ON GENERALIZED SUBHARMONIC FUNCTIONS

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1. *Introduction.* In a recent paper (1) I studied a class of generalized convex functions of a single real variable which I called sub-(L) functions. Given an ordinary linear differential equation of the second order $L(y) = 0$, a function $f(x)$ is sub-(L) in (a, b) if it is majorized there by the solutions of the equation. More precisely, for every x_1, x_2 in (a, b), $f(x) \leqslant F_{12}(x)$ in (x_1, x_2), where F_{12} is that solution of $L(y) = 0$ (supposed unique) which takes the values $f(x_i)$ at x_i. It was found that sub-(L) functions are characterized in a manner closely analogous to ordinary convex functions.

In the present paper, subharmonic functions are generalized in a similar way, the Laplace equation $\Delta u = 0$ being replaced, as 'majorizing equation', by a general partial differential equation of elliptic type, $L(u) = 0$. The discussion is restricted to the case of two independent variables for the sake of simplicity. It is found that sub-(L) functions of two variables are characterized by best possible differential and integral inequalities which reduce to the familiar inequalities for subharmonic functions in the case when $L \equiv \Delta$.

It is well known that certain functions of a single variable related to a subharmonic $f(x, y)$ are convex, $\left[\sup_{-\infty<y<\infty} f(x, y), \overline{\lim_{y\to\infty}} f(x, y)\right]$. Similar properties are found to hold for a wide class of sub-(L) $f(x, y)$. The related functions of a single variable are sub-(L^*), where L^* is an ordinary differential operator related to L.

The characterization theorems are applied to the modulus of an analytic function $f(z)$. $|f(z)|$ is sub-(L), with $L \equiv \Delta - \lambda^2$ in any domain where

$$\left|\frac{f'(z)}{f(z)}\right| \geqslant \lambda.$$

A 'three-lines theorem' is obtained for a function schlicht in a half plane.

2. *Definitions and notation.* I consider, throughout, an elliptic partial differential equation, reduced to its canonical form:

$$\left.\begin{aligned} L(w) &\equiv \left\{\Delta + p_1\frac{\partial}{\partial x} + p_2\frac{\partial}{\partial y} + p_3\right\} w = 0, \\ \Delta &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \end{aligned}\right\} \tag{2·1}$$

where the p_i are *regular* functions of (x, y) in some domain $\mathscr{D}$; i.e. they are continuous and have continuous derivatives of the first and second orders. This restriction on L is not strictly necessary, but it is perhaps hardly worth while to attempt to obtain full generality in this respect†. All point sets considered will be supposed to lie in $\mathscr{D}$.

† It is sufficient that p_1, p_2 have continuous first derivatives and p_3, $\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y}$ satisfy a Hölder condition.

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A real function $f(x,y)$ is denoted by $f(z)$ $(z = x+yi)$, whenever this is convenient. D denotes a domain, D' its closure, $F(D)$ its frontier. I have followed, as far as possible, the notation and definitions of J. E. Littlewood (2).

A function u will be called an L-frontier majorant of a function g for the bounded domain D_1, if u is a solution of $L(u) = 0$ regular in D_1, is continuous in D_1', and

$$u(z) \geqslant \overline{\lim_{\zeta \to z}}\, g(\zeta) \quad (\zeta \to z \text{ in } D_1), \tag{2·2}$$

for all points z of $F(D_1)$.

A function $w(z)$ will be called *sub-*(L) in a bounded D if it satisfies the conditions A, B:

(A) (1) $$w(z) \geqslant \overline{\lim_{\zeta \to z,\ \zeta \neq z}}\, w(\zeta) \text{ for all } z \text{ of } D, \tag{2·3}$$

(A) (2) $$w(z) < \infty \text{ in } D, \tag{2·4}$$

(B) $w \leqslant u$ in D_1 for every domain D_1 and function u which are such that D_1' is contained in D, and u is an L-frontier majorant of w for D_1. (2·5)

A function $w(z)$ is sub-(L) in an unbounded D if it is sub-(L) in every bounded subdomain of D; $w(z)$ will be called super-(L) if $-w(z)$ is sub-(L).

