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A geometric criterion for decomposition and multivalence

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Abstract

We give a quite general geometric criterion for a function analytic in the unit disc to be a polynomial of a univalent function, and hence a criterion for multivalence. We believe that this is the essence why multivalent close-to-convex functions enjoy the latter decomposition property. As another application, we study, as suggested by T. Sheil-Small [**9**], the geometry of classes of analytic functions which arise from his recent investigation of multivalent harmonic mappings.

1. *Preliminaries*

Let $\mathbb{B}$ be the open unit disc $\{z;|z|<1\}$, and let $\bar{\mathbb{B}}$ and T be its closure and boundary respectively. For $a, b \in T$ we let $[a, b]$ denote the closed subarc of T obtained by going positively along T from a to b. We say that $f\colon \mathbb{B}\to\mathbb{C}$ is p-valent if f takes on every complex number at most p times and some number exactly p times, counting multiplicity.

Suppose that $f\colon \bar{\mathbb{B}}\to\mathbb{C}$ is a continuous function analytic in $\mathbb{B}$. There is no ambiguity in saying that f is locally topologically (conformally) z^n, where $n\in\mathbb{N}$, at points $z\in\mathbb{B}$; however, for $z\in T$ we say f is *locally topologically* z^n *at* z if we can find a relatively open neighbourhood $N(z)$ of z in $\bar{\mathbb{B}}$, and homeomorphisms $h_1\colon N(z)\to S$, where $S=\{z;|z|<1, \operatorname{Im} z\geqslant 0\}$, and $h_2\colon \mathbb{C}\to\mathbb{C}$ such that $h_1(z_0)=h_2\circ f(z_0)=0$ and $h_2\circ f\circ h_1^{-1}(z)=z^n$ for $z\in S$. Let Y be the Riemann surface formed from pairs $(z, f(z))$, for $z\in\mathbb{B}$, with the topological and conformal structures induced from $\mathbb{B}$ via f, and let $\pi\colon Y\to\mathbb{C}$ be the *projection map* defined by $\pi((z,f(z)))=f(z)$. Then (Y,π) is an analytic covering over the image plane of f which is called the *image surface* of f. The surface Y is the interior of the *bordered Riemann surface* $\bar{Y}$ which is constructed like Y from points $(z,f(z))$ for $z\in\bar{\mathbb{B}}$, and inherits via f both the topological and conformal structures of $\bar{\mathbb{B}}$. If we extend π to $\bar{Y}$ by $\pi((z,f(z)))=f(z)$ for $z\in T$, and assume that f is locally topologically z^n at every $z\in T$, where n depends on z, then we call $(\bar{Y},\pi)$ the *bordered image surface* of f. Let $\tilde{f}$ be the *lift* of f to $\bar{Y}$; that is $f(z)=(z,f(z))$. Then $f=\pi\circ\tilde{f}$, $\tilde{f}\colon \bar{\mathbb{B}}\to Y$ is a homeomorphism, $\tilde{f}\colon \mathbb{B}\to Y$ is a conformal map, and $\tilde{f}(T)$ is the border of $\bar{Y}$ or Y. We call the *positive orientation* of $\bar{Y}$ the orientation inherited from the positive orientation of $\bar{\mathbb{B}}$ via $\tilde{f}$. This orientation induces positive orientations on both Y and its border $\tilde{f}(T)$.

Note that the surface-theoretic concepts, notations and theory appearing in this paper are adaptations of the corresponding ones in Ahlfors and Sario [**1**] (see pp. 23–27 and 117–119).

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Now, suppose that $f: \bar{\mathbb{B}} \to \mathbb{C}$ is a continuous function. We say that f is *locally univalent* on T if every $z \in T$ has a neighbourhood $N(z)$ such that f is a homeomorphism in $N(z) \cap \bar{\mathbb{B}}$; and that f is *locally Jordan on* T if every $z \in T$ belongs to an open subarc of T on which f is injective.

The primary objective of this paper is to give a geometric condition for a continuous function $f: \bar{\mathbb{B}} \to \mathbb{C}$, analytic in $\mathbb{B}$, to be a polynomial of a univalent function. We do this in Theorems 1, 2 and 3. Theorems 1 and 3 deal with those functions satisfying respectively the local univalence property on T and the local Jordan property on T. Theorem 2 generalizes Theorem 1 to functions that may not be continuous on $\bar{\mathbb{B}}$. It turns out that a function with the local Jordan property on T is locally univalent on T with the exception of at most finitely many points. Hence Theorem 3 yields Theorem 1 which is left independent for the sake of clarity.

