,s])$ for all $s > r$. ∎

Corollary 3.14 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. The following are equivalent:*

- $\mathcal{O}(R) = \cup_{r\in R}\mathcal{O}(r)$.
- *R has no right-sided accumulation point.*

Proof Just combine Proposition 3.13 with Lemma 3.7. ∎

Proposition 3.15 *Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Then,*

$$\mathcal{O}(R, r^+), \quad r \in R,$$

is a partition of $\mathcal{O}(R)$.

Proof Just combine Lemma 3.7, Proposition 3.8, and Lemma 3.10. ∎

Corollary 3.16 $\mathcal{O}(r^-) := \cap_{s<r}\mathcal{O}([s,r]) = \mathcal{O}(r)$ *for all $r \in (0,\infty]$.*

Proof Let $F \in \mathcal{O}(r^-)$. By Lemma 3.15, for each $s < r$, there is $p(s) \in [s,r]$ such that $F \in \mathcal{O}(p(s)^+)$. By Lemma 3.7, $p(s) = r$ for all $s < r$. Applying Lemma 3.6(i) puts an end to the proof. ∎

Corollary 3.17 *Let $q \in (0,\infty]$. Let $R \subseteq (0,\infty]$ be closed, nonempty, and bounded away from zero. Then $\mathbf{\Psi}_q \in \mathcal{O}(R)$ if and only if $q \in R$.*

Proof Suppose that $\mathbf{\Psi}_q \in \mathcal{O}(R)$. By Proposition 3.15, there is $r \in R$ such that $\mathbf{\Psi}_q \in \mathcal{O}(r^+)$. By Lemma 3.7, $q = r$. ∎

Example 3.18 Let $0 < p < \infty$ and the functions $F_{p,a}$ of Example 3.9 with $a < 0$. Then, $F_{p,a}$ does not belong to $\mathcal{O}(R)$ for any set $R \subseteq (0,\infty]$. Otherwise, by Proposition 3.15 and Lemma 3.5, there would be $r \in (0,\infty)$ and $(C_s)_{s\geq r}$ in $(0,\infty)$ such that

$$\frac{1}{C_s}\boldsymbol{\psi}_s(t) \leq F_{p,a}(t) \leq C_r\boldsymbol{\psi}_r(t), \quad s \in (r,\infty],\ t \in [0,1].$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

The right-hand inequality would yield $p > r$. Then, applying the left-hand inequality with $r < s < p$, we would reach an absurdity.

Theorem 3.19 *Let $R \subseteq (0, \infty]$ be closed, nonempty, and bounded away from zero. Let F be a convex Orlicz function such that $F \in \mathcal{O}(R)$ and $F^* \in \mathcal{O}((0, \infty])$. Then, there is $r \in R \cap [1, \infty]$ such that $F \in \mathcal{O}(r)$.*

Proof By Proposition 3.15, there are $r \in R$ and $q \in (0, \infty]$ such that $F \in \mathcal{O}(r^+)$ and $F^* \in \mathcal{O}(q^+)$. Since F and F^* are convex, there are $0 < c < 1$ and $0 < C < \infty$ such that $\max\{F(t), F^*(t)\} \leq Ct$ for all $t \in [0, c]$. By Lemma 3.5, $r, q \in [1, \infty]$.

By Lemma 3.5, there are $0 < c_1 < 1$ and $0 < C_1 < \infty$ such that $F^*(t) \leq C_1 \boldsymbol{\Psi}_q(t)$ for all $0 \leq t \leq c_1$. Hence,

$$uv \leq F(u) + F^*(v) \leq F(u) + C_1 \boldsymbol{\Psi}_q(v), \quad 0 \leq v \leq c_1.$$

By Lemmas 2.5 and 2.6 and Corollary 3.11, there are $0 < c_2 < 1$ and $0 < C_2 < \infty$ such that $\boldsymbol{\Psi}_{q'}(t) \leq C_2 F(t)$ for all $t \in [0, c_2]$. By Lemma 3.5, $r \leq q'$.

Pick $s > q$. By Lemma 3.5, there are $0 < c_3 < 1$ and $0 < C_3 < \infty$ such that $\boldsymbol{\Psi}_s(t) \leq C_3 F^*(t)$ for all $0 \leq t \leq c_3$. By Lemma 2.5,

$$uv \leq \boldsymbol{\Psi}_{s'}(u) + \boldsymbol{\Psi}_s(v) \leq \boldsymbol{\Psi}_{s'}(u) + C_3 F^*(v), \quad 0 \leq v \leq c_3.$$

By Lemma 2.6 and Corollary 3.11, there are $0 < c_4 < 1$ and $0 < C_4 < \infty$ such that $F(t) \leq C_4 \boldsymbol{\Psi}_{s'}(t)$ for all $t \in [0, c_4]$. By Lemma 3.5, $s' \leq s_1$ for all $s_1 > r$. Consequently, $q' \leq r$.

We have proved that $q' = r$ and that $\boldsymbol{\Psi}_r(t) \leq C_2 F(t)$ for all $t \in [0, c_2]$. By Lemma 3.5, a reverse inequality holds, that is, there are $0 < c_5 < 1$ and $0 < C_5 < \infty$ such that $F(t) \leq C_5 \boldsymbol{\Psi}_r(t)$ for all $t \in [0, c_5]$. ∎

4 Subsymmetric sequences in variable exponent Lebesgue spaces

A sequence space, say $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$, will be a function space over $\mathbb{N}$ endowed with the counting measure. Let $(\boldsymbol{e}_n)_{n=1}^{\infty}$ denote the unit vectors of $\mathbb{F}^{\mathbb{N}}$. If $\|\boldsymbol{e}_n\|_{\mathbb{S}} \approx 1$ for $n \in \mathbb{N}$, we say that $\mathbb{S}$ is semi-normalized. If the mapping T_π given by

$$(a_n)_{n=1}^{\infty} \mapsto (a_{\pi(n)})_{n=1}^{\infty}$$

defines an isometry of $\mathbb{S}$ for any permutation π of $\mathbb{N}$, we say that $\mathbb{S}$ is a symmetric sequence space. In turn, if T_π defines an isometry from

$$\{f \in \mathbb{S} \colon \operatorname{supp}(f) \subseteq \pi(\mathbb{N})\}$$

onto $\mathbb{S}$ for every increasing map $\pi \colon \mathbb{N} \to \mathbb{N}$, we say that $\mathbb{S}$ is a subsymmetric sequence space. Any symmetric sequence space is subsymmetric.