It is known (2) that a function $w(z)$ satisfying (A) has an attained upper bound in any bounded closed subset E of D, and is the limit function of a decreasing sequence $\{w_n(z)\}$ of continuous functions in such E.

To avoid tedious repetition, I use the following notation:

(*a*) L, of (2·1), is of *type* M in D if $p_3 < 0$ in D.

(*b*) L is of *type* M^* in D if there exists a $\phi(z)$, positive and continuous in D', regular in D, and such that $L(\phi) < 0$ in D.

3. *Preliminary lemmas.* It is evident that an equation of type M is of type M^* (take $\phi \equiv 1$). Also, if $p_3 \leqslant 0$ in D it is easily established, by taking $\phi = A - e^{\alpha x}$, that L is of type M^*. The following simple lemmas are presumably known:

LEMMA 1. *Every L is of type M^* in the intersection of any sufficiently narrow strip, $\alpha < x < \beta$, with any bounded closed set E contained in $\mathscr{D}$.*

This lemma is readily proved by taking $\phi = \cos k(x - x_0)$ and suitably adjusting the values of k, x_0.

LEMMA 2. (i) *If L is of type M in D, a solution regular in D cannot attain a positive maximum or a negative minimum in D.*

(ii) *If L is of type M^* in a bounded D, it has the following uniqueness property: the only solution regular in a D_1 contained in D which is continuous in D_1' and vanishes identically on $F(D_1)$ is the solution identically zero in D_1.*

The proof of (i) is well known (3) and (ii) is a simple consequence. In fact, if $L(w) = 0$ in D_1, w is regular in D_1 and continuous in D_1', and $w = 0$ on $F(D_1)$; then writing $w = \phi v$ and $L(w) = \phi L^*(v)$ we see that $L^*(v) = 0$ in D_1 and L^* is of type M in D_1. The result now follows from (i).

4. *Characterization of regular sub-(L) functions.*

THEOREM 1. *If,* (i) *L is of type M^* in D,* (ii) *$w(x,y)$ is regular in D,* (iii) *$L(w) \geqslant 0$ in D; then w is sub-(L) in D.*

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Proof. Consider first the case when L is of type M in D and D is bounded. Let u be any L-frontier majorant† of w for a subdomain D_1 of $D(D_1' \subset D)$. Then $v = w-u$ satisfies the conditions of the theorem in D_1, is continuous in D_1' and is non-positive on $F(D_1)$. Thus, as in the proof of Lemma 2 (i), $v \leqslant 0$ in D_1.

Next, suppose that L is of type M^*, and set $w = \phi w^*$, $u = \phi u^*$. Then $L(w) = \phi L^*(w^*)$, so that $L^*(w^*) \geqslant 0$, $L^*(u^*) = 0$, L^* is of type M. Since $w^* \leqslant u^*$ on $F(D_1)$, it now follows from the first case that $w^* \leqslant u^*$ in D_1. Since $\phi > 0$, this completes the theorem for bounded D and the extension to unbounded D is trivial.

The following slight extension of Theorem 1 is required in order to apply the theorem to the modulus of an analytic function:

THEOREM 1*. *Let L be of type M^* in D; let $w(z)$ be non-negative and satisfy conditions* (A) *in D, and satisfy conditions* (ii) *and* (iii) *of Theorem* 1 *in $D-E$, where E is any bounded closed subset of D. Then, if $w = 0$ at all points of E, w is sub-(L) in D.*

Proof. Consider first an L of type M and a bounded D containing E. Let u be an L-frontier majorant of w for D_1, $(D_1' \subset D)$. Then $u(z) \geqslant \overline{\lim_{\zeta \to z}}\, w(\zeta) \geqslant 0$ on $F(D_1)$, and hence $u \geqslant 0$ in D_1. Suppose that $w-u$ attains a positive maximum in D_1, then as before this cannot be at a point of $D-E$, and since $w = 0$ in E it cannot be at a point of E. Hence $w \leqslant u$ in D_1. The theorem is now completed as in the proof of Theorem 1.