As applications, first we conclude an earlier result of the second author: every multivalent close-to-convex function is a polynomial of a univalent function [**6**]. Next, we introduce some geometric classes of functions and show that they also satisfy the above decomposition property. Included are the classes of functions arising recently in the study of T. Sheil-Small [**9**] of multivalent harmonic maps. In his paper, Sheil-Small has expressed interest in the geometry of such classes. We analyse this geometry and deduce using Theorem 3 some of the results obtained by Sheil-Small [**9**].

2. *Statement and proof of Theorem* 1

Let $f: \bar{\mathbb{B}} \to \mathbb{C}$ be a continuous function, analytic in $\mathbb{B}$, and locally Jordan on T; and let $f(T)$ inherit the positive orientation of T under f. Then for every point $w = f(z)$ where $z \in T$ there is a Jordan arc $\gamma(w)$ from w to infinity such that for some (non-degenerate) subarc $\delta(w)$ of $\gamma(w)$ and open subarc $\sigma(w)$ of $f(T)$ containing w we have (*a*) $\sigma(w) \cap \delta(w) = \{w\}$, and (*b*) $\delta(w)$ lies on the right of $\sigma(w)$. We call $\gamma(w)$ an *outer Jordan arc of* $f(T)$ *at* w.

Let $z_1, \ldots, z_n$ be distinct points of T, and $w_j = f(z_j)$ and $\gamma_j = f([z_j, z_{j+1}])$, with $z_{n+1} = z_1$. We say that f *satisfies property* (A) if we can find $z_1, \ldots, z_n$, and outer Jordan arcs $\gamma(w_1)$, $\ldots$, $\gamma(w_n)$ of $f(T)$ at $w_1, \ldots, w_n$, respectively, such that every $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ is a Jordan arc in $\mathbb{C}$ with $w_{n+1} = w_1$.

THEOREM 1. *Let* $f: \bar{\mathbb{B}} \to \mathbb{C}$ *be a continuous function, analytic in* $\mathbb{B}$, *locally univalent on* T *and satisfying property* (A). *If* f' *has* $p-1$ *zeros (counting multiplicity) in* $\mathbb{B}$, *then* $f = P \circ \phi$ *where* P *is a polynomial of degree* p *and* $\phi: \bar{\mathbb{B}} \to \mathbb{C}$ *is a homeomorphism into* $\mathbb{C}$. *In particular,* f *is at most* p*-valent.*

The idea of the proof is based on embedding the image surface of f in the Riemann sphere $\mathbb{P}$ which is viewed as a p-sheeted covering of itself. This method was used earlier in [**6**] and [**7**].

3. *Proof of Theorem* 1

Let $\gamma = f(T)$. Suppose that γ and every subarc γ_j of γ carry the positive orientation inherited from T under f.

Every Jordan arc $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ divides the sphere into two Jordan domains. We distinguish these domains according to their location relative to γ_j. Let G_j denote

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the domain lying on the right of γ_j. Since the arcs $\gamma(w_j)$ and $\gamma(w_{j+1})$ leave γ off to the right, it follows that γ_j has an open neighbourhood 0 and $[z_j, z_{j+1}]$ has an open neighbourhood W such that $G_j \cap 0$ and $f(W \cap \mathbb{B}) \cap 0$ lie adjacently on the opposite sides of γ_j.

Let $\bar{Z}_j$ denote the bordered Riemann surface obtained by taking the product space $\bar{G}_j \times \{j\}$, together with the topological and conformal structures of $\bar{G}_j$ inherited from $\bar{G}_j$ via the map $\pi_j: \bar{Z}_j \to \bar{G}_j$ defined by $\pi_j((w,j)) = w$ for all $w \in \bar{G}_j$. Then every pair $(\bar{Z}_j, \pi_j)$ constitutes a bordered covering surface of $\bar{G}_j$ whose border is given by $B_j = \tilde{\gamma}_j(w_j) \cup \tilde{\gamma}_j \cup \tilde{\gamma}_j(w_{j+1})$, where $\tilde{\gamma}_j(w_j) = \pi_j^{-1}(\gamma_j(w_j))$, $\tilde{\gamma}_j = \pi^{-1}(\gamma_j)$ and $\tilde{\gamma}_j(w_{j+1}) = \pi^{-1}(\gamma_j(w_{j+1}))$.

