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Long-Range Percolation Mixing Time

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We provide an estimate, sharp up to poly-logarithmic factors, of the asymptotic almost sure mixing time of the graph created by long-range percolation on the cycle of length N ($\mathbb{Z}/N\mathbb{Z}$). While it is known that the asymptotic almost sure diameter drops from linear to poly-logarithmic as the exponent s decreases below 2 [4, 9], the asymptotic almost sure mixing time drops from N^2 only to N^{s-1} (up to poly-logarithmic factors).

1. Introduction

In this note we study mixing time of simple random walks on the random graph obtained by adding edges to a graph of a cycle, where edges are added between any two vertices with probability decaying with their distance, and independently for any two vertices (in fact, this probability is $1-\exp(-\beta\|x-y\|^{-s})$, where β and the exponent s are parameters). See below for definitions of the model and of mixing times, as well as further remarks.

The bounds on the mixing time are presented in the following sections.

Let us mention two interesting findings. First, the mixing time undergoes a phase transition as the exponent s decreases below 2. Second, in this natural model the almost sure diameter is a.a.s. (asymptotically almost surely) poly-logarithmic, for certain ranges of the parameters, yet the a.a.s. mixing time is polynomial. Such a gap between the diameter and the mixing time cannot exist for vertex-transitive graphs or in graphs where the isoperimetric dimension is determined by the volume growth function.

Long-range percolation graphs have received some recent interest: see [14]. From an algorithmic viewpoint it is useful to consider the mixing times of these graphs.

1.1. The model

The model we discuss is the finite long-range percolation model with polynomial decay. Let N be a positive integer, let $\beta > 0, 1 < s < 2$, and consider the following random graph.

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Start with the cycle on N vertices ($\mathbb{Z}/N\mathbb{Z}$). Define $\|x-y\| = \min\{|x-y|, N-|x-y|\}$ (which is the regular graph-theoretical distance). The following edges are randomly added. If $x \neq y$, then x and y will be attached with probability $1-\exp\left(-\beta\|x-y\|^{-s}\right)$. The different edges are all independent of each other. The probability of an edge between two (distant enough) vertices is very close to $\beta\|x-y\|^{-s}$. We call the graph created in this way $G_{s,\beta}(N)$. For updated background on long-range percolation, see [9, 10].

1.2. Mixing time

The *variational distance* of the random walk on a graph G is defined by

$$\Delta_x(t) = \frac{1}{2}\sum_{y\in V(G)} |P^t(x,y) - \pi(x)|,$$

where P^t is the tth power of the transition matrix of the walk, and π is the stationary distribution, *i.e.*, $\pi(x) = \frac{d(x)}{2|E(G)|}$ ($d(x)$ is the degree of the vertex x). $\Delta_x(t)$ measures how close the distribution of the walk starting at x, at time t, is to the stationary distribution π.

Define

$$\tau_x(\varepsilon) = \min\{t \mid \forall\ t' \geqslant t\ \Delta_x(t') \leqslant \varepsilon\}.$$

Since we are interested in the time it takes the walk to converge to the stationary distribution, the *mixing time* is defined by

$$\tau(G) = \max_{x\in V(G)} \tau_x(e^{-1}).$$

It is well known that, using the second eigenvalue of the transition matrix, one can bound the mixing time of the graph. Formally (though not in full generality), Diaconis and Stroock [11] prove that

$$\tau(G) \leqslant \frac{3\log(N)}{1-\lambda_G},$$

where λ_G is the second eigenvalue of G. For more on mixing, see [1, 17].

Sinclair [17] provides a tool for bounding the mixing time of a graph, called the *canonical paths method*. In this method, we consider oriented edges, *i.e.*, every edge $\{x,y\}$ in G is considered as two directed edges (x,y) and (y,x). For any pair of vertices $x \neq y$, we need to specify a single path γ_{xy} from x to y. Sinclair shows that, for any collection of such paths,

$$\frac{1}{1-\lambda_G} \leqslant \frac{1}{|E(G)|}\max_{e\in E(G)} \sum_{\gamma_{xy}\ni e} d(x)d(y)|\gamma_{xy}|,$$

where $E(G)$ is now the set of oriented edges, $|\gamma|$ is the length of γ, and the sum is over all paths γ_{xy} in the collection containing the edge e. So, to bound the mixing time, it is enough to specify a set of paths, one for each pair of vertices, while making sure that any path is not too long, and that no edge is overloaded with paths.