5. So far, the condition that the p_i in L be regular has not been required and continuity would have sufficed. However, in proving the converse of Theorem 1 some restriction on the p_i seems to be required. In fact we appear to need a restriction which will ensure the existence of a solution of the Dirichlet problem for L relative to sufficiently small circles.

THEOREM 2. *If $w(z)$ is regular and sub-(L) in D, then $L(w) \geqslant 0$ in D.*

Proof. $L(w)$ is continuous, and hence if $L(w) < 0$ at z_0, $L(w) < 0$ in any sufficiently small circle $C_\rho(z_0)$, centre z_0. Every L is locally of type M^* by Lemma 1; hence $-w$ is sub-(L) in a domain containing $C_\rho(z_0)$. Let u be the solution of $L(u) = 0$ regular in the interior of C_ρ and continuous in the closed circle, such that $u = w$ on the circumference‡. Then since w is sub-(L), $w \leqslant u$ in C_ρ; and since $-w$ is sub-(L), $-w \leqslant -u$ in C_ρ. Thus $w = u$ and $L(w) = 0$ in C_ρ, contrary to hypothesis.

6. *Characterization of general sub-(L) functions.* In order to obtain inequalities for sub-(L) functions corresponding to the well-known inequality

$$f(z_0) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0 + \rho e^{\theta i})\, d\theta \tag{6·1}$$

for subharmonic f, we require a *non-negative* kernel

$$k(\rho, \theta) = k(\rho, \theta; z_0, L),$$

such that

$$v(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(\rho, \theta)\, v(z_0 + \rho e^{\theta i})\, d\theta, \tag{6·2}$$

for every solution $v(z)$ of $L(v) = 0$ regular in the open circle $A_\rho(z_0)$, $(|z - z_0| < \rho)$ and continuous in its closure $C_\rho(z_0)$, and for all sufficiently small ρ.

† If no L-frontier majorant exists, there is nothing to prove.

‡ That such a solution exists is well known (4).

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Let $\bar{L}$ denote the operator adjoint to L, i.e.

$$\bar{L} \equiv \Delta - p_1 \frac{\partial}{\partial x} - p_2 \frac{\partial}{\partial y} + \left(p_3 - \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial y} \right), \tag{6·3}$$

and suppose first that $\bar{L}$ is of type M^* in D. Also let $z = z_0 + re^{\theta i}$ and let $C_\rho(z_0)$ be contained in D. Since $\bar{L}$ is of type M^*, there exists no solution (other than the identically zero solution) of $\bar{L}(u) = 0$ regular in A_ρ, continuous in C_ρ and vanishing on the circumference. It is then known (4) that there exists a solution $G(z)$ of $\bar{L}(u) = 0$ in $0 < r < \rho$, with the properties:

(i) $G(z) = U \log r + V$, U, V regular in A_ρ and continuous in C_ρ;

(ii) $U(z_0) = -1$;

(iii) $G(z_0 + \rho e^{\theta i}) = 0$, for all θ.

It is also known (4) that $G(z)$ has continuous first derivatives in C_ρ, except at z_0, that

$$\frac{\partial G}{\partial n} = \left(-\frac{\partial G}{\partial r} \right)_{r=\rho} \geqslant 0; \tag{6·4}$$

and that

$$v(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\rho \frac{\partial G}{\partial n} \right) v(z_0 + \rho e^{\theta i})\, d\theta, \tag{6·5}$$

for every solution $v(z)$ of $L(v) = 0$, regular in A_ρ, and continuous in C_ρ.

If we write

$$k(\rho, \theta) = \rho \frac{\partial G}{\partial n}, \tag{6·6}$$

and observe that $\bar{L}$ is always locally of type M^* (6·2) follows.

7. THEOREM 3. *If $w(z)$ is sub-(L) in D, and $k(\rho, \theta)$ is defined by* (6·6), *then*

$$w(z_0) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, w(z_0 + \rho e^{\theta i})\, d\theta \tag{7·1}$$

for every z_0 of D and all sufficiently small ρ.