Now we weld the bordered surfaces $\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_n$ together to a single bordered Riemann surface by identifying closed arcs of their borders (see [1], pp. 118–119). First, let $\bar{Z}$ be the topological sum of $\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_n$, then identify for all $1 \leqslant j < n$ the points of $\tilde{\gamma}_j(w_{j+1})$ and $\tilde{\gamma}_{j+1}(w_{j+1})$ lying over the same point in $\gamma(w_{j+1})$ under π_j and π_{j+1}, respectively, with $\gamma_{n+1}(w_{n+1}) = \gamma_1(w_1)$. Note that the positive orientations of $\bar{Z}_j$ and $\bar{Z}_{j+1}$, with $\bar{Z}_{n+1} = \bar{Z}_1$, induce opposite orientations on $\tilde{\gamma}_j(w_{j+1})$ and $\tilde{\gamma}_{j+1}(w_{j+1})$ due to the choice of the domains G_j. Then it is immediate (see [1], pp. 118–119) that the identification results in a bordered Riemann surface, which we again denote $\bar{Z}$, for convenience. This surface has border $B_{\bar{Z}} = \tilde{\gamma}_n \cup \tilde{\gamma}_{n-1} \cup \ldots \cup \tilde{\gamma}_1$, it is simply connected, and by defining $\pi_{\bar{Z}}: \bar{Z} \to \mathbb{P}$ by $\pi_{\bar{Z}}(Q) = \pi_j(Q)$ for all $Q \in \bar{Z}_j$ and all j, the pair $(\bar{Z}, \pi_{\bar{Z}})$ becomes a bordered covering surface over $\mathbb{P}$. Moreover, $\bar{Z}$ has a single point over infinity.

Now let $(\bar{Y}, \pi_{\bar{Y}})$ be the bordered covering surface of f, and let $B_{\bar{Y}}$ be its border.

Next, we weld the surfaces $\bar{Y}$ and $\bar{Z}$ along their borders. Again, let X be the topological sum of $\bar{Y}$ and $\bar{Z}$, and let $\phi: B_{\bar{Y}} \to B_{\bar{Z}}$ be the homeomorphism that associates to every point $(z, f(z)) \in B_{\bar{Y}}$ the point of $\tilde{\gamma}_j$ lying over $f(z)$ if z belongs to $[z_j, z_{j+1}]$. It follows directly that ϕ is analytic and sense-reversing. Hence, identifying points of $B_{\bar{Y}}$ and $B_{\bar{Z}}$ that correspond to each other under ϕ yields a compact Riemann surface (without boundary), denoted X, for convenience. This surface is simply connected since it is the double of the unit disc; and by viewing $\bar{Y}$ and $\bar{Z}$ as embedded in X, the map $\pi: X \to \mathbb{P}$ defined by $\pi|_{\bar{Y}} = \pi_{\bar{Y}}$ and $\pi|_{\bar{Z}} = \pi_{\bar{Z}}$ turns the pair (X, π) into a compact analytic covering of $\mathbb{P}$.

It is immediate that (X, π) is a complete covering of $\mathbb{P}$ (see [1], pp. 41–43); hence (X, π) covers all points of $\mathbb{P}$ the same number, n, of times (counting multiplicity). Since (X, π) has only one point over infinity, this point would be a branch point of order $n-1$. In addition, the only other branch points of X are those of Y, since the identifications leading to X resulted locally in single sheets, namely over the initial points of the curves $\gamma(w_j)$ which is due to the manner they leave γ.

It is given that f' has $p-1$ zeros (counting multiplicity), so the sum of the orders of the branch points of Y is $p-1$. Then by the Riemann–Hurwitz formula (see [3], p. 18) we have

$$0 = g = \tfrac{1}{2}(p+n-2) - n + 1,$$

where g is the genus of X, and this gives $n = p$.

