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Topological entropy of polygon exchange transformations and polygonal billiards

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Abstract. We study the topological entropy of a class of transformations with mild singularities: the generalized polygon exchanges. This class contains, in particular, polygonal billiards. Our main result is a geometric estimate, from above, on the topological entropy of generalized polygon exchanges. One of the applications of our estimate is that the topological entropy of polygonal billiards is zero. This implies the subexponential growth of various geometric quantities associated with a polygon. Other applications are to the piecewise isometries in two dimensions, and to billiards in rational polyhedra.

1. *Introduction*

Let $T : X \to X$ be a continuous selfmapping of a compact metric space. The topological entropy is an important invariant that characterizes by a number, $h(T) \geq 0$, the exponential complexity of the orbits of T (see, e.g., [**22, 39**]). Suppose that $T = T^t$ is the geodesic flow on a compact Riemannian manifold, N, of negative curvature. Then there are explicit bounds on $h(T)$ from above and from below in terms of the average curvature of N, its volume, diameter, etc. [**19, 28, 30, 36**]. Theorems of another type provide estimates on $h(T)$, where $T : M \to M$ is a diffeomorphism of a compact manifold, in terms of the growth rates of various geometric quantities on M, under the iterations of T [**10, 21, 22, 29, 31, 34, 41**]. The following result of S. Newhouse served as a partial motivation for the present work.

Let $T : M \to M$ be as above. For a compact, piecewise C^1 curve, γ, in M, denote by $|\gamma|$ its length with respect to a Riemannian metric on M. Then

$$\lambda(T) = \sup_{\gamma} \limsup_{n\to\infty} \log\left(\sum_{k=0}^{n} |T^k\gamma|\right)/n \tag{1}$$

does not depend on the choice of a metric, and we say that $\lambda(T)$ is the *growth rate for the length of curves* in M.

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THEOREM 1.1. ([**31**]) *Let M be a compact C^2 surface, and let $T : M \to M$ be a $C^{1+\epsilon}$ diffeomorphism. Then $h(T) \leq \lambda(T)$.*

A metric entropy version of this estimate (for smooth invariant measures) goes back to [**25**]. Let $T : M \to M$ be a continuous selfmapping of a compact closed C^1 manifold. A. Manning has obtained a lower bound on $h(T)$.

THEOREM 1.2. ([**29**]) *Let the setting be as above, and let A be the induced transformation on the first homology. Then the topological entropy of T is bounded from below by the logarithm of the spectral radius of A.*

Relatively little is known about the topological entropy of the transformations with singularities, although they arise naturally, e.g. in the billiard dynamics. We briefly survey the literature. Y. Pesin and B. Pitskel [**33**] have worked on the variational principle for a particular class of singular transformations. Their approach is topological (continuous selfmaps of noncompact metric spaces), and goes back to R. Bowen [**4**]. There are a few results on the topological entropy (suitably interpreted) of singular billiard dynamics. A. Katok [**20**] has shown that the topological entropy of polygonal billiards is zero (see also [**3, 9**]). N. Chernov has obtained estimates on the topological entropy of Sinai billiards [**6**]. Chernov's results extend to the hyperbolic billiards the classical connection between the entropy of a diffeomorphism and the growth rate of the number of its periodic points [**21**]. The metric (as opposed to topological) entropy of transformations with singularities has received more attention (see, e.g., [**23, 26**]).

In the present work we introduce a class of transformations with mild singularities: the *generalized polytope exchanges* (Definition 2.3). In this setting, the space X (we assume X is a manifold with a boundary) is partitioned into a finite number of *generalized polytopes*, $P_i, 1 \leq i \leq n$. The transformation $T : X \to X$ is a diffeomorphism on each P_i, and, in general, T is not well defined on the singular set, $\partial\mathcal{P} = \cup_{1\leq i\leq n}\partial P_i$. A simple example of a generalized polytope exchange is a *generalized interval exchange*. In this case, X is an interval, and P_i are closed subintervals.

The topological entropy, $h(T) \geq 0$, of a generalized polytope exchange measures the exponential complexity of the regular orbits of T. The emphasis of our work is on the geometric bounds for the entropy. Our techniques are especially suitable for the two-dimensional case, i.e. for the *generalized polygon exchanges* (GPEs). Theorem 3.6 and Theorem 3.11 estimate from above the topological entropy of a GPE. Theorem 3.11 is a direct analog, for generalized polygon exchanges, of Theorem 1.1.