Proof. If $w(z)$ is sub-(L) in D, given any z_0 and small ρ, $w(z)$ is the limit of a decreasing sequence of functions w_n, continuous in $C_\rho(z_0)$. Let v_n be that solution of $L(v_n) = 0$ regular in $A_\rho(z_0)$, continuous in $C_\rho(z_0)$, and such that $v_n = w_n$ on the circumference (these solutions all exist for sufficiently small ρ). Then $v_n(z)$ is an L-frontier majorant of $w(z)$ for $A_\rho(z_0)$, and hence

$$\begin{aligned} w(z_0) \leqslant v_n(z_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, v_n(z_0 + \rho e^{\theta i})\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, w_n(z_0 + \rho e^{\theta i})\, d\theta. \end{aligned} \tag{7·2}$$

Now since $k(\rho, \theta) \geqslant 0$, the functions kw_n form a decreasing sequence and (7·1) follows in the limit as $n \to \infty$.

8. LEMMA 4. *If $w(z)$ is sub-(L) in D, then*

$$w(z) = \overline{\lim_{\zeta \to z,\, \zeta \neq z}} w(\zeta) \tag{8·1}$$

for all z of D.

Proof. If $v(z)$ is any regular solution of $L(v) = 0$ in a neighbourhood of z_0 and $v(z_0) \neq 0$, then for sufficiently small ρ, and any $\epsilon > 0$,

$$\frac{1}{2\pi} \{v(z_0) - \epsilon\} \int_{-\pi}^{\pi} k(\rho, \theta)\, d\theta \leqslant v(z_0) \leqslant \frac{1}{2\pi} \{v(z_0) + \epsilon\} \int_{-\pi}^{\pi} k(\rho, \theta)\, d\theta.$$

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Thus
$$\lim_{\rho \to 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, d\theta = 1. \tag{8·2}$$

By Theorem 3, if $\overline{\lim}_{\zeta \to z,\ \zeta \neq z} w(\zeta)$ is finite and ρ is small,

$$w(z) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, w(z + \rho e^{\theta i})\, d\theta$$

$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, d\theta \left\{ \overline{\lim_{\zeta \to z,\ \zeta \neq z}} w(\zeta) + \epsilon \right\}, \tag{8·3}$$

ϵ being arbitrary, (8·2) shows that

$$w(z) \leqslant \overline{\lim_{\zeta \to z,\ \zeta \neq z}} w(\zeta),$$

and this combined with (A)(1) yields (8·1). If $\overline{\lim}_{\zeta \to z,\ \zeta \neq z} w(\zeta) = -\infty$, (8·1) follows at once from (7·1).

THEOREM 4. *If* (i) *L is of type M^* in D,* (ii) *for each z_0 in D and each sufficiently small ρ, $k(\rho, \theta) > 0$ except perhaps in a set of measure zero where $k = 0$,* (iii) *$w(z)$ satisfies conditions* (A) *in D and satisfies* (7·1) *for all z_0 of D and some† arbitrarily small ρ; then $w(z)$ is sub-(L) in D.*

Proof. This is based on the proof(2) of the corresponding theorem for subharmonic functions. Let D_1 ($D_1' \subset D$) be a bounded domain, and $v(z)$ an L-frontier majorant of $w(z)$ for D_1. Suppose the theorem is false, i.e. we suppose that $t(z) = w(z) - v(z)$ attains a positive maximum M at points of D_1. Now the set E of points where $t(z) = M$ is closed, and we can select a point Q of $F(E)$, and therefore of E, in such a way that every small circle with centre Q and radius ρ has a point R of the complement of E on its circumference. Now $t(z) < M$ at R and so, by (A), in a neighbourhood of R. Hence if Q is z_0,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, t(z_0 + \rho e^{\theta i})\, d\theta < \frac{M}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, d\theta \tag{8·4}$$

for all sufficiently small ρ.