Therefore (X, π) is a p-sheeted simply connected analytic covering of $\mathbb{P}$ with a single branch point of order $p-1$ over infinity. It follows directly by the uniformization theorem (see [1], p. 181) that (X, π) is the image surface of a polynomial P of degree

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p. Using the embedding of $\bar{Y}$ in X, we obtain a homeomorphism $\phi: \bar{\mathbb{B}} \to \mathbb{C}$ analytic in $\mathbb{B}$ such that $f = P \circ \phi$. Therefore f is p-valent and the proof is complete.

4. *A generalization of Theorem* 1

For $0 < r < 1$, let $\mathbb{B}_r = \{z; |z| < r\}$. We replace the continuity condition of f on $\bar{\mathbb{B}}$ in Theorem 1 by the assumption that the hypotheses of the theorem hold on every compact disc $\bar{\mathbb{B}}_r$ of some exhaustion of $\mathbb{B}$. This yields a slight generalization of Theorem 1 as follows.

THEOREM 2. *Let $f: \mathbb{B} \to \mathbb{C}$ be an analytic function such that f' has $p-1$ zeros (counting multiplicity) in $\mathbb{B}$. Suppose that we can find an increasing sequence $(r(n))_{n=1}^{\infty}$ of positive reals such that $r(n) \to 1^-$ and f satisfies the assumptions of Theorem 1 on every $\bar{\mathbb{B}}_{r(n)}$. Then there exist a polynomial P of degree p and a univalent function ϕ in $\mathbb{B}$ such that $f = P \circ \phi$ in $\mathbb{B}$.*

Proof. Let $f_n(z) = f(r(n)z)$ for every $n \in \mathbb{N}$. Then each f_n satisfies the hypotheses of Theorem 1 and $f_n = P_n \circ \phi_n$, where P_n is a polynomial of degree p and ϕ_n is a univalent function in $\mathbb{B}$. Without loss of generality suppose that each $\phi_n \in S$; that is, $\phi_n(0) = 0$ and $\phi_n'(0) = 1$. Since S is compact, it follows by lemma 9·1 of [**6**] that there exist an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, a polynomial P of degree p and $\phi \in S$ such that $P_{n_k} \to P$ and $\phi_{n_k} \to \phi$ locally uniformly in $\mathbb{C}$ and $\mathbb{B}$ respectively, and $f = P \circ \phi$.

5. *Statement and proof of Theorem* 3

In this section we generalize Theorem 1 by replacing the local univalence of f and T by the local Jordan property of f and T.

For this purpose, we need the following result of Stoïlow [**10**]:

LEMMA 1. *Let f be a function continuous and open in a neighbourhood of some $z_0 \in \mathbb{C}$. Then f at z_0 is locally topologically z^n for some $n \in \mathbb{N}$.*

Using this lemma we prove

LEMMA 2. *Let G^+ and G denote topologically the open semi-disc $S^+ = \{z; |z| < 1, \operatorname{Im} z > 0\}$ and the half-closed semi-disc $S = \{z; |z| < 1, \operatorname{Im} z \geqslant 0\}$, respectively. Suppose that $f: G \to \mathbb{C}$ is a continuous function open in G^+ and topological on $G \backslash G^+$. Then for every $z_0 \in G \backslash G^+$ there is $n \in \mathbb{N}$ such that f at z_0 is locally topologically z^{2n-1}.*

Proof. Without loss of generality let $G^+ = S^+$, $G = S$, and $f((-1, 1)) = (-1, 1)$. It is immediate that the function $F: \mathbb{B} \to \mathbb{C}$ defined by $F(z) = f(z)$ for $z \in S^+$ and $F(z) = \overline{f(\bar{z})}$ otherwise, is continuous. Let $\zeta \in (-1, 1)$ and $\eta = f(\zeta) \in (-1, 1)$. Since F is open in $\mathbb{B} \backslash (-1, 1)$ and topological between $(-1, 1)$ and itself, it follows by Lemma 1 that there is a disc D centred at ζ with positively oriented boundary γ such that the image curve $\Gamma = F(\gamma)$ misses η. Let B be an open disc centred at η and disjoint from Γ, and let $w \in B \backslash (-1, 1)$. It follows by Lemma 1 and the continuity of F that $F^{-1}(w)$ consists of finitely many points $z_1, \ldots, z_m \in D$. Note that none of these points belongs to $(-1, 1)$. It is not difficult to see from the definition of F that the degrees $d(F, z_1), \ldots,$