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1.3. Remarks

- In the following sections we will provide upper and lower bounds on the mixing time of $G_{s,\beta}(N)$, that match up to poly-logarithmic factors. It is interesting to note that, using two different methods, we obtain matching bounds.
- As will be seen in Section 4, a phase transition occurs in the mixing time when s passes from below 2 to above 2. Two open questions regarding the mixing time are as follows.

  (1) What is the mixing time at $s = 2$? We note that even the diameter is not known in this case.
  (2) When s drops below 1, it is shown in [5] that the diameter is bounded. We conjecture that the mixing time is constant (independent of N) in this case.

- Long-range percolation gives natural examples of graphs with small diameter (poly-logarithmic: see [9]) yet large polynomial mixing time. Long-range percolation is a natural model of some social networks, in which the probability that you know a person decays with distance. This suggests that while the diameter of such networks might be small, sampling from such network via random walk might take a long time.
- For almost sure expansion of other models of random graphs, see [2, 16]. Regarding mixing for random walks on other models of random graphs, see [6], [13] and [8].
- Another question related to mixing on random graphs is the following. Let $\{G_n\}$ be a family of vertex-transitive graphs such that $\lim_{n\to\infty} |G_n| = \infty$. Assume that the average degree of the giant component G_n' of Bernoulli percolation on G_n is uniformly bounded in n. Prove

$$\tau(G_n') \leqslant O\big(\max\{\tau(G_n),\ \log^2 |G_n|\}\big).$$

- Consider a uniform spanning tree on the long-range percolation graph over $\mathbb{Z}$. Our mixing time estimates show that the mixing time of long-range percolation at $s = 3/2$ is like that of a 4 dimensional torus. Since the transition from tree to forest in the uniform spanning tree on $\mathbb{Z}^d$ occurs at $d = 4$, this suggests that perhaps the uniform spanning tree on the long-range percolation graph over $\mathbb{Z}$ is supported on a tree a.s. if and only if $s \geqslant 3/2$. See [7] for background.

2. Upper bound

Proposition 2.1. *Let $G_{s,\beta}(N)$ be the graph obtained by long-range percolation on the cycle of length N. Then there exists $c = c(s,\beta) > 0$ such that*

$$\lim_{N\to\infty} \mathbb{P}\big[\tau(G_{s,\beta}(N)) \leqslant \log^c(N) \cdot N^{s-1}\big] = 1.$$

Proof. Set $G = G_{s,\beta}(N)$. With hindsight, set $L = \lceil \sqrt{N^s \xi} \rceil$ for $\xi = \xi(N) = \frac{1}{\beta 2^s} \log(N)$. Let $\ell = N \pmod{L}$, and set $k = \frac{N-\ell}{L}$. Divide the cycle into k intervals, $S_1, \ldots, S_k$, so that $S_1, \ldots, S_{k-1}$ are of length L, and S_k is of length $L + \ell \geqslant L$.

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For any two intervals, S_i, S_j, we wish to bound the probability that there is no edge (not in the cycle) connecting S_i, S_j. Since $\|x-y\| \leqslant \frac{N}{2}$, this probability is bounded by

$$\prod_{x\in S_i}\prod_{y\in S_j}\exp\left(-\beta\|x-y\|^{-s}\right) \leqslant \exp\left(-\frac{\beta 2^s L^2}{N^s}\right).$$

Thus, the probability that there exist i, j such that there is no edge (not in the cycle) connecting S_i, S_j is bounded by

$$\sum_i \sum_{j\neq i} \exp\left(-\frac{\beta 2^s L^2}{N^s}\right) < k^2 \exp\left(-\frac{\beta 2^s L^2}{N^s}\right),$$

which tends to 0 as N tends to infinity, by our choice of L, and because $s > 1$. So with probability tending to 1, we have that any two intervals S_i, S_j are connected with at least one edge not in the cycle.

In order to use Sinclair's canonical paths method, we define a canonical path γ_{xy} for each pair of vertices x, y.

Let x, y be vertices in G. If x, y are both in the same interval S_i, define the path γ_{xy} from x to y as the shortest path on this interval. If $x \in S_i$ and $y \in S_j$ for $i \neq j$, define γ_{xy} to be the following path. Let e_{ij} be an oriented edge connecting the intervals S_i and S_j. Take the shortest path in the interval S_i from x to e_{ij}, then cross e_{ij}, and finally take the shortest path in the interval S_j from e_{ij} to y. This defines a canonical path between any pair of vertices.