Suppose now that in L the coefficient $p_3 \leqslant 0$, then by Theorems 1 and 3,

$$-1 \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, (-1)\, d\theta,$$

i.e.
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, d\theta \leqslant 1, \tag{8·5}$$

for all sufficiently small ρ. Hence in this case (8·4) becomes

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, t(z_0 + \rho e^{\theta i})\, d\theta < M = t(z_0), \tag{8·6}$$

which establishes a contradiction, since $t(z)$ satisfies (7·1) for some arbitrarily small ρ.

Finally, if L is of type M^* in D the identity

$$v(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\rho, \theta)\, v(z_0 + \rho e^{\theta i})\, d\theta,$$

may be written
$$v^*(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k^*(\rho, \theta)\, v^*(z_0 + \rho e^{\theta i})\, d\theta \tag{8·7}$$

† In general, depending on z_0.

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where $v^* = v/\phi$ is a solution of $L^*(v^*) = 0$ (L^* of type M in D), and

$$k^*(\rho,\theta) = \frac{\phi(z_0+\rho e^{\theta i})}{\phi(z_0)}\,k(\rho,\theta). \tag{8·8}$$

Also on writing $w^* = w/\phi$, (7·1) becomes

$$w^*(z_0) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} k^*(\rho,\theta)\,w^*(z_0+\rho e^{\theta i})\,d\theta. \tag{8·9}$$

It follows from (8·7) and (8·9) that w^* is sub-(L^*) in D and hence w is sub-(L).

As in the case of subharmonic functions, slightly more than Theorem 4 is true if D is a bounded domain. By considering the function

$$\tilde{t}(z) = \overline{\lim_{\zeta\to z}}\, w(\zeta) - v(z),$$

it may be proved that if $v(z)$ is an L-frontier majorant of $w(z)$ for D itself, then $w \leqslant v$ in D.

Without the condition that L be of type M^*, $w(z)$ will be locally sub-(L), but this or some equivalent uniqueness condition is required for the truth of the theorem. For suppose that v_1, v_2 are two solutions of $L(v) = 0$ regular in D_1, continuous in D_1' and such that $v_1 = v_2$ on $F(D_1)$ but $v_1 \neq v_2$ in D_1. We may suppose that $v_1(z_0) > v_2(z_0)$ for some z_0 in D_1; then since v_2 is an L-frontier majorant of v_1 for D_1, v_1 is not sub-(L). But v_1 satisfies (7·1).

9. The interesting applications of the characterization theorems arise in those cases when $k(\rho,\theta)$ can be explicitly calculated and especially when k is independent of θ. This occurs, about the origin, for L of the form

$$L \equiv \Delta - f(r), \tag{9·1}$$

where f is holomorphic†, or has a simple pole, at the origin. In this case, solutions independent of θ are solutions of

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - r^2 f(r)\,u = 0. \tag{9·2}$$

There exists a solution $W(r)$ of (9·2) holomorphic at the origin and satisfying $W(0) = 1$. To obtain the logarithmic solution $G(r)$ of § 6, take $G(r) = Wv$, then

$$\frac{v''}{v'} + \frac{2W'}{W} + \frac{1}{r} = 0,$$

and hence

$$v'(r) = -1/(rW^2). \tag{9·3}$$

Choosing the arbitrary constant in v so that $v(\rho) = 0$, we obtain

$$\left(\frac{dG}{dr}\right)_{r=\rho} = W(\rho)\,v'(\rho) = -1/\rho W(\rho). \tag{9·4}$$

Hence

$$k(\rho,\theta) = -\rho\left(\frac{dG}{dr}\right)_{r=\rho} = 1/W(\rho). \tag{9·5}$$

10. *Strip theorems.* It is a simple matter to construct theorems exhibiting sub-(L^*) properties of sub-(L) functions, L^* being an ordinary differential operator related to L. For example, consider an L of the following special type in the strip $\alpha < x < \beta$, (D):

$$L \equiv \Delta + p_1(x)\frac{\partial}{\partial x} + p_2(x,y)\frac{\partial}{\partial y} + p_3(x), \tag{10·1}$$

† The word regular is avoided here since it has already been used in another sense.