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$d(F, z_m)$ of F at $z_1, \ldots, z_m$ respectively, are non-zero and have the same sign. Then by the degree theorem we have

$$n(\Gamma; w) = \sum_{j=1}^{m} d(F, z_j).$$

However $\eta(\Gamma; w)$ is constant for all $w \in B$; hence by the same argument every $w \in B \setminus (-1, 1)$ is covered by F in D exactly $\sum_{j=1}^{m} |d(F, z_j)|$ times (counting multiplicity). Therefore F is open in $\mathbb{B}$, and consequently by Lemma 1 at every $z \in (-1, 1)$ F is locally topologically z^n for some $n \in \mathbb{N}$. Since $F: (-1, 1) \to (-1, 1)$ is topological and $F(\bar{z}) = \overline{F(z)}$ it follows that n is odd. This completes the proof.

As a consequence of this result we have

LEMMA 3. *Let $f: \bar{\mathbb{B}} \to \mathbb{C}$ be a continuous function, analytic in $\mathbb{B}$, and locally Jordan on T. Then f is locally univalent on T except possibly at finitely many points $\zeta \in T$, where f is locally topologically z^{2n-1} for some $n \in \mathbb{N}$ depending on ζ.*

Proof. It is immediate from Lemma 2 that for every $z \in T$ there is $n \in \mathbb{N}$ such that f is locally topologically z^{2n-1} at z. This implies that the points $z \in T$ for which $n \geqslant 2$ are isolated. Since T is compact, the desired conclusion follows.

Lemmas 2 and 3 lead naturally to the following

Definition 1. Let $f: \bar{\mathbb{B}} \to \mathbb{C}$ be a continuous function analytic in $\mathbb{B}$ and locally Jordan on T. Then f is locally topologically $z^{2n(\zeta)-1}$, where $n(\zeta) \in \mathbb{N}$, for all $\zeta \in T$. We call $n(\zeta) - 1$ the *order of f at ζ*, and the finite sum $b = \sum_{\zeta \in T} (n(\zeta) - 1)$ the *boundary order of f*. (See Lemma 3.)

For instance, the order of f at the points $\zeta \in T$ of local univalence is zero; and the boundary order of f is zero if and only if f is locally univalent on T.

Now we have

THEOREM 3. *Let $f: \bar{\mathbb{B}} \to \mathbb{C}$ be a continuous function analytic in $\mathbb{B}$, locally Jordan on T, having boundary order $b \geqslant 0$, and satisfying property (A). If f' has $p-1$ zeros (counting multiplicity), then $f = P \circ \phi$ where P is a polynomial of degree $b+p$ and ϕ is a homeomorphism into $\mathbb{C}$ in $\bar{\mathbb{B}}$. In particular, f is at most $(b+p)$-valent.*

Proof. If $b = 0$, then the theorem coincides with Theorem 1 and we are done.

Suppose that $b \neq 0$. It follows at once from Lemma 3 that f induces a positively oriented bordered image surface $\bar{Y}$, together with the associated $\tilde{f}, \pi_{\bar{Y}}$, and $\gamma = \tilde{f}(T)$ which are defined in the same manner as in the locally univalent case.

Let A be the set of all points $\zeta \in T$ at which f is locally topologically $z^{2n(\zeta)-1}$, where $n(\zeta) \geqslant 2$. For every $\zeta \in A$, choose a subarc $[a, b]$ of T such that ζ belongs to the interior of $[a, b]$, f is injective on $[a, b]$, and the arcs $[a, b]$ are mutually disjoint for all $\zeta \in A$.

For a given $\zeta \in A$, let $\eta = f(\zeta)$, $\tilde{\eta} = \tilde{f}(\zeta)$, $\sigma(\eta) = f([a, b])$ and $\tilde{\sigma}(\eta) = \tilde{f}([a, b])$. Then choose a closed topological disc $\bar{W}$ such that the boundary of $\bar{W}$ contains $\sigma(\eta)$ and the orientations of $\sigma(\eta)$ inherited from the positive orientations of $\bar{W}$ and γ are opposite. Using the conformal structure of $\mathbb{C}$, $\bar{W}$ becomes a bordered Riemann surface. There is no ambiguity in viewing $\sigma(\eta)$ as a subarc of the border of $\bar{W}$, and defining the projection map $\pi_{\bar{W}}: \bar{W} \to \mathbb{C}$ as the identity map.