As shown above, the mixing time is bounded by

$$\tau(G) \leqslant 3\log(N)\cdot\rho \qquad \rho = \frac{1}{|E(G)|}\max_{e\in E(G)}\sum_{\gamma_{xy}\ni e} d(x)d(y)|\gamma_{xy}|,$$

where $E(G)$ is the set of oriented edges of G (each edge counts as two oriented edges), $d(x)$ is the degree of x in G, and $|\gamma_{xy}|$ is the length of the path γ_{xy}.

First we would like to bound the length of the paths in our collection. In [9], the diameter of long-range percolation on the interval of length n is shown to be $\log^{\delta}(n)$, with probability tending to 1, where $\delta = \frac{\log 2}{\log(2/s)} + o(1)$. Long-range percolation on the cycle of length N and the interval of length N can be coupled so that the cycle has at least those edges appearing in the interval. Thus, the diameter of long-range percolation on the cycle is dominated by the diameter of long-range percolation on the interval. So all our paths are bounded by $|\gamma_{xy}| \leqslant 2\log^{\delta}(N) + 1$. (This is the reason we want the shortest paths in each interval.)

Next, we show that with probability tending to 1, the maximal degree in G is bounded by $2\log(N)$. This follows from the following considerations.

Fix a vertex x in G. Let $d(x)$ be the degree of x. We can write $d(x) = 2 + \sum_{y\neq x} Z_{xy}$, where Z_{xy} is the indicator function of the event that x and y are connected by an edge not in the cycle. The random variables Z_{xy} are all independent. Note that $\mu_{xy} = \mathbb{E}[Z_{xy}] = 1 - e^{-\beta\|x-y\|^{-s}}$, and for any $t > 0$, $\mathbb{E}[\exp(tZ_{xy})] = 1 + \mu_{xy}(e^t - 1) \leqslant \exp\{(e^t-1)\mu_{xy}\}$. So we

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can calculate using Markov's inequality, for all $\lambda > 0$,

$$\begin{aligned}\mathbb{P}[d(x) > 2 + \lambda] &\leqslant \exp(-\lambda) \prod_{y \neq x} \exp\big((e-1)\mu_{xy}\big) \\ &\leqslant \exp\left(-\lambda + (e-1)\beta \sum_{y \neq x} \|x-y\|^{-s}\right) \\ &\leqslant \exp\left(-\lambda + (e-1)2\beta \sum_{k=1}^{N/2} k^{-s}\right) \\ &\leqslant \exp\big(-\lambda + c(s,\beta) N^{1-s}\big).\end{aligned}$$

Thus, taking $\lambda = 2\log(N) - 2$ and using a union bound, the probability that there exists a vertex x with $d(x) > 2\log(N)$ is bounded by $N\frac{1+o(1)}{N^2} \leqslant \frac{1+o(1)}{N}$. Thus, with probability tending to 1 we have that the maximal degree in G is bounded by $2\log(N)$.

Finally, by our choice of the paths, the number of paths passing through any oriented edge is bounded by $\max_{i,j} |S_i| \cdot |S_j| \leqslant (2L-1)^2 \leqslant 4L^2$.

Thus,

$$\tau(G) \leqslant O\left(\log(N)\frac{L^2}{N}\big(\max_x d(x)^2\big)\log^{\delta}(N)\right) \leqslant \log^{\delta+4}(N) N^{s-1},$$

with probability tending to 1 as N tends to infinity. □

3. Lower bound

In Corollary 5.2 of [4], it is shown that $\tau(G) \geqslant \Omega(N^{s-1})$ with probability tending to 1. This miraculously matches our upper bound up to poly-logarithmic factors. In [4] there is a typo in the parameters, so for completeness we provide the proof here.

Proposition 3.1. *Let $G_{s,\beta}(N)$ be the graph obtained by long-range percolation on the cycle of length N. Then*

$$\lim_{N\to\infty} \mathbb{P}\big[\tau(G_{s,\beta}(N)) \leqslant cN^{s-1}\big] = 0,$$

where $c = c(s,\beta)$ is a constant independent of N.