We say that a sequence space $\mathbb{S}$ embeds into a function space $\mathbb{X}$ if there is an isomorphic embedding $T \colon \mathbb{S} \to \mathbb{X}$. If $(T(\boldsymbol{e}_n))_{n=1}^{\infty}$ are pairwise disjointly supported, we say that $\mathbb{S}$ disjointly embeds into $\mathbb{X}$. Note that the following are equivalent:

- $\mathbb{S}$ disjointly embeds into $\mathbb{X}$.
- There is an isomorphic embedding $T_0 \colon \mathbb{S}_0 \to \mathbb{X}$ such that $(T(\boldsymbol{e}_n))_{n=1}^{\infty}$ is disjointly supported, that is, the unit vector system of $\mathbb{S}$ is equivalent to a disjointly supported sequence in $\mathbb{X}_0$.
- There is a pairwise disjointly supported sequence $\mathfrak{X}$ in $\mathbb{X} \backslash \{0\}$ such that $\mathbb{S} = \mathbb{S}[\mathbb{X}, \mathfrak{X}]$.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Proposition 4.1 *Let $\boldsymbol{P}$ be a variable exponent on a σ-finite measure space (Ω, Σ, μ). Assume that $\boldsymbol{P}^- > 0$. Let $\mathcal{X} = (x_n)_{n=1}^{\infty}$ be a disjointly supported semi-normalized sequence in $L_{\boldsymbol{P}}$. Then, $\mathcal{X}$ has a subsequence $\mathcal{Y}$ such that $\mathbb{S}[L_{\boldsymbol{P}}, \mathcal{Y}] = \ell_F$ for some Orlicz function $F \in \mathcal{O}(R(\boldsymbol{P}))$.*

Proof We have $L_{\boldsymbol{P}}[\mathcal{X}] = \ell_F$, where $\boldsymbol{F} = (F_n)_{n=1}^{\infty}$ is given by

$$F_n(t) = \int_{\Omega} \boldsymbol{\Psi}_{P(\omega)}\left(t\,|x_n(\omega)|\right) d\mu(\omega), \quad t \in [0, \infty).$$

We have $b = \sup_n \rho_M^L\left(|x_n|\right) < \infty$. For each $n \in \mathbb{N}$, we define

$$G_n\colon [0, \infty) \to [0, \infty], \quad G_n(t) = \int_{\Omega \setminus \Omega_\infty^{\boldsymbol{P}}} \frac{t^{P(\omega)}}{\boldsymbol{P}(\omega)} \left(\frac{|x_n(\omega)|}{b}\right)^{P(\omega)} d\mu(\omega).$$

Since $\left\| x_n|_{\Omega_\infty^{\boldsymbol{P}}} \right\|_\infty \leq b$ for all $n \in \mathbb{N}$, $F_n(t) = G_n(tb)$ for all $t \in [0, 1/b]$. Besides,

$$c_n := \int_{\Omega \setminus \Omega_\infty^{\boldsymbol{P}}} \left(\frac{|x_n(\omega)|}{b}\right)^{P(\omega)} d\mu(\omega) \leq 1, \quad n \in \mathbb{N}.$$

Consider the following dichotomy:

- $\limsup_n c_n > 0$.
- $\limsup_n c_n = 0$.

In the former case, passing to a subsequence, we can assume that $\inf_n c_n > 0$. By Proposition 3.8 and Lemma 3.3, passing to a further subsequence, we can assume that there exists an Orlicz function $F \in \mathcal{O}(R(\boldsymbol{P}))$ such that

$$\sum_{n=1}^{\infty} \sup_{0 \leq t \leq 1} \left| \frac{G_n(t)}{c_n} - F(t) \right| < \infty.$$

By Theorem 2.3, we deduce that $\ell_F = \ell_F$.

In the latter case, passing to a subsequence, we can assume that $\sum_{n=1}^{\infty} c_n < \infty$. Since

$$G_n(t) \leq \frac{c_n}{\boldsymbol{P}^-}, \quad 0 \leq t \leq 1,\ n \in \mathbb{N},$$

$\ell_F = \ell_\infty$ by Theorem 2.3. It remains to be shown that $\infty \in R(\boldsymbol{P})$. Assume by contradiction that it is not the case. Then, $\boldsymbol{P}^+ < \infty$ and $\Omega_\infty^{\boldsymbol{P}}$ is a null set. If $0 < a < \inf_n \rho_M^L\left(|x_n|\right)$,

$$1 \leq \int_{\Omega} \left(\frac{|x_n(\omega)|}{a}\right)^{P(\omega)} d\mu(\omega) \leq \left(\frac{b}{a}\right)^{\boldsymbol{P}^+} c_n, \quad n \in \mathbb{N}.$$

We reach an absurdity, and the proof is over. ∎

The following result generalizes [11, Proposition 4.5].

Lemma 4.2 *Let $\boldsymbol{P}$ be a variable exponent over a σ-finite nonatomic measure space (Ω, Σ, μ). For each $n \in \mathbb{N}$, let A_n be a finite subset of $R(\boldsymbol{P})$. Set*

$$\mathcal{N} = \left\{(n, p)\colon (n, p) \in \mathbb{N} \times (0, \infty],\ p \in A_n\right\}.$$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

For each $(n,p)\in\mathcal{N}$, let $U_{n,p}$ be a neighborhood of p. Then, there is a family $(\Omega_{n,p})_{(n,p)\in\mathcal{N}}$ of pairwise disjoint measurable sets such that $\mu(\Omega_{n,p})>0$ and $\boldsymbol{P}(\Omega_{n,p})\subseteq U_{n,p}$ for all $(n,p)\in\mathcal{N}$.

