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A theorem about hyperspaces

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Recently Isbell (4) and Ward (6) have given examples of pairs of uniformities inducing the same topology on the hyperspace of subsets of a uniform space. Smith (5) has shown that, if two uniformities induce the same topology on the hyperspace, they must be equal in proximity. The author, in (2), has introduced a relation, 'height', between uniformities which is in a sense dual to proximity. It is the purpose of this paper to show that, if two uniformities are equal in proximity and height, then they induce the same topology on the hyperspace. Although Ward's uniformities are not equal in height, an example of two distinct uniformities equal in proximity and height is given in (3). In fact, a slight modification of that example will give an example of a countably infinite number of distinct uniformities, all of which are equal in proximity and height.

We will use terminology and notation similar to that in Bourbaki (1). As in (2), we say $\mathfrak{U} \leqslant {}^{h}\mathfrak{V}$ ('$\mathfrak{U}$ less than or equal in height to $\mathfrak{V}$') if, for every $U \in \mathfrak{U}$, there is $V \in \mathfrak{V}$ and a finite cover, $A_1, \ldots, A_n$ of X (the space considered) such that $V \cap (A_i \times A_i) \subset U$. One way of defining equality in proximity is to write $\mathfrak{U} \leqslant {}^{p}\mathfrak{V}$ if, for every $A \subset X$ and $U \in \mathfrak{U}$ there is $V \in \mathfrak{V}$ such that $V(A) \subset U(A)$. Then $\mathfrak{U} = {}^{p}\mathfrak{V}$ if the inequality goes both ways.

It will be convenient to have it understood that all entourages considered in the remainder of this paper are symmetric.

We begin by extracting a lemma from Ward's paper. If $U \in \mathfrak{U}$, we say that $A \subset X$ is U-discrete if, for every $a \in A$, $U(a) \cap A = \{a\}$.

Lemma. *If* $\mathfrak{U} \leqslant {}^{p}\mathfrak{V}$, *and for every* $U \in \mathfrak{U}$ *and every* U*-discrete set* A *there is* $V \in \mathfrak{V}$ *such that* $V(a) \subset U(a)$ *for all* $a \in A$; *then the topology induced on the hyperspace by* $\mathfrak{V}$ *is finer than that induced by* $\mathfrak{U}$.*

Proof. We need only show that, for every $A \subset X$ and every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{V}$ such that the V-induced neighbourhood of A is contained in the U-induced neighbourhood of A. To see this, let $W \in \mathfrak{U}$ such that $W^2 \subset U$. Let B be a maximal W-discrete subset of A. Let $V \in \mathfrak{V}$ be such that $V(A) \subset U(A)$; and $V(b) \subset W(b)$ for all $b \in B$. If C is a member of the V-induced neighbourhood of A, then $C \subset V(A)$ and $A \subset V(C)$. Consequently $C \subset U(A)$; and, if $a \in A$, there exists $b \in B$ such that $(a, b) \in W$. Since $b \in V(C)$, there is $c \in C$ such that $c \in V(b) \subset W(b)$. This implies $(c, a) \in U$, so $A \subset U(C)$.

* The referee advises that this lemma is explicitly stated 'in a slightly different but equivalent form' in (7).

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THEOREM. *If* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are uniformities such that* $\mathfrak{U} = {}^{p}\mathfrak{V}$ *and* $\mathfrak{U} = {}^{h}\mathfrak{V}$ *then* $\mathfrak{U}$ *and* $\mathfrak{V}$ *induce the same topology on the hyperspace.*

Proof. It follows from equality in proximity that, given $S \in \mathfrak{V}$ and a finite cover $D_1, \ldots, D_n$ of $A \subset X$, there exists $U \in \mathfrak{U}$ such that $a \in A$ and $(a, x) \in U$ implies

$$(a, x) \in \bigcup_i (D_i \times S(D_i)).$$

Thus, in view of the lemma, it will suffice to show that, given $V \in \mathfrak{V}$ and A a V-discrete set, there exist $T \in \mathfrak{U}$, $S \in \mathfrak{V}$ and a finite cover $D_1, \ldots, D_n$ of A such that

$$(a, x) \in \bigcup_i (T \cap (D_i \times S(D_i)))$$

implies $(a, x) \in V$.

Let $W \in \mathfrak{V}$ be such that $W^3 \subset V$. Let $T \in \mathfrak{U}$, and $K_1, \ldots, K_r$, a finite cover of X, be such that $(T^3 \cap (K_i \times K_i)) \subset W$. Let $S \in \mathfrak{V}$, and $E_1, \ldots, E_s$, a finite cover of X, be such that $S^2 \subset W$ and $(S^2 \cap (E_i \times E_i)) \subset T$. Then:

(1) *Given* $x \in X$, *there exist at most* r *distinct members* $a \in A$ *such that* $(a, x) \in T$.

Proof. If there are more than r, then at least two of them, say a_1 and a_2, are members of the same K_i, so $(a_1, a_2) \in (T^2 \cap (K_i \times K_i)) \subset W \subset V$. A contradiction.

(2) *Given* $a_0 \in A$, *there exist at most* r *distinct members,* $a_k \neq a_0$ *of* A *such that, for each* k *there exists* $x_k \in X$ *such that* $(a_0, x_k) \in T$ *and* $(a_k, x_k) \in S$.

Proof. Since $x_k = x_j$, $k \neq j$ implies $(a_k, a_j) \in S^2 \subset V$, the x_k's are distinct. If there are more than r of them, at least two of them, say x_1 and x_2 are in the same K_i which implies $(x_1, x_2) \in (T^2 \cap (K_i \times K_i)) \subset W$, which in turn implies $(a_1, a_2) \in W^3 \subset V$. A contradiction.

(3) *Given* $a_0 \in A$, *there exist at most* sr^2 *distinct members* $a_k \neq a_0$ *of* A *such that for each* k *there exists* $x_k \in X$ *such that* $(a_k, x_k) \in T$ *and* $(a_0, x_k) \in S$.

Proof. Suppose not. Then, by (1), more than sr of the x_k's involved are distinct. Consequently, more than r of them, say $x_1, x_2, \ldots, x_{r+1}$, are in the same E_i. Thus, $(x_k, x_j) \in (S^2 \cap (E_i \times E_i)) \subset T$ for all j and k between 1 and $r+1$. Thus, for the same j's and k's $(a_k, a_j) \in T^3$. But, at least two of these, say a_1 and a_2 are in the same K_i. Thus

$$(a_1, a_2) \in (T^3 \cap (K_i \times K_i)) \subset W \subset V.$$

A contradiction.

(4) *Given* $a_0 \in A$ *there exist at most* $n = r(1 + sr)$ *distinct members* $a_k \neq a_0$ *of* A *such that for each* k *there exists* $x_k \in X$ *such that either* $(a_0, x_k) \in T$ *and* $(a_k, x_k) \in S$, *or* $(a_0, x_k) \in S$ *and* $(a_k, x_k) \in T$.

Proof. Combine (2) and (3).

Now let $D_1, \ldots, D_{n+1}$ be maximal disjoint subsets of A such that, for $a, a' \in D_i$ and any $x \in X$, $(a, x) \in T$ and $(a', x) \in S$ implies $a = a'$. By (4), the D_i's exhaust A.

Now suppose $a \in A$, $x \in X$, and $(a, x) \in (T \cap (D_i \times S(D_i)))$. Then, there is $a' \in D_i$ such that $(a', x) \in S$. By the construction of D_i, $a = a'$, and $(a, x) \in S \subset V$. This completes the proof.

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