The framework of GPEs is wide enough to include transformations arising in very different situations. For instance, the 'baker transformation' [**1**], on one hand, and the polygonal inner and outer billiards [**11, 16, 17**], on the other hand, are GPEs. There are many open questions about the (generalized) polytope exchanges. The interval exchange transformations have attracted much attention, partly because of the connection with the billiards in rational polygons (see, e.g., [**24, 38**]). A direct analog of interval exchange transformations in two dimensions are the *rectangle exchange transformations*. The only works on the subject known to us [**18, 32**], discuss the minimality and the topological transitivity of certain rectangle exchange transformations. Note that Theorem 4.2 answers, in particular, a question of [**18**].

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A few words about our techniques. With a generalized polytope exchange, $T : X \to X$, we associate an increasing tower, $\mathcal{P}_n, n \geq 1$, of partitions of X. A crucial point is to relate the growth rates (h and λ) of the cardinality, $|\mathcal{P}_n|$, and the length, $|\partial\mathcal{P}_n|$, of the singular set. Roughly speaking, the singular length limits the number of atoms in a partition. In order to implement this idea, we need to get a handle on the growth rate, β, of the maximal number of edges at a vertex of $\mathcal{P}_n$ (the *vertex complexity*). A combinatorial argument, devised for two dimensions, allows us to obtain an efficient estimate on β (Proposition 3.8). This estimate enables us to show that $h \leq \lambda$ (Theorem 3.6).

Getting an efficient bound on vertex complexity has been central in a variety of problems in dynamics. Some of them are closely related to our context [**20**], others are quite far away [**5, 35**].

To illustrate our results, in §4 we apply them to a variety of transformations: piecewise isometries on surfaces, polygonal billiards, and directional billiards in rational polyhedra. We prove that the topological entropy is zero (Theorems 4.2, 4.4, and 4.9). This implies, in particular, the subexponential growth of various geometric quantities associated with a polygon or a polyhedral surface (Corollary 4.5). Other proofs of the subexponential growth in this context involved a detailed analysis of the topology of billiard orbits [**9, 20**]. Our proof exploits the elementary observation that the length of singular sets for polygonal billiards grows quadratically.

Some of the results of the present work were announced in [**14, 15**]. We thank N. Chernov, A. Katok, and M. Rychlik for discussions and remarks.

2. *Generalized polytope exchanges*

2.1. *The transformations.* By a *polytope*, P, $\dim P = n$, we mean a compact Euclidean polytope [**8**] in $\mathbf{R}^n$, with a nonempty interior, $\operatorname{int} P$. The m-dimensional faces of P are polytopes of dimension $m < n$. *Polygons (polyhedra)* are the polytopes of dimension 2 (3).

Definition 2.1.

(i) A *partition* $\mathcal{P}$ *of a polytope* P is a representation $P = \bigcup_{1 \leq i \leq n} P_i$, where P_i are *subpolytopes* of P, and $\operatorname{int} P_i \cap \operatorname{int} P_j = \emptyset$ for $i \neq j$.

(ii) A *generalized polytope* X of dimension n is a closed, with nonempty interior, subset of a ($C^r, r \geq 1$) manifold, M^n, and a mapping $f : X \mapsto \mathbf{R}^n$ such that: (1) f extends to a diffeomorphism of an open neighborhood of X; (2) $f(X)$ is a polytope.

The mappings $f : X \to \mathbf{R}^n$ will be implicit in what follows.

Definition 2.2. A *space* X (of dimension d) *with a (generalized) polytope partition* $\mathcal{P}$ is a closed, with nonempty interior, subset of a manifold M^d, and a representation, $X = \bigcup_{1 \leq i \leq n} X_i$, satisfying the following conditions:

(i) the sets X_i are generalized polytopes of dimension d, for $1 \leq i \leq n$;

(ii) for $i \neq j$, $\operatorname{int} X_i \cap \operatorname{int} X_j = \emptyset$;

(iii) if $I \subset \{1, \ldots, n\}$ is such that $\bigcap_{i \in I} X_i \neq \emptyset$, then $\bigcup_{i \in I} X_i$ is a generalized polytope.

If $(X, \mathcal{P})$ is as above, we say that X_i are the *atoms* of $\mathcal{P}$, and that $\partial\mathcal{P} = \bigcup_{1 \leq i \leq n} \partial X_i$ is the *boundary* of $\mathcal{P}$. We use the notation $\operatorname{int}\mathcal{P} = X \setminus \partial\mathcal{P}$. A partition $\mathcal{Q}$ is a *refinement*

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of a partition $\mathcal{P}$ if every atom of $\mathcal{Q}$ is a subpolytope of an atom of $\mathcal{P}$. We denote this by $\mathcal{P} < \mathcal{Q}$. If $\mathcal{P} : X = \bigcup_{i\in I} Y_i$ and $\mathcal{Q} : X = \bigcup_{j\in J} Z_j$ are two partitions of X, their *join* $\mathcal{P} \vee \mathcal{Q}$ is the partition formed by the intersections $Y_i \cap Z_j$ such that $\operatorname{int} Y_i \cap \operatorname{int} Z_j \neq \emptyset$. Then $\mathcal{P}, \mathcal{Q} < \mathcal{P} \vee \mathcal{Q}$.