Proof Assume without loss of generality that $(A_n)_{n=1}^{\infty}$ is nondecreasing. Set $A_0=\varnothing$, $A=\cup_{n=1}^{\infty}A_n$,

$$B=\{p\in A\colon \boldsymbol{P}(\mu)(\{p\})=0\},\quad \mathcal{M}=\{(n,p)\in\mathcal{N}\colon p\in B\}.$$

Let n_0 be the smallest $n\in\mathbb{N}$ such that $B_n:=A_n\cap B\neq\varnothing$. We recursively construct for each $n\in\mathbb{N}$ a pairwise disjoint family

$$(V_{n,p})_{p\in B_n}$$

such that $V_{n,p}\subseteq U_{n,p}$ is a measurable neighborhood of p for all $p\in B_n$, and the set $D_n:=\bigcup_{p\in B_n}V_{n,p}$ satisfies

$$\boldsymbol{P}(\mu)(D_n)<\frac{1}{2}\min_{p\in B_{n-1}}\boldsymbol{P}(\mu)\left(V_{n-1,p}\right)$$

as long as $n\geq n_0+1$. Set $E_n=\cup_{k=n}^{\infty}D_k$ for all $n\in\mathbb{N}$. Since $\boldsymbol{P}(\mu)(D_{n+1})\leq\boldsymbol{P}(\mu)(D_n)/2$ for all $n\in\mathbb{N}$, $n\geq n_0$, $\mu\circ\boldsymbol{P}^{-1}(E_n)\leq 2\mu\circ\boldsymbol{P}^{-1}(D_n)$ for all $n\in\mathbb{N}$, $n\geq n_0$. Consequently, $\mu\circ\boldsymbol{P}^{-1}(E_{n+1})<\mu\circ\boldsymbol{P}^{-1}\left(V_{n,p}\right)$ for all $n\in\mathbb{N}$ and $p\in B_n$. Set

$$W_{n,p}=V_{n,p}\backslash E_{n+1},\quad (n,p)\in\mathcal{M}.$$

The family $(W_{n,p})_{(n,p)\in\mathcal{M}}$ is pairwise disjoint, and $\mu\circ\boldsymbol{P}^{-1}(W_{n,p})>0$ for all $(n,p)\in\mathcal{M}$. Consider the set

$$\mathcal{L}=\left\{(n,p)\in\mathcal{M}\colon W_{n,p}\cap(A\backslash B)\right\}=\varnothing.$$

Let $\pi\colon\mathcal{N}\backslash\mathcal{L}\to A\backslash B$ be such that $\pi(n,p)=p$ for all $(n,p)\in\mathcal{N}\backslash\mathcal{M}$, and $\pi(n,p)\in W_{n,p}$ for all $(n,p)\in\mathcal{M}\backslash\mathcal{L}$. Since μ is nonatomic, for each $q\in A\backslash B$, there is a pairwise disjoint family

$$(\Omega_{n,p})_{(n,p)\in\pi^{-1}(q)}$$

of nonnull measurable subsets of $\boldsymbol{P}^{-1}(p)$. If for each $(n,p)\in\mathcal{L}$, we choose

$$\Omega_{n,p}=\boldsymbol{P}^{-1}(W_{n,p}),$$

the family $(\Omega_{n,p})_{(n,p)\in\mathcal{N}}$ satisfies the desired conditions. ∎

Proposition 4.3 *Let $\boldsymbol{P}$ be a variable exponent on a nonatomic σ-finite measure space (Ω,Σ,μ). Assume that $\boldsymbol{P}^{-}>0$. Let $\mathbb{S}$ be a subsymmetric sequence space. The following are equivalent:*

- *$\mathbb{S}$ disjointly embeds into $L_{\boldsymbol{P}}$.*
- *$\mathbb{S}=\ell_F$ for some $F\in\mathcal{O}(R(\boldsymbol{P}))$.*

Proof Bearing in mind Proposition 4.1, it suffices to prove that for any Orlicz function F such that $F|_{[0,1]}\in\mathcal{K}(R(\boldsymbol{P}))$, there exists a disjointly supported sequence $\mathcal{X}$ in $L_{\boldsymbol{P}}^0$ with $L_{\boldsymbol{P}}[\mathcal{X}]=\ell_F$. Choose a positive sequence $(\varepsilon_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. Let

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

$(F_n)_{n=1}^{\infty}$ be such that, for all $n \in \mathbb{N}$, F_n belongs to convex hull of $\{\boldsymbol{\Psi}_p : p \in R(\boldsymbol{P})\}$, and

$$\max_{0\le t\le 1} |F(t) - F_n(t)| \le \varepsilon_n.$$

Let $(A_n)_{n=1}^{\infty}$ be a sequence of finite subsets of $R(\boldsymbol{P})$ such that for each $n \in \mathbb{N}$, there is $(a_{n,p})_{p\in A_n}$ in $(0,\infty)$ such that

$$F_n = \sum_{p\in A_n} a_{n,p}\boldsymbol{\Psi}_p, \qquad \sum_{p\in A_n} a_{n,p} = 1.$$

By Lemma 3.1, for each $n \in \mathbb{N}$ and $p \in A_n$, there is a neighborhood $U_{n,p}$ of p such that $\left\|\boldsymbol{\psi}_p - \boldsymbol{\psi}_q\right\| \le \varepsilon_n$ for all $q \in U_{n,p}$. Let $(\Omega_{n,p})_{(n,p)\in\mathcal{N}}$ be the family of sets provided by Lemma 4.2. Choose, for each $(n,p) \in \mathcal{N}, f_{n,p}\colon \Omega \to [0,\infty)$ with $\operatorname{supp}(f_{n,p}) = \Omega_{n,p}$ and $\int_\Omega f_{n,p}\, d\mu = 1$. Set

$$G_{n,p}\colon [0,\infty) \to [0,\infty), \quad t \mapsto \int_\Omega \boldsymbol{\Psi}_{P(\omega)}(t) f_{n,p}(\omega)\, d\mu(\omega).$$

