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Erratum to 'Holomorphic foliations with Liouvillian first integrals' (*Ergodic Theory and Dynamical Systems* 21 (2001), 717–756)

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We draw attention to the following mistakes which appear in the above-mentioned paper (pp. 732–734). We are grateful to J.-F. Mattei who pointed out with an example that the notion of singular holonomy should be enlarged and consequently the Structure Theorem restated. The main result (Theorem 1) remains untouched. We refer to the same notation and terminology of the paper.

Singular holonomy. By a *circuit* on D ending at D_{j_k} we shall mean an ordered sequence of components of D, say $\gamma = D_{j_1} \dots D_{j_k}$, such that if $\ell \in \{1, \dots, k-1\}$, then $p_l = D_{j_\ell} \cap D_{j_{\ell+1}} \neq \emptyset$ and:

(i) if p_ℓ is resonant then it is linearizable (cf. §3.1);

(ii) if p_ℓ is a saddle-node then its strong manifold is contained in $D_{j\ell}$.

Given two circuits $\gamma = D_{j1} \dots D_{j_k}$ and $\delta = D_{j_k} \dots D_{j_m}$, the union is a circuit $\gamma * \delta = D_{j_1} \dots D_{j_k} \dots D_{j_m}$ with end at D_{j_m}. Notice that is always possible to consider the adjunction of holonomies from D_{j_ℓ} to $D_{j_{\ell+1}}$ in a circuit. In particular, we can define an adjunction holonomy group associated with the circuit γ and supported on the divisor D_{j_k} as follows. The *complete adjunction* of holonomy to G_{j_k} is the group $\gamma^* G_{j_k} = (\dots((G_{j_1}) * G_{j_2}) * \dots * G_{j_{k-1}}) * G_{j_k}$.

Definition 4. The singular holonomy group of D_j is the subgroup $\mathrm{Hol}^{\mathrm{sing}}(\mathcal{F}, D_j, \mathcal{S}_j) = G_j^{\mathrm{sing}} \subset \mathrm{Diff}(\mathcal{S}_j, p_j)$ generated by the groups $\gamma^* G_j$, where γ runs through all the possible circuits with end at D_j. In particular, we may have repetitions of components of a given circuit γ as well as closed circuits. Also, for $\gamma = D_j$ the group $\gamma^* G_j$ is, by definition, the holonomy group of D_j.

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THEOREM 3. (Structure Theorem) *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{C}P(n)$ and suppose that one of its algebraic leaves, say S, has only non-dicritical infinitesimally hyperbolic singularities. Then the following two assertions are equivalent:*

(a) *$\mathcal{F}$ has a Liouvillian first integral on $\mathbb{C}P(n)$;*

(b) *$\mathcal{F}$ has a Liouvillian resolution relative to S.*

The statement involves the notion of Liouvillian resolution (Definition 5 on p. 734) which must be completed with the addition of an extra condition (iv) below. Let $\mathcal{F}$ be a foliation on a compact complex surface M and let $S \subset M$ be an irreducible invariant analytic curve. Denote by $D = \bigcup_j D_j$ the divisor of the resolution of $\operatorname{sing}\mathcal{F} \cap S$, which is assumed to be invariant and by $\tilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ after the resolution process. Denote by G_j (respectively G_j^{sing}) the holonomy group (respectively singular holonomy group) of the component D_j of D.

Definition 5. (Liouvillian resolution) The foliation $\mathcal{F}$ has a *Liouvillian resolution (relative to S)* if the divisor D satisfies the following conditions.

(i) Each holonomy group G_j and each singular holonomy group G_j^{sing} is either Abelian linearizable or solvable analytically conjugate to a non-Abelian subgroup of $\mathbb{H}_{k_j}$.

(ii) If $q = D_i \cap D_j$ is a saddle-node corner whose strong manifold is contained in D_i, then $G_i \subset \mathbb{H}_{k_i}$ is a solvable non-Abelian and G_j is Abelian.

(iii) If $q = D_i \cap D_j$ is a corner such that $G_i \subset \mathbb{H}_{k_i}$ and $G_j \subset \mathbb{H}_{k_j}$ are non-Abelian, then q is a resonant singularity and we can find local coordinates $(x, y) \in U$ centered at q such that $D_i \cap U = (y = 0)$, $D_j \cap U = (x = 0)$ and $\tilde{\mathcal{F}}|_U : k_i x\,dy + k_j y\,dx + \text{higher order terms} = 0$.

(iv) Let D_i be a component with Abelian holonomy group G_i. For any non-saddle-node corner $q_j = D_i \cap D_j$ for which the group $G_j \subset \mathbb{H}_{k_j}$ is not Abelian, we can find local coordinates $(x, y) \subset U$ centered at q_j such that $D_j \cap U = (x = 0)$, $D_i \cap U = (y = 0)$ and $\tilde{\mathcal{F}}|_U : x\,dy - \lambda_j y\,dx = 0$. Moreover, the quotient k_j/λ_j is the same for two such singularities: if $q_l = D_i \cap D_l$ is another non-saddle-node singularity with $G_l \subset \mathbb{H}_{k_l}$ non-Abelian, then $k_j/\lambda_j = k_l/\lambda_l$.