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Now we weld $\bar{Y}$ and the surfaces $\bar{W}$ together by identifying the corresponding points of each $\tilde{\sigma}(\eta)$ and $\sigma(\eta)$. This yields at once a bordered Riemann surface $\bar{X}$. We view $\bar{Y}$ and every $\bar{W}$ as contained in $\bar{X}$, and define the projection map $\pi: \bar{X} \to \mathbb{C}$ as $\pi_{|\bar{Y}} = \pi_{\bar{Y}}$ and $\pi_{|\bar{W}} = \pi_{\bar{W}}$. Then $(\bar{X}, \pi)$ is a bordered analytic covering surface of $\mathbb{C}$.

It is immediate from the construction of $\bar{X}$ that every $\tilde{\eta} = \tilde{f}(\zeta)$, where $\zeta \in A$, is a branch point of X of order $n(\zeta)-1$. In addition to these points, there are only the branch points of Y. Hence the sum of the orders of all the branch points of X is

$$p-1+\sum_{\zeta\in A}(n(\zeta)-1) = p+b-1.$$

Clearly X and $\bar{X}$ are simply connected. Using the uniformization theorem (see [1], p. 181) and the theorem of Carathéodory about the extension of conformal mappings of Jordan domains (see [8], p. 281), it follows that there exists a continuous function $g: \bar{\mathbb{B}} \to C$, analytic in $\mathbb{B}$ and locally univalent on T, such that $(\bar{X}, \pi)$ is the bordered image surface of g.

Next we show that for a careful choice of the discs $\bar{W}$, the function g also satisfies property (A). Since f satisfies this property, there exist distinct points $z_1, \ldots, z_n$ of T and outer Jordan arcs $\gamma(w_1), \ldots, \gamma(w_n)$ of γ at $w_1, \ldots, w_n$, where $w_j = f(z_j)$ for each j, such that every $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ is a Jordan arc, where $\gamma_j = f([z_j, z_{j+1}])$ and $w_{n+1} = w_1$. Now we choose the discs $\bar{W}$ as follows:

(a) Suppose $\zeta \in A$ is an interior point of $[z_j, z_{j+1}]$ for some j. Let $\eta = f(\zeta) \in \gamma_j$ and G_j be the domain bounded by $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ and on the right of γ_j. Then we choose the corresponding $\bar{W}$ so that $W \backslash \sigma(\eta)$ is contained in G_j.

(b) Suppose $\zeta \in A$ coincides with some z_j. Let $w_j = f(z_j)$, let σ_{j-1} be a subarc of γ_{j-1}, where $\gamma_0 = \gamma_n$, let σ_j be a subarc of γ_j, and let δ be a subarc of $\gamma(w_j)$ satisfying $\sigma_{j-1} \cap \sigma_j = \delta \cap \sigma_{j-1} = \delta \cap \sigma_j = \{w_j\}$. Let D be an open disc centred at w_0 such that its boundary meets σ_{j-1}, σ_j and δ. Suppose without loss of generality that the arcs σ_{j-1}, σ_j and δ lie in D with the exception of one endpoint of each, a, b and c respectively; and let α be a common cross-cut of G_j and D joining b and c, and β be a common cross-cut of G_{j-1}, where $G_0 = G_n$, and D joining a and c. Then we choose the corresponding $\bar{W}$ to ζ as the closed topological disc bounded by $\sigma_1 \cup \sigma_2 \cup \alpha \cup \beta$.

In view of this, it follows at once that g satisfies property (A). Hence by Theorem 1 we have $g = P \circ \psi$, where P is a polynomial of degree $b+p$ and $\psi: \bar{\mathbb{B}} \to \mathbb{C}$ is a homeomorphism into $\mathbb{C}$ analytic in $\mathbb{B}$. But $\bar{Y}$ is embedded in the bordered image surface of g, which gives $f = g \circ h$ where $h: \bar{\mathbb{B}} \to \bar{\mathbb{B}}$ is a homeomorphism into $\bar{\mathbb{B}}$ analytic in $\mathbb{B}$. Therefore $f = P \circ \phi$, where $\phi = \psi \circ h: \bar{\mathbb{B}} \to \mathbb{C}$ is as desired, and f is $(b+p)$-valent.