Note that, if $0 \le t \le 1$,

$$G_{n,p}(t) = \int_{\Omega\setminus\Omega_\infty^P} \boldsymbol{\Psi}_{P(\omega)}(t) f_{n,p}(\omega)\, d\mu(\omega).$$

Since

$$\max_{0\le t\le 1} \left|G_{n,p}(t) - \boldsymbol{\psi}_p(t)\right| \le \varepsilon_n,$$

the Musielak–Orlicz sequence $\boldsymbol{G} = (G_n)_{n=1}^{\infty}$ given by

$$G_n = \sum_{p\in A_n} a_{n,p} G_{n,p}$$

satisfies $\max_{0\le t\le 1} |G_n(t) - F(t)| \le 2\varepsilon_n$ for all $n \in \mathbb{N}$. By Theorem 2.3, $\ell_F = \ell_{\boldsymbol{G}}$. If we set

$$x_n = \sum_{p\in A_n} \left(a_{n,p} f_{n,p}\right)^{1/P},$$

then for every $n \in \mathbb{N}$ and $0 \le t \le 1$, we have

$$\int_\Omega M_P\left(t x_n(\omega)\right) d\mu(\omega) = \sum_{p\in A_n} \int_{\Omega\setminus\Omega_\infty^P} \boldsymbol{\Psi}_{P(\omega)}(t) a_{n,p} f_{n,p}(\omega)\, d\mu(\omega) = G_n(t).$$

Consequently, $\mathcal{X} = (x_n)_{n=1}^{\infty}$ is a pairwise disjointly supported sequence with $L_P[\mathcal{X}] = \ell_{\boldsymbol{G}} = \ell_F$. ∎

Corollary 4.4 (See [11, Theorem 3.5]) *Let $q \in (0,\infty]$ and $\boldsymbol{P}$ be a variable exponent over a nonatomic σ-finite measure space. Assume that $\boldsymbol{P}^- > 0$. Then, $L_{\boldsymbol{P}}$ has a disjointly supported sequence equivalent to the unit vector system of ℓ_q if and only if $q \in R(\boldsymbol{P})$.*

Proof Just combine Proposition 4.3 with Corollary 3.17. ∎

Given a function space $\mathbb{X}$, the following are equivalent:

- $\mathbb{X}$ fails to be absolutely continuous;
- ℓ_∞ embeds into $\mathbb{X}$;
- ℓ_∞ disjointly embeds into $\mathbb{X}$

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

(see [20, Proposition 1.a.7]). Since any separable Banach space linearly embeds into ℓ_∞ by the Hahn–Banach theorem, the subsymmetric embedding problem for locally convex function spaces splits into two issues. Namely,

- determining which nonseparable subsymmetric sequence spaces embed into a given non absolutely continuous function space, and
- determining which separable subsymmetric sequence spaces embed into a given absolutely continuous function space.

We give now a result for variable exponent Lebesgue spaces framed in the latter problem.

Theorem 4.5 *Let $\boldsymbol{P}$ be a variable exponent over a σ-finite nonatomic measure space μ. Suppose that $\boldsymbol{P}^- \geq 2$ and $q := \boldsymbol{P}^+ < \infty$. Let $\mathbb{S}$ be a subsymmetric sequence space. Then, $\mathbb{S}$ embeds into $L_{\boldsymbol{P}}(\mu)$ if and only if $\mathbb{S} = \ell_2$ or $\mathbb{S} = \ell_F$ for some $F \in \mathcal{O}(R(\boldsymbol{P}))$.*

Proof Assume without loss of generality that μ is finite. Suppose that a symmetric sequence space $\mathbb{S}$ other than ℓ_2 embeds into $L_{\boldsymbol{P}}(\mu)$. By [2, Theorem 4.2], $L_{\boldsymbol{P}}(\mu)$ contains a disjointly supported sequence equivalent to the canonical basis of $\mathbb{S}$, and, by Proposition 4.3, $\mathbb{S} = \ell_F$ for some $F \in \mathcal{O}(R(\boldsymbol{P}))$.

Proposition 4.3 also gives that ℓ_F embeds into $L_{\boldsymbol{P}}(\mu)$ for any $F \in \mathcal{O}(R(\boldsymbol{P}))$. We close the proof by noticing that, since $L_q(\mu) \subseteq L_{\boldsymbol{P}}(\mu) \subseteq L_2(\mu)$, any Rademacher sequence over Ω is equivalent to the canonical ℓ_2-basis by Khintchine's inequalities. ∎

The behavior of variable exponent Lebesgue spaces $L_{\boldsymbol{P}}$ in the case where $\boldsymbol{P}^- < 2$ dramatically opposes that described in Theorem 4.5.

Proposition 4.6 *Let $\boldsymbol{P}$ be a variable exponent over a σ-finite nonatomic measure space μ. Suppose that $\boldsymbol{P}^- < 2$. Let F be an Orlicz function satisfying (1.1) for some $p \in (\boldsymbol{P}^-, \infty) \cap [1, 2]$. Then ℓ_F isomorphically embeds into $L_{\boldsymbol{P}}$.*

Proof Just combine Proposition 2.9 with Theorem 1.1. ∎

We now pass to the discrete case. To compare the behavior of non-atomic variable exponent Lebesgue spaces with that of atomic ones, we adapt Peirats–Ruiz's results from [24] on the latter spaces to our language. Given a sequence $\boldsymbol{P}\colon \mathbb{N} \to (0, \infty]$, we set $\boldsymbol{P}_m = (p_{n+m-1})_{n=1}^\infty$ for all $m \in \mathbb{N}$. Consider the set of Orlicz functions

$$\mathcal{K}(\boldsymbol{P}) = \bigcap_{m=1}^{\infty} \mathcal{K}(R(\boldsymbol{P}_m)).$$

We denote by $\mathcal{O}(\boldsymbol{P})$ the set of all Orlicz functions for which there is $\varphi \in \mathcal{K}(\boldsymbol{P})$ and $0 \leq c \leq 1$ such that $F(t) \approx \varphi(t)$ for all $0 \leq t \leq c$.