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where p_1, p_3 are functions of x only, p_2 is bounded and $p_3 < 0$ in D. Associated with L is an ordinary differential operator:

$$L^* \equiv \frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_3(x). \tag{10·2}$$

Since $p_3 < 0$, there exists a unique solution of $L^*(u) = 0$ taking given values at any x_1, x_2 in the open interval (α, β). Thus L^* is of the kind considered in (1).

THEOREM 5. *Let L, L^* be of the type defined above* (10·1), (10·2). *If $f(x, y)$ is sub-(L), not identically $-\infty$, and bounded above in the strip $\alpha < x < \beta$, then*

$$\psi(x) = \sup_{-\infty < y < \infty} f(x, y) \tag{10·3}$$

is sub-(L^*) in (α, β).

Proof. Let x_1, x_2 be any two points in the open interval (α, β), and let $u(x)$ be the solution of $L^*(u) = 0$ which satisfies $u(x_i) = \psi(x_i)$, $i = 1, 2$, ($\psi(x_i)$ supposed finite). Since p_1, p_3 are functions of x only, $u(x)$ is also a solution of $L(u) = 0$ in the strip. Hence

$$g(x, y) = f(x, y) - u(x) \tag{10·4}$$

satisfies the conditions of the theorem and

$$g(x_i, y) \leqslant 0 \quad (i = 1, 2; -\infty < y < \infty). \tag{10·5}$$

Choosing x_3, x_4 for which $\alpha < x_3 < x_1 < x_2 < x_4 < \beta$ let $v = -\{e^{\kappa y} + e^{-\kappa y}\}$, where κ is chosen so small that $L(v) > 0$ for all (x, y) in the strip $x_3 \leqslant x \leqslant x_4$. Then v is sub-(L) in this strip. If R is any rectangular subdomain of the strip $x_3 < x < x_4$ and $\epsilon < 0$,

$$g_\epsilon(x, y) = g(x, y) + \epsilon v \leqslant w_\epsilon(x, y) \tag{10·6}$$

in R, w_ϵ being any L-frontier majorant of g_ϵ for R. Moreover, $v < 0$ for all y and $v \to -\infty$ as $|y| \to \infty$. Thus η can be chosen so large that

$$g_\epsilon(x, \eta) < 0, \quad g_\epsilon(x, -\eta) < 0 \quad \text{in} \quad \langle x_1, x_2 \rangle. \tag{10·7}$$

It follows from (10·5) and (10·7) that the identically zero solution of $L(w) = 0$ is an L-frontier majorant of $g_\epsilon(x, y)$ for the rectangle bounded by the lines $x = x_1$, $x = x_2$, $y = \pm\eta$. Hence

$$g_\epsilon(x, y) \leqslant 0 \tag{10·8}$$

in this rectangle. Since η is as large as we please (10·8) holds for all (x, y) in the strip $x_1 \leqslant x \leqslant x_2$, and since ϵ is arbitrary

$$g(x, y) \leqslant 0 \quad \text{for all} \quad (x, y) \quad \text{in} \quad x_1 \leqslant x \leqslant x_2. \tag{10·9}$$

Finally, taking the sup. in $-\infty < y < \infty$, we obtain

$$\psi(x) \leqslant u(x) \quad \text{in} \quad (x_1, x_2), \tag{10·10}$$

which establishes the theorem in the case when the $\psi(x_i)$ are finite. Otherwise some slight modification of the argument is required.

The following is another example of the same class of theorems:

THEOREM 6. *Let $L \equiv \Delta - p_3(x, y)$, where p_3 decreases to a continuous positive $p^*(x)$ as $|y| \to \infty$; $L^* \equiv d^2/dx^2 - p_3^*$. If $f(x, y)$ is sub-(L), positive and bounded above in the strip $\alpha < x < \beta$, then*

$$\psi(x) = \sup_{-\infty < y < \infty} f(x, y)$$

is sub-(L^*) in (α, β).