Proof. Let $G = G_{s,\beta}(N)$. It is well known (see, *e.g.*, [1, 11, 17]) that it is enough to bound from above the Cheeger constant of the graph G, which is defined as

$$\mathscr{C}(G) = \min_{\substack{\emptyset \neq A \subset V(G) \\ |A| \leqslant N/2}} \frac{|\partial A|}{|A|} \qquad \partial A = \{\{x,y\} \in E(G) \mid x \in A, y \notin A\}.$$

The natural subset to choose is $A = \{1, 2, \ldots, N/2\}$ (any arc of length $N/2$ will suffice). For $x \in A$ and $y \notin A$, let Z_{xy} be the indicator function of the event that x and y are connected by an edge not in the cycle. Then, Z_{xy} are all independent. Set $\mu_{xy} = \mathbb{E}[Z_{xy}] = 1 - \exp\big(-\beta\|x-y\|^{-s}\big)$. For any $t > 0$,

$$\mathbb{E}\big[\exp(tZ_{xy})\big] = 1 + \mu_{xy}(e^t - 1) \leqslant \exp(\mu_{xy}(e^t - 1)).$$

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We have that $|\partial A| = 2 + \sum_{x\in A}\sum_{y\notin A} Z_{xy}$, and that

$$\sum_{x\in A}\sum_{y\notin A}\mu_{xy} \leqslant \sum_{x\in A}\sum_{y\neq x}\mu_{xy} \leqslant \sum_{x\in A} c(s,\beta)N^{1-s} \leqslant c(s,\beta)N^{2-s}.$$

Thus, there exists $c = c(s,\beta)$ such that, for any $t > 0$ and any $\lambda > 0$,

$$\begin{aligned}\mathbb{P}\big[|\partial A| > 2 + \lambda N^{2-s}\big] &\leqslant \frac{\mathbb{E}\big[\exp(t(|\partial A| - 2))\big]}{\exp(t\lambda N^{2-s})} \\ &= \exp\big(-t\lambda N^{2-s}\big)\prod_{x\in A}\prod_{y\notin A}\mathbb{E}\big[\exp(tZ_{xy})\big] \\ &= \exp\left((e^t - 1)\sum_{x\in A}\sum_{y\notin A}\mu_{xy}\right)\exp\big(-t\lambda N^{2-s}\big) \\ &\leqslant \exp\big(N^{2-s}\cdot(c(s,\beta)(e^t - 1) - t\lambda)\big).\end{aligned}$$

Choosing t small enough, we obtain, for some fixed $c = c(s,\beta)$,

$$\lim_{N\to\infty}\mathbb{P}\big[|\partial A| > cN^{2-s}\big] = 0,$$

which implies that with probability tending to 1 as N tends to infinity, $\mathscr{C}(G) \leqslant cN^{1-s}$ for some (possibly different) $c = c(s,\beta)$ independent of N.

This gives a bound on the mixing time (see [17]):

$$\tau(G) \geqslant \left(\frac{1-\log 2}{2}\right)\left(\frac{1}{2\mathscr{C}(G)} - 1\right) \geqslant cN^{s-1},$$

with probability tending to 1, and $c = c(s,\beta)$ independent of N. □

4. A phase transition

In the previous sections we have shown that the mixing time of $G_{s,\beta}$ is N^{s-1} (disregarding poly-logarithmic factors) for $1 < s < 2$. When s tends to 2, this quantity tends to N. We will show that a phase transition occurs at $s = 2$, meaning that for $s > 2$ the mixing time will 'jump' to N^2.

Proposition 4.1. *Let $G = G_{s,\beta}(N)$ for $s > 2$. Then the mixing time of G satisfies*

$$\lim_{N\to\infty}\mathbb{P}\big[\tau(G) \geqslant cN^2\big] = 1$$

for some constant $c = c(s,\beta)$, independent of N.

Proof. For simplicity we assume that N is divisible by 8. For other N the proof is similar. Set

$$A = \left\{1, 2, \dots, \frac{N}{2}\right\}, \quad B = \left\{\frac{N}{2} + 1, \dots, \frac{3N}{4}\right\}, \quad C = \left\{\frac{3N}{4} + 1, \dots, N\right\}.$$

Also, for $i = 1, 2, \dots, 8$ set $K_i = \{(i-1)\frac{N}{8} + 1, \dots, i\frac{N}{8}\}$.