Denote by $A(\boldsymbol{P})$ the set of all limit points of $\boldsymbol{P}$, that is,

$$A(\boldsymbol{P}) = \bigcap_{m=1}^{\infty} R(\boldsymbol{P}_m).$$

Note that any accumulation point of $R(\boldsymbol{P})$ is a limit point of $\boldsymbol{P}$, but the converse does not hold.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Theorem 4.7 (cf. [24, Proposition 2.3]) *Let $\boldsymbol{P} = (p_n)_{n=1}^{\infty}$ be a variable exponent with $\boldsymbol{P}^- > 0$ and $\boldsymbol{P}^+ < \infty$. Let $\mathbb{S}$ be a subsymmetric sequence space. Then, $\mathbb{S}$ isomorphically embeds into $\ell_{\boldsymbol{P}}$ if and only if $\mathbb{S} = \ell_F$ for some Orlicz function $F \in \mathcal{O}(\boldsymbol{P})$. Further, for each $F \in \mathcal{O}(\boldsymbol{P})$, there is $r \in A(\boldsymbol{P})$ such that $F \in \mathcal{O}(r^+)$.*

Proof Applying [24, Proposition 2.3] to the case where the Musielak–Orlicz sequence consists of power Orlicz functions yields the wished-for characterization of isomorphic embeddability. Assume that $F \in \mathcal{O}(\boldsymbol{P})$. By Proposition 3.15, for each $m \in \mathbb{N}$, there is $r_m \in R(\boldsymbol{P}_m)$ such that $F \in \mathcal{O}(R(\boldsymbol{P}_m), r_m^+)$. By Lemma 3.7, there is $r \in (0, \infty]$ such that $r = r_m$ for all $m \in \mathbb{N}$. ∎

Theorem 4.7 implies that if a subsymmetric sequence space $\mathbb{S}$ isomorphically embeds into the Nakano space $\ell_{\boldsymbol{P}}$, then there are $r \in A(\boldsymbol{P})$ and $F \in \mathcal{O}(R(\boldsymbol{P}), r^+)$ such that $\mathbb{S} = \ell_F$. The following example shows that this condition does not characterize isomorphic embeddability.

Example 4.8 Let $\boldsymbol{P} = (p_n)_{n=1}^{\infty}$ be a variable exponent for which there exists $\lim_n p_n = p \in (0, \infty)$. Lemma 3.12 implies that, despite $A(\boldsymbol{P}) = \{p\}$, $\mathcal{O}(R(\boldsymbol{P}))$ contains Orlicz functions that are not equivalent to the potential Orlicz function $\boldsymbol{\Psi}_p$ near the origin. However, ℓ_p is the unique subsymmetric space that embeds into $\ell_{\boldsymbol{P}}$. Indeed, if $\varphi \in \mathcal{K}(\boldsymbol{P})$, then, by Lemma 3.5,

$$\frac{t^{p_m}}{p_m} \le \varphi(t) \le \frac{t^p}{p}, \quad 0 \le t \le 1,\ m \in \mathbb{N}.$$

Letting m tend to infinity, we get $\varphi(t) = \boldsymbol{\Psi}_p(t)$ for all $0 \le t \le 1$.

To close this section, we provide a Nakano space that exhibits a behavior opposed to that of the Nakano space in Example 4.8.

Example 4.9 Let $\boldsymbol{Q} = (q_j)_{j=1}^{\infty}$ be a sequence in $(0, \infty)$ decreasing to $q \in (0, \infty)$. We have

$$R := R(\boldsymbol{Q}) = \{q_j : j \in \mathbb{N}\} \cup \{q\}.$$

Let $\sigma : \mathbb{N}^2 \to \mathbb{N}$ be a bijection. Let $\beta : \mathbb{N} \to \mathbb{N}$ be the second component of the inverse of σ. Consider the variable exponent $\boldsymbol{P} = (p_n)_{n=1}^{\infty}$ given by

$$p_n = q_{\beta(n)}.$$

We have $\mathcal{K}(\boldsymbol{P}) = \mathcal{K}(R)$. Hence, $\ell_{\boldsymbol{P}}$ contains an isomorphic copy of ℓ_F for every $F \in \mathcal{O}(R)$. Further, $A(\boldsymbol{P}) = R$.

5 Complemented sequences in variable exponent Lebesgue spaces

Let $\mathbb{X}$ be a function space over a σ-finite measure space (Ω, Σ, μ). We say that a sequence space $\mathbb{S}$ *complementably embeds* into $\mathbb{X}$ if there are bounded linear maps $T : \mathbb{S} \to \mathbb{X}$ and $P : \mathbb{X} \to \mathbb{S}$ such that $P \circ T = \mathrm{Id}_{\mathbb{S}}$. If, $\mathbb{S}$ complementably embeds into $\mathbb{X}$, then there are $\mathcal{X} = (x_n)_{n=1}^{\infty}$ in $\mathbb{X}$ and $\mathcal{X}^* = (x_n^*)_{n=1}^{\infty}$ in $\mathbb{X}^*$, called projecting functionals for $\mathcal{X}$, such that the unit vector system of $\mathbb{S}$ is equivalent to $\mathcal{X}$, $(\mathcal{X}, \mathcal{X}^*)$ is a biorthogonal

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

system, and the mapping

$$f \mapsto \sum_{n=1}^{\infty} x_n^*(f)x_n$$

defines a bounded operator from $\mathbb{X}$ into $\mathbb{X}$. Besides, if $\mathbb{S}$ is absolutely continuous or $\mathcal{X}$ is disjointly supported, then the converse also holds. If $\mathcal{X}$ is disjointly supported and $\mathcal{X}^*$ is a disjointly supported sequence in $\mathbb{X}'$, we say that $\mathbb{S}$ *disjointly complementably embeds* into $\mathbb{X}$.