6. *Application* 1

The purpose of Theorems 1, 2 and 3 is to show that the decomposition property holds for classes of analytic functions larger than the classes of close-to-convex functions. In this regard, property (A) serves as the alternative for the strict property of close-to-convexity.

In [6], the proof of the decomposition property for the class $K(p)$, which yields that of $K_w(p)$, uses the following result which we state in the terminology of this paper.

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LEMMA 4. *Let $f \in K(p)$. Then there exists r_0 with $0 < r_0 < 1$ such that, for every $r \in [r_0, 1)$, the function $f_r : \bar{\mathbb{B}} \to \mathbb{C}$ defined by $f_r(z) = f(rz)$ for $z \in \bar{\mathbb{B}}$ is analytic, locally univalent on T and satisfies property (A).*

For the proof see Lyzzaik [**6**], pp. 183–186.

Recall that for every $f \in K(p)$, f' has $p-1$ zeros (counting multiplicity) in $\mathbb{B}$. Then using the lemma, Theorem 2 yields at once the decomposition property of $K(p)$.

THEOREM 4. *Let $f \in K(p)$. Then $f = P \circ \phi$, where P is a polynomial of degree p and ϕ is a univalent function.*

7. *Application* 2

Here we introduce and study geometric classes of analytic functions in $\mathbb{B}$ which contain those that have arisen recently in the work of Sheil-Small [**9**].

Definition 1. Let $H(p)$, where $p \in \mathbb{N}$, be the class of all continuous functions $f : \bar{\mathbb{B}} \to \mathbb{C}$, analytic in $\mathbb{B}$, such that the curve $\gamma = f(T)$ subdivides into $2p$ cross-cuts of some strip $S(c, d) = \{z : c < \operatorname{Im} z < d\}$ which join the sides of the strip.

First we have the following

LEMMA 5. *Every $f \in H(p)$ is locally Jordan.*

Proof. Let $f \in H(p)$, let $S(c, d)$ be the associated strip of f, and L_1 and L_2 the sides $\operatorname{Im} z = c$ and $\operatorname{Im} z = d$ of $S(c, d)$, respectively. Then there exist points $z_1, \ldots, z_{2p}$ of T such that $w_{2j-1} \in L_1$ and $w_{2j} \in L_2$, where $w_j = f(z_j)$ for $1 \leqslant j \leqslant n$, and the arcs $\gamma_j = f([z_j, z_{j+1}])$, where $z_{2p+1} = z_1$, are Jordan for $1 \leqslant j \leqslant 2p$. So we only need to show that f is locally Jordan at every z_j. It is not hard to see that this follows by an orientation consideration of γ, and the proof is complete.

Using Theorem 3 and this lemma we infer

THEOREM 5. *Let $f \in H(p)$. Then $f = P \circ \phi$ where P is a polynomial of degree p and $\phi : \bar{\mathbb{B}} \to \mathbb{C}$ is a homeomorphism into $\mathbb{C}$ analytic in $\mathbb{B}$.*

Proof. Let $S(c, d)$ be the associated strip of f, and let z_j, w_j and γ_j be as in Lemma 5 for $1 \leqslant j \leqslant 2p$. Consider the semi-lines $\gamma(w_j)$ defined by $z(t) = w_j + t$, where $t > 0$ for odd j and $t < 0$ for even j. Then evidently each $\gamma(w_j)$ is an outer curve of γ and each $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ is a Jordan curve.

Since f is locally Jordan by Lemma 5, Lemma 3 implies that f' has a finite number, n, of zeros (counting multiplicity) and a finite boundary order b. Then by Theorem 3 we have $f = P \circ \phi$, where P is a polynomial of degree $b+n+1$ and $\phi : \bar{\mathbb{B}} \to \mathbb{C}$ is a homeomorphism into $\mathbb{C}$ analytic in $\mathbb{B}$.

On the other hand, it is clear from the proofs of Theorems 1 and 3 that the image surface X of P is a p-sheeted covering of $\mathbb{P}$ since it is so near infinity. For X is obtained by welding the bordered image surface of f together with the Jordan regions G_j bounded by $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ and lying on the right of γ_j. Thus $b+n+1 = p$, and the proof is complete.