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Proof. If $u(x)$ is the solution of $L^*(u) = 0$ for which $u(x_i) = \psi(x_i)$, $i = 1, 2$, then $u(x) \geqslant 0$ in an interval (x_3, x_4) containing $\langle x_1, x_2 \rangle$. Thus

$$L(u) = \frac{d^2u}{dx^2} - p_3 u = (p_3^* - p_3)\, u \leqslant 0. \tag{10·11}$$

It follows that $-u$ is sub-(L) in the strip $x_3 < x < x_4$ and so if we write

$$g(x, y) = f(x, y) - u(x),$$

from this point the proof is similar to that of Theorem 5.

11. *Applications.* Let $f(z)$ be holomorphic and have no zeros in a domain D and let $g(x, y) = |f(z)|$; then straightforward differentiation yields

$$\Delta g = g\left|\frac{f'}{f}\right|^2. \tag{11·1}$$

If now
$$O \leqslant \lambda \leqslant \left|\frac{f'}{f}\right| \leqslant \mu \quad \text{in} \quad D, \tag{11·2}$$

then
$$L_\lambda(g) = (\Delta - \lambda^2)\, g \geqslant 0, \tag{11·3}$$

$$L_\mu(g) = (\Delta - \mu^2)\, g \leqslant 0. \tag{11·4}$$

It follows that g is sub-(L_λ) and super-(L_μ) in D. By Theorem 1*, the conclusion that g is sub-(L_λ) also holds when f has a finite number of zeros in D.

In the case when λ, μ are constants the kernels $k(\rho, \theta)$ may be calculated by the method of § 9 and turn out to be modified Bessel functions of the first kind. In fact

$$\begin{aligned} 1/k(\rho, \theta) &= I_0(\lambda\rho) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{\lambda\rho}{2}\right)^{2n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda\rho\cos\theta}\, d\theta. \end{aligned} \tag{11·5}$$

We have now established

THEOREM 7. *If* (i) $f(z)$ *is holomorphic and has no zeros in* D, (ii) $0 \leqslant \lambda \leqslant \left|\frac{f'}{f}\right| \leqslant \mu$ *in* D; *then*

$$\frac{1}{I_0(\lambda\rho)}\int_{-\pi}^{\pi} |f(z_0 + \rho e^{\theta i})|\, d\theta \geqslant 2\pi |f(z_0)| \geqslant \frac{1}{I_0(\mu\rho)}\int_{-\pi}^{\pi} |f(z_0 + \rho e^{\theta i})|\, d\theta$$

for every z_0 *in* D *and all* ρ *such that* $z \in D$ *if* $|z - z_0| \leqslant \rho$. *The left-hand inequality holds also if* f *has a finite number of zeros in* D.

The function $e^{\alpha z}$ shows this theorem to be best possible.

12. Let $f(z)$ be holomorphic and schlicht in the half plane $x = R(z) > 0$; then by mapping the half plane on the unit circle, it may be proved that

$$\left|\frac{f''(z)}{f'(z)}\right| \leqslant \frac{3}{x}, \tag{12·1}$$

from which we may at once deduce

THEOREM 8. *If* $f(z)$ *is holomorphic and schlicht in the half plane* $x = R(z) > 0$, *then* $|f'(z)|$ *is super-*(L) *in the half plane with*

$$L \equiv \Delta - 9/x^2. \tag{12·2}$$

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It now follows from Theorem 5 that

$$\phi(x) = \inf_{-\infty<y<\infty} |f'(x+yi)| \tag{12·3}$$

is super-(L^*) for $x>0$, with $$L^* \equiv \frac{d^2}{dx^2} - \frac{9}{x^2}. \tag{12·4}$$

Thus $$\phi(x) \geqslant Ax^\alpha + Bx^\beta \quad \text{in} \quad (x_1, x_2), \tag{12·5}$$

where α, β are the roots of $t^2-t-9=0$ and A, B are chosen to give equality in (12·5) when $x = x_i$, $i = 1, 2$.

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