Lemma 5.1 *Let $\mathbb{X}$ be a function space and $\mathbb{S}$ be a sequence space. Suppose that $\mathbb{X}$ is L-convex. Suppose that there is a disjointly supported sequence $\mathcal{X} = (x_n)_{n=1}^{\infty}$ in $\mathbb{X}$ with projecting functionals $\mathcal{X}^*$ in $\mathbb{X}'$ such that $\mathbb{S} = \mathbb{S}[\mathbb{X}, \mathcal{X}]$. Then, $\mathbb{S}$ complementably disjointly embeds into $\mathbb{X}$.*

Proof Let P be the endomorphism of $\mathbb{X}$ given by

$$f \mapsto \sum_{n=1}^{\infty} x_n^*(f)x_n.$$

By [14, Theorem 3.3], there is a constant C such that

$$\left\| \left(\sum_{j\in J} \left|P(f_j)\right|^2 \right)^{1/2} \right\| \leq C \left\| \left(\sum_{j\in J} \left|f_j\right|^2 \right)^{1/2} \right\|$$

for every family $(f_j)_{j\in J}$ in $\mathbb{X}$. Pick a partition $(A_n)_{n=1}^{\infty}$ of (Ω, Σ, μ) so that $\operatorname{supp}(x_n) \subseteq A_n$ for all $n \in \mathbb{N}$. Set $y_n^* = x_n^* \chi_{A_n}$ for all $n \in \mathbb{N}$. Given $f \in \mathbb{X}$, applying the above estimate to $(f\chi_{A_n})_{n=1}^{\infty}$, we obtain

$$\left\| \sum_{n=1}^{\infty} y_n^*(f)x_n \right\| \leq C \|f\|.$$

Consequently, $(y_n^*)_{n=1}^{\infty}$ are projecting functionals for $\mathcal{X}$. ∎

Proposition 5.2 *Let $\boldsymbol{P}$ be a variable exponent over a σ-finite measure space (Ω, Σ, μ). Suppose that $1 \leq \boldsymbol{P}^- \leq \boldsymbol{P}^+ < \infty$ and that a subsymmetric sequence space $\mathbb{S}$ complementably disjointly embeds into $\boldsymbol{L}_{\boldsymbol{P}}$. Then, $\mathbb{S} = \ell_q$ for some $q \in R(\boldsymbol{P})$.*

Proof There are pairwise disjointly supported sequences $\mathcal{X} = (x_n)_{n=1}^{\infty}$ in $\boldsymbol{L}_{\boldsymbol{P}}$ and $\mathcal{X}' = (x_n')_{n=1}^{\infty}$ in $\boldsymbol{L}_{\boldsymbol{P}'}$ such that $\mathcal{X}$ is equivalent to the unit vector system of $\mathbb{S}$ and $\mathcal{X}'$ is equivalent to the unit vector system of $\mathbb{S}'$. By Proposition 4.1, there are $F \in \mathcal{O}(R(\boldsymbol{P}))$ and $G \in \mathcal{O}(R(\boldsymbol{P}'))$ such that $\mathbb{S} = \ell_F$ and $\mathbb{S}' = \ell_G$. We have $\ell_{F^*} = \ell_G$, whence, by Corollary 3.11, $F^* \approx G$ near the origin. Consequently, $F^* \in \mathcal{O}(R(\boldsymbol{P}'))$. By Theorem 3.19, $F \in \mathcal{O}(q)$ for some $q \in R(\boldsymbol{P})$. Hence, $F \approx t^q$. ∎

Lemma 5.3 *Let $\boldsymbol{P}$ be a variable exponent over a nonatomic σ-finite measure space (Ω, Σ, μ). Suppose that $\boldsymbol{P}^- > 0$. Let $\mathbb{S}$ be a subsymmetric sequence space. Set $\boldsymbol{P}_c = \boldsymbol{P}|_{\Omega_c^{\boldsymbol{P}}}$. Then, $\mathbb{S}$ complementably embeds into $L_{\boldsymbol{P}}$ if and only if it complementably embeds into $L_{\boldsymbol{P}_c}$.*

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Proof Let $T\colon \mathbb{S} \to L_{\boldsymbol{P}}$ and $P\colon L_{\boldsymbol{P}} \to \mathbb{S}$ be such that $P \circ T = \mathrm{Id}_{\mathbb{S}}$. Let J be the canonical embedding of $L_{\boldsymbol{P}_c}$ into $L_{\boldsymbol{P}}$, and Q be the canonical projection from $L_{\boldsymbol{P}}$ into $L_{\boldsymbol{P}_c}$. By Proposition 2.7, $P = P \circ J \circ Q$. Consequently, $\mathbb{S}$ complementably embeds into $L_{\boldsymbol{P}_c}$. ∎

Theorem 5.4 *Let $\boldsymbol{P}$ be a variable exponent over a nonatomic σ-finite measure space (Ω, Σ, μ). Suppose that $\boldsymbol{P}^- > 0$ and $q := \boldsymbol{P}^+ < \infty$ and that $\boldsymbol{P}^{-1}((1,\infty))$ is not null. Let $\mathbb{S}$ be a subsymmetric sequence space. Set $\boldsymbol{P}_c = \boldsymbol{P}|_{\Omega_c^{\boldsymbol{P}}}$. The following are equivalent:*

(a) *$\mathbb{S}$ complementably embeds into $L_{\boldsymbol{P}}$.*
(b) *$\mathbb{S} = \ell_2$ or $\mathbb{S}$ complementably disjointly embeds into $L_{\boldsymbol{P}}$.*
(c) *$\mathbb{S} = \ell_r$ for some $r \in \{2\} \cup R(\boldsymbol{P}_c)$.*

Proof There is $p > 1$ such that $\Omega_0 := \boldsymbol{P}^{-1}([p,q])$ is not null. We can, without loss of generality, assume that μ is finite and $\mu(\Omega_0) = 1$.