This proof yields at once the following

COROLLARY 1. *Let $f \in H(p)$. Then f is at most p-valent, and $b+n = p-1$ where n is the number of zeros of f' (counting multiplicity) in $\mathbb{B}$, and b the boundary order of f.*

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In view of this corollary, it is not difficult to see that the degree of the polynomial P in Theorem 5 cannot be less than p.

Now we introduce the afore-mentioned classes of Sheil-Small [9] as follows:

Definition 2. Let $SH(p)$, where $p \in \mathbb{N}$, be the class of all continuous functions $f: \bar{\mathbb{B}} \to \mathbb{C}$, analytic in $\mathbb{B}$, such that the curve $\gamma = f(T)$ subdivides into $2p$ simple arcs descending and ascending alternately between the sides of some strip $S(c, d)$.

In order to study these classes we need the following definitions:

(1) The class $S_a(p)$ of *annular p-valent starlike* functions consists of all functions f, analytic in $\mathbb{B}$ and with p zeros (counting multiplicity), with $\operatorname{Re}(zf'(z)/f(z)) > 0$ for all z in some annulus $A_\rho = \{z: \rho < |z| < 1\}$. Replacing this inequality by the weaker inequality

$$\liminf_{r \to 1^-} [\min_{|z|=r} \operatorname{Re}(zf'(z)/f(z))] \geqslant 0,$$

we obtain the larger class $S_w(p)$ of *p-valent weakly starlike functions* due to Hummel [4].

(2) The class $K(p)$ of multivalent *close-to-convex* functions consists of all functions f, analytic in $\mathbb{B}$, such that there exists a function $g \in S_a(p)$ with $g(0) = 0$ and with $\operatorname{Re}(zf'(z)/g(z)) > 0$ for every z in some annulus A_ρ. This class is due to Livingston [5]. If we relax the latter conditions by requiring that $g \in S_w(p)$ and

$$\liminf_{r \to 1^-} [\min_{|z|=r} \operatorname{Re}(zf'(z)/g(z))] \geqslant 0,$$

then we obtain the class $K_w(p)$ of *weakly close-to-convex* functions *of order p* due to Styer [11].

Livingston [5] has shown that if $f \in K(p)$ then f is at most p-valent and f' has exactly $p-1$ zeros (counting multiplicity).

Let us now return to the class $SH(p)$. Clearly $SH(p)$ is contained in $H(p)$ for all p. Then by Theorem 5 for every $f \in SH(p)$, $f = P \circ \phi$, where P is a polynomial of degree p and $\phi: \bar{\mathbb{B}} \to \mathbb{C}$ is a homeomorphism into $\mathbb{C}$ analytic in $\mathbb{B}$. Furthermore, in view of the proofs of Theorems 1 and 3 and in terms of the notation there, the bordered image surface $\bar{Y}$ of f is embedded in the image surface X of P in such a way that $X \setminus Y$ is the bordered surface $\bar{Z}$ obtained by the identifications of the edges of the Jordan domains G_j bounded by the curves $\gamma(w_j) \cup \gamma_j \cup \gamma(w_{j+1})$ and lying on the right of γ_j. Note that in this case every G_j is a disjoint union of horizontal (straight) lines and semi-lines, and so also $\bar{Z}$. Hence $\mathbb{C} \setminus \phi(\mathbb{B})$ is a disjoint union of arcs which map under P univalently to horizontal lines and semi-lines. This implies by theorem 4·1 of [6] that $f \in K_w(p)$. (See also [7].)

We summarize this discussion as follows:

THEOREM 6. *Let $f \in SH(p)$. Then $f \in K_w(p)$ and $f = P \circ \phi$, where P is a polynomial of degree p and $\phi: \bar{\mathbb{B}} \to \mathbb{C}$ is a homeomorphism into $\mathbb{C}$ analytic in $\mathbb{B}$ such that $\mathbb{C} \setminus \phi(\mathbb{B})$ is a disjoint union of arcs which map under P univalently to horizontal lines and semi-lines.*

The set-inclusion $SH(p) \subset K_w(p)$ is due to Sheil-Small [9]. The theorem implies that every $f \in SH(1)$ is convex in the direction of the real axis, a result due to Clunie and Sheil-Small [2].

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