Proof of the Structure Theorem. First we prove that (a) $\Longleftrightarrow$ (b). It remains to show that (iv) is fulfilled. Thus we may assume that the foliation $\tilde{\mathcal{F}}$ admits a Liouvillian first integral and, as in the proof of Theorem 3 on p. 744, we may find 1-forms Ω and η, where Ω defines the foliation and η is a closed rational logarithmic derivative of Ω adapted to S. In a neighborhood of a non-saddle-node corner singularity $q_j = D_i \cap D_j$ for which the group $G_j \subset \mathbb{H}_{k_j}$ is not Abelian, it follows that we can find local coordinates $(x, y) \subset U$ centered at q_j such that $D_j \cap U = (x = 0)$, $D_i \cap U = (y = 0)$ and $\tilde{\mathcal{F}}|_U : x\,dy - \lambda_j y\,dx = 0$. In this case $\Omega(x, y) = g_j(x\,dy - \lambda_j y\,dx)$ for some meromorphic function g_j in U and necessarily we have

$$\eta(x, y) = a_j \frac{dx}{x} + b_j \frac{dy}{y} = \frac{d_{g_j}}{g_j}$$

for some $a_j, b_j \subset \mathbb{C}$. Now, the condition $d\Omega = \eta \wedge \Omega$ implies $a_j + b_j\lambda_j = 1 + \lambda_j$ and,

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therefore,

$$\mathrm{Res}_{D_i}\,\eta = b_j = 1 + \frac{1-a_j}{\lambda_j}.$$

For any other such singularity $q_l = D_i \cap D_l$ we therefore have

$$\frac{1-a_j}{\lambda_j} = \frac{1-a_l}{\lambda_l}.$$

Now, since $G_j \subset \mathbb{H}_{k_j}$ is non-Abelian it follows from part (c) of the theorem on p. 724 that $a_j = k_j + 1$ and, therefore, in the case $G_l \subset \mathbb{H}_{k_l}$ is non-Abelian we conclude that $k_j/\lambda_j = k_l/\lambda_l$.

Now we prove that (b) $\Longleftrightarrow$ (a). We have already seen that conditions (i), (ii) and (iii) imply the existence, for each j, of closed logarithmic derivatives η_j, with simple poles, defined in neighborhoods V_j of the components D_j. Each η_j is adapted to Ω relative to $D \cap V_j$. Moreover, according to §4 and the proof of Proposition 3, if $q = D_i \cap D_j$ is a corner, then we can choose $\eta_i = \eta_j$ in a neighborhood of q except, maybe, in the case where q is linearizable non-resonant, $G_j \subset \mathbb{H}_{k_j}$ is non-Abelian and G_i is Abelian. In this situation we can construct a 1-form η_{ij} defined in a neighborhood U_{ij} of $D_i \cup D_j$ by $\eta_{ij}|_{U_{ij}\cap V_j} = \eta_j$ and $\eta_{ij}|_{U_{ij}\cap V_i} = \eta_i + \omega_i$, where w_i is a closed meromorphic 1-form which defines $\tilde{\mathcal{F}}$ in V_i. Indeed, since the holonomy group G_i is Abelian linearizable it follows that $\tilde{\mathcal{F}}$ is given in a neighborhood of D_i by a closed meromorphic 1-form ω_i. We may assume that ω_i is defined in V_i. We have $\Omega = h_i w_i$ for some meromorphic function h_i in V_i. On the other hand, we have, in a neighborhood W of $q_j = D_i \cap D_j$, $\eta_j - \eta_i = f\Omega$ for some meromorphic function f defined in W. Therefore, $\eta_j - \eta_i = f h_i \omega_i$ in W. Since η_i, η_j and ω_i are closed, it follows that $f h_i$ is a meromorphic first integral for $\tilde{\mathcal{F}}$ in a neighborhood of q, but q is non-resonant so that necessarily $f h_i$ is a constant and, therefore, we may assume that $\eta_j = \eta_i + \omega_i$ in $V_i \cap V_j$.

Two of these 1-forms η_{ij} and η_{il} also glue. In fact, G_i is Abelian, $G_j \subset \mathbb{H}_{k_j}$ and $G_l \subset \mathbb{H}_{k_l}$ are non-Abelian and $q_j = D_i \cap D_j$ and $q_l = D_i \cap D_l$ are linearizable. In this case we claim that $\mathrm{Res}_{D_i}\,\eta_{ij} = \mathrm{Res}_{D_i}\,\eta_{il}$. This is a straightforward consequence of the fact that the residue b along D_i of any closed logarithmic derivative η for Ω adapted to D in a neighborhood of a singularity q_j as above is given by

$$\mathrm{Res}_{D_i}\,\eta = 1 + \frac{1-a_j}{\lambda_j}$$

where, as we have seen above, $a_j = k_j + 1$. Therefore, since by condition (iv) we have $k_j/\lambda_j = k_l/\lambda_l$, we conclude that $\mathrm{Res}_{D_i}\,\eta_{ij} = \mathrm{Res}_{D_i}\,\eta_{il}$. Therefore, the difference $\eta_{ij} - \eta_{il} = \omega$ is a closed 1-form with simple poles, such that $\omega \wedge \Omega = 0$, defined in a neighborhood of D_i and which is holomorphic along $D_i \setminus \mathrm{sing}\,\mathcal{F}$. Since the holonomy group of the leaf $D_i \setminus \mathrm{sing}\,\mathcal{F}$ is non-trivial (by hypothesis $\mathrm{sing}\,\mathcal{F} \cap D_i$ contains some hyperbolic singularity), it follows that actually $\omega \equiv 0$. This completes the proof of the Structure Theorem. □

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