Let $(r_n)_{n=1}^\infty$ be a Rademacher sequence over Ω_0. Let S be the associated embedding of ℓ_2 into $L_q(\Omega_0)$, and Q the associated projection from $L_p(\Omega_0)$ onto ℓ_2. Let J be the canonical map from $L_q(\Omega_0)$ into $L_{\boldsymbol{P}}$, and P be the canonical map from $L_{\boldsymbol{P}}$ into $L_p(\Omega_0)$. The maps $J \circ S$ and $Q \circ P$ witness that ℓ_2 complementably embeds into $L_{\boldsymbol{P}}$.

Suppose a subsymmetric space $\mathbb{S}$ other than ℓ_2 complementably embeds into $\mathbb{X} := L_{\boldsymbol{P}}$. Set $\Omega_1 = \boldsymbol{P}^{-1}([1,2))$ and $\Omega_2 = \boldsymbol{P}^{-1}([2,\infty))$ and $\boldsymbol{P}_j = \boldsymbol{P}|_{\Omega_j}$, $j = 1, 2$. By Lemma 5.3 and [2, Theorem 5.2], there is $a \in \{1,2\}$ such that Ω_a is nonnull and $\mathbb{S}$ complementably disjointly embeds into $L_{\boldsymbol{P}_a}$. By Proposition 5.2, $F \in \mathcal{O}(r)$ for some $r \in R(\boldsymbol{P}_a)$.

In [11, Proposition 4.4] it is proved that ℓ_r complementably embeds into $L_{\boldsymbol{P}}$ for every $r \in R(\boldsymbol{P}_c)$. For the sake of completeness and clarity, we use our approach to variable exponent Lebesgue spaces to reprove this result. Choose $(\varepsilon_n)_{n=1}^\infty$ in $(0,\infty)$ with $\sum_{n=1}^\infty \varepsilon_n < \infty$. By Lemma 3.1, there is a sequence $(V_n)_{n=1}^\infty$ of neighborhoods of r such that

$$\left\| \boldsymbol{\psi}_p - \boldsymbol{\psi}_r \right\|_\infty \le \varepsilon_n, \quad \left\| \boldsymbol{\psi}_{p'} - \boldsymbol{\psi}_{r'} \right\|_\infty \le \varepsilon_n$$

for all $n \in \mathbb{N}$ and $p \in V_n$. Use Lemma 4.2 to choose pairwise disjoint measurable sets $(\Omega_n)_{n=1}^\infty$ such that $0 < \mu(\Omega_n) < \infty$ and $\boldsymbol{P}(\Omega_n) \subseteq V_n \cap [1,\infty]$ for all $n \in \mathbb{N}$. Use Lemma 2.8 to pick sequences $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ of nonnegative measurable functions such that $\operatorname{supp}(f_n) = \operatorname{supp}(g_n) \subseteq \Omega_n$ and

$$\int_\Omega f_n g_n \, d\mu = \rho_P^L(f_n) = \rho_Q^L(g_n) = 1$$

for all $n \in \mathbb{N}$. By Proposition 4.3, there are bounded linear maps

$$T\colon \ell_r \to L_{\boldsymbol{P}}, \quad S\colon \ell_{r'} \to L_{\boldsymbol{P}'}$$

given by

$$(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n f_n, \quad (a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n g_n,$$

respectively. The map S is a kernel operator whose conjugate map $S'\colon L_{\boldsymbol{P}} \to \ell_r$ is given by

$$S'(g)(n) = \int_\Omega g_n(\omega) g(\omega) \, d\mu(\omega)$$

for $n \in \mathbb{N}$. Since $S' \circ T = \mathrm{Id}_{\ell_r}$, we are done. ∎

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Theorem 5.5 *Let $\boldsymbol{P}$ be a variable exponent over a nonatomic σ-finite measure space (Ω, Σ, μ). Suppose $0 < \boldsymbol{P}^- \le \boldsymbol{P}^+ \le 1$. Given a semi-normalized subsymmetric sequence space $\mathbb{S}$, the following are equivalent:*

- *$\mathbb{S}$ complementably embeds into $L_{\boldsymbol{P}}$.*
- *$\mathbb{S}$ complementably disjointly embeds into $L_{\boldsymbol{P}}$.*
- *$\mathbb{S} = \ell_1$ and $\Omega_1 := \boldsymbol{P}^{-1}(1)$ is not null.*

Proof Set $\mu_1 = \mu|_{\Omega_1}$ and $\boldsymbol{P}_1 = \boldsymbol{P}|_{\Omega_1}$. If Ω_1 is nonnull, then $L_{\boldsymbol{P}_1} = L_1(\mu_1)$.

Suppose that $\mathbb{S}$ complementably embeds into $L_{\boldsymbol{P}}$. By Lemma 5.3, Ω_1 is nonnull and $\mathbb{S}$ complementably embeds into $L_1(\mu_1)$. By [17], $\mathbb{S} = \ell_1$.

Conversely, if Ω_1 is not null, it is well-known that ℓ_1 complementably disjointly embeds into $L_1(\mu_1)$. ∎

Lemma 5.6 *Suppose a subsymmetric sequence space $\mathbb{S}$ complementably embeds into an absolutely continuous sequence space $\mathbb{X}$ over $\mathbb{N}$. Then, $\mathbb{S}$ complementably disjointly embeds into $\mathbb{X}$.*

Proof Let $\mathcal{X} = (x_n)_{n=1}^{\infty}$ in $\mathbb{X}$ with projecting functionals $\mathcal{X}^* = (x_n^*)_{n=1}^{\infty}$ be equivalent to the unit vector system of $\mathbb{S}$. By the Cantor diagonal technique, passing to a subsequence, we can assume that $\mathcal{X}$ converges pointwise. Since $(x_{2n-1})_{n=1}^{\infty}$ is equivalent to $(x_{2n})_{n=1}^{\infty}$, the mapping

$$f \mapsto \sum_{n=1}^{\infty} x_{2n-1}^*(f) x_{2n-1} - \sum_{n=1}^{\infty} x_{2n-1}^*(f) x_{2n}$$

defines an endomorphism of $\mathbb{X}$. Consequently, we can replace $\mathcal{X}$ by

$$(x_{2n-1} - x_{2n})_{n=1}^{\infty}$$

and $\mathcal{X}^*$ by

$$(x_{2n-1}^*)_{n=1}^{\infty}.$$

In this way, we can assume that $\mathcal{X}$ converges to zero pointwise. By the gliding-hump technique (see [19, Proposition 1.a.9] or [2, Lemma 3.5]), passing to a subsequence, we can assume that $\mathcal{X}$ is finitely disjointly supported. We conclude the proof by applying Lemma 5.1. ∎

Our latest result improves upon a Peirats–Ruiz's achievement from [24].