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By the proof of Theorem 3.1(A) in [4], there exists $c_1 = c_1(s,\beta)$ such that, with probability $1 - o(1)$, we have that all sets $K_1, \dots, K_8$ each contain at least $c_1 N$ vertices of degree 2.

Further, by rotating the cycle, without loss of generality we can assume that $\pi(A) \geqslant \pi(B \cup C)$, and that $\pi(B) \geqslant \pi(C)$. This implies that $\pi(A \cup B) \geqslant \frac{3}{4}$.

Fix a vertex $x \notin A \cup B$, and let $(S_t \,;\, t \geqslant 0)$ be a simple random walk on G starting at $S_0 = x$. Let T be the hitting time of the set $A \cup B$. Note that at any time $t \geqslant \tau(G)$, we have that

$$\frac{3}{4} - \mathbb{P}[S_t \in A \cup B] \leqslant \sum_{y \in A \cup B} |\pi(y) - P^t(x,y)| \leqslant 2\Delta_x(t) \leqslant \frac{2}{e}.$$

So we conclude that, for any $x \notin A \cup B$ and $t \geqslant \tau(G)$,

$$\mathop{\mathbb{P}}_{x}[T \leqslant t] \geqslant \mathop{\mathbb{P}}_{x}[S_t \in A \cup B] > 0.01.$$

This implies that for any $x \notin A \cup B$ and for any real s, $\mathbb{P}_x[T > s] \leqslant 0.99^{(s/\tau - 1)}$. Thus, there exists $c_2 > 0$ independent of N, such that

$$\mathop{\mathbb{E}}_{x}[T] = \sum_{t=0}^{\infty} \mathop{\mathbb{P}}_{x}[T > t] \leqslant c_2 \tau.$$

Set $u = \frac{3}{8}N$. Recall that there are at least $c_1 N$ vertices of degree 2 separating u from $A \cup B$ (on each side of u). We will show that this implies that $\mathbb{E}_u[T] \geqslant c_3 N^2$ for some $c_3 = c_3(s,\beta)$ independent of N.

We use the language of electrical networks: see [12, 15] for background. We remark that for the reader not familiar with these notions, one can use the Varopoulos–Carne bounds (see, *e.g.*, [15]) to show that a linear diameter implies that the mixing time is at least $\frac{cN^2}{\log N}$.

We can write $T = \sum_{x \notin A \cup B} V_x$, where V_x is the number of visits to the vertex x, up to time T. Ground the set $A \cup B$ (so that its voltage is 0), and set a potential to u so that there is a unit current flowing into u. We get that for any x, we have the identity $\mathbb{E}_u[V_x] = v(x)d(x)$, where $v(x)$ is the voltage at x, and $d(x)$ is the degree of x (this follows from noting that $\mathbb{E}_u[V_x]/d(x)$ is harmonic).

Let $x_1, x_2, \dots, x_{c_1 N}$ be the vertices of degree 2 on the side of u with at least $1/2$ the current (without loss of generality, say, in the interval $\{\frac{3}{8}N + 1, \dots, N\}$). Fix $1 \leqslant i \leqslant c_1 N$. Since x_i is a cut point, the current flowing into and out of x_i is at least $1/2$. Thus, the voltage at x_i, $v(x_i)$, is at least $1/2$ the resistance between x_i and $A \cup B$. This resistance is bounded from below by the number of cut *edges* (which are resistors of resistance 1 connected serially), which in turn is bounded by the number of vertices of degree 2 between x_i and the set $A \cup B$. That is, $v(x_i) \geqslant i/2$. Thus,

$$\mathop{\mathbb{E}}_{u}[T] \geqslant \sum_{\substack{x \notin A \cup B \\ d(x)=2}} \mathop{\mathbb{E}}_{u}[V_x] = \sum_{\substack{x \notin A \cup B \\ d(x)=2}} d(x)v(x) \geqslant \sum_{i=1}^{c_1 N} i = c_3 N^2,$$

for some $c_3 = c_3(s,\beta)$ independent of N.

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We have shown that with probability $1 - o(1)$ there is a linear number of vertices of degree 2 separating a vertex u from a set of high weight under the stationary distribution. Thus, $\tau(G) \geqslant c_4 \mathbb{E}_u[T] \geqslant c_5 N^2$, where c_4, c_5 depend only on s and β. □

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