Theorem 5.7 (cf. [24, Theorem 2.9]) *Let $\boldsymbol{P}\colon \mathbb{N} \to (0, \infty)$ be a variable exponent with $\boldsymbol{P}^- > 0$ and $\boldsymbol{P}^+ < \infty$. Let $\mathbb{S}$ be a subsymmetric sequence space. The following are equivalent:*

(i) *$\mathbb{S}$ complementably embeds into $\ell_{\boldsymbol{P}}$.*
(ii) *$\mathbb{S} = \ell_r$ for some $r \in A(\boldsymbol{P})$.*
(iii) *There is a subsequence $\boldsymbol{Q}$ of $\boldsymbol{P}$ such that $\mathbb{S} = \ell_{\boldsymbol{Q}}$.*

Proof (i) implies (ii) by Lemma 5.6 and Proposition 5.2. In turn, (ii) implies (iii) by Corollary 4.4 and Lemma 3.1. Since it is obvious that (iii) implies (i), we are done. ∎

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

Data availability statement Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Competing interests The authors have no competing interests to declare that are relevant to the content of this article.

References

[1] J. L. Ansorena and G. Bello, *Toward an optimal theory of integration for functions taking values in quasi-Banach spaces*. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **116**(2022), no. 2, Article no. 85, 38 pp. MR4396837.

[2] J. L. Ansorena and G. Bello, *Unconditional basic sequences in function spaces with applications to Orlicz spaces*. Positivity **29**(2025), no. 1, Article no. 1, 36 pp. MR4822119.

[3] J. L. Ansorena and A. Marcos, *A characterization of complete topological vector spaces with applications to spaces of measurable functions*. Involve (2026). Accepted for publication.

[4] T. Aoki, *Locally bounded linear topological spaces*. Proc. Imp. Acad. Tokyo **18**(1942), 588–594. MR14182.

[5] S. Banach, *Théorie des opérations linéaires*, Éditions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original. MR1357166.

[6] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, 129, Academic Press, Inc., Boston, MA, 1988. MR928802.

[7] D. G. Bourgin, *Linear topological spaces*. Amer. J. Math. **65**(1943), 637–659. MR9104.

[8] J. Bretagnolle and D. Dacunha-Castelle, *Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces L^p*. Ann. Sci. École Norm. Sup. (4) **2**(1969), 437–480. MR265930.

[9] C. Carathéodory, *Die homomorphieen von Somen und die Multiplikation von Inhaltsfunktionen*. Ann. Scuola Norm. Super. Pisa Cl. Sci. (2) **8**(1939), no. 2, 105–130. MR1556820.

[10] J. Flores, F. L. Hernández, C. Ruiz, and M. Sanchiz, *On the structure of variable exponent spaces*. Indag. Math. (N.S.) **31**(2020), no. 5, 831–841. MR4143506.

[11] F. L. Hernández and C. Ruiz, *l_q-structure of variable exponent spaces*. J. Math. Anal. Appl. **389**(2012), no. 2, 899–907. MR2879267.

[12] M. I. Kadets, *Linear dimension of the spaces L^p and l^q*. Uspehi Mat. Nauk **13**(1958), no. 6(84), 95–98. MR101486.

[13] M. I. Kadets and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*. Stud. Math. **21**(1961/1962)), 161–176. MR0152879.

[14] N. J. Kalton, *Convexity conditions for nonlocally convex lattices*. Glasg. Math. J. **25**(1984), no. 2, 141–152. MR752808.

[15] M. Kanter, *Stable laws and the embedding of L^p spaces*. Amer. Math. Mon. **80**(1973), 403–407. MR319277.

[16] A. Y. Khintchine, *Über dyadische Brüche*. Math. Z. **18**(1923), no. 1, 109–116. MR1544623.

[17] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in L_p-spaces and their applications*. Stud. Math. **29**(1968), 275–326. MR0231188.

[18] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces. III*. Israel J. Math. **14**(1973), 368–389. MR322476.

[19] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I—Sequence spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], Springer-Verlag, Berlin, 1977. MR0500056.

[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II—Function spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97, Springer-Verlag, Berlin, 1979. MR540367.

[21] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, 1983. MR724434.

[22] H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen Co., Ltd, Tokyo, 1950. MR0038565.

[23] R. E. A. C. Paley, *Some theorems on abstract spaces*. Bull. Amer. Math. Soc. **42**(1936), no. 4, 235–240. MR1563277.

[24] V. Peirats and C. Ruiz, *On l^p-copies in Musielak-Orlicz sequence spaces*. Arch. Math. **58**(1992), no. 2, 164–173. MR1143166.

[25] A. Pełczyński, *Projections in certain Banach spaces*. Stud. Math. **19**(1960), 209–228. MR126145.

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press

[26] S. Rolewicz, *On a certain class of linear metric spaces*. Bull. Acad. Polon. Sci. Cl. III **5** (1957), 471–473. MR0088682.
[27] C. Schütt, *On the embedding of 2-concave Orlicz spaces into* L^1. Stud. Math. **113**(1995), no. 1, 73–80. MR1315523.

Universidad de La Rioja, Spain
e-mail: joseluis.ansorena@unirioja.es

Universidad de Zaragoza, Spain
e-mail: gbello@unizar.es

https://doi.org/10.4153/S0008414X26102272 Published online by Cambridge University Press