The terminology above is similar to the standard terminology used in measure theory [**39**]. Since we are usually dealing with generalized polytope partitions, we often delete 'generalized polytope', and speak simply of partitions.

Definition 2.3. A *generalized polytope exchange*, $T : X \to X$, on a partition $\mathcal{R}$, is a collection of diffeomorphisms, $T|_P : P \to X$ $(P \in \mathcal{R})$, such that $X = \bigcup_{P\in\mathcal{R}} T(P)$ is a partition, $\mathcal{R}_1 = T\mathcal{R}$. Then T^{-1} is also a generalized polytope exchange.

The *defining partition*, $\mathcal{R}$, plays an essential role in the definition above. In what follows we make a simplifying assumption (without loss of generality) that both T, T^{-1} are defined on $\mathcal{R}$. Note it does not mean that $\mathcal{R}_1 = \mathcal{R}$. When $\dim X = 2$ (3) we will speak of a *generalized polygon (polyhedron) exchange*.

Example 1. A partition of $X = [0, 1]$ is given by n intervals $P_i = [a_{i-1}, a_i]$, where $0 = a_0 < a_1 < \cdots < a_n = 1$. An *interval exchange* on the intervals $P_i, 1 \leq i \leq n$, (see, e.g., [**7**]) is a special case of Definition 2.3. We obtain a *generalized interval exchange* by taking n nondegenerate affine transformations T_i, such that the intervals $Q_i = T_i(P_i), 1 \leq i \leq n$, form a partition of $[0, 1]$.

Example 2. Let X be a rectangle in $\mathbf{R}^2$, e.g. $X = [0, 1] \times [0, 1]$. Let the atoms of the partition $\mathcal{P} : X = \bigcup_{1\leq i\leq n} P_i$ be rectangles (with vertical and horizontal sides). Let t_i be n vectors such that the rectangles $Q_i = P_i + t_i, 1 \leq i \leq n$, form a partition of X. This defines a *rectangle exchange*.

Example 2′. Here $X \subset \mathbf{R}^2$ is an arbitrary polygon, the atoms of a partition $\mathcal{P} : X = \bigcup_{1\leq i\leq n} P_i$ are subpolygons. We define an affine polygon exchange on X by n nondegenerate affine transformations T_i such that $\mathcal{Q} = \{Q_i = T_i(P_i), 1 \leq i \leq n\}$ is a partition of X.

Example 3. An obvious analog of Example 2 in three dimensions features the unit cube $X = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbf{R}^3$ partitioned by n *rectangular parallelepipeds* P_i. The mappings T_i are parallel translations $x \to x + t_i$ such that the parallelepipeds $Q_i = P_i + t_i$ form a partition of X.

Example 3′. The 3D version of Example 2′. Here X is a polyhedron, $\mathcal{P} : X = \bigcup_{1\leq i\leq n} P_i$ is a polyhedral partition of X, and T_i are affine transformations of $\mathbf{R}^3$ such that the polyhedra $Q_i = T_i(P_i), 1 \leq i \leq n$, form a partition of X. This defines an affine polyhedron exchange.

2.2. *A regularization.* We will associate with any generalized polytope exchange a homeomorphism of a compact space. For brevity, we suppress the adjective 'generalized', and will often use 'transformation' for a polytope exchange.

Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$. For any pair of indices, $m, n \geq 0, m + n > 0$, we define, by induction, the partition $\mathcal{R}_{(-m,n)} =$

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$T^{-m+1}\mathcal{R} \vee \cdots \vee \mathcal{R} \vee \cdots \vee T^{n-1}\mathcal{R}$. For $n > 0$ set $\mathcal{P}_n = \mathcal{R}_{(-n,0)}$, $\mathcal{Q}_n = \mathcal{R}_{(0,n)}$, and $\mathcal{R}_{(0,0)}$ is the empty partition. Let $A \subset \mathbf{Z}^2$ be the set of pairs: $-m \leq 0 \leq n$. For $\alpha = (-m, n), \alpha' = (-m', n')$, we set $\alpha < \alpha'$ if $-m' \leq -m$ and $n \leq n'$. This defines a partial order on A, and A is *directed*, i.e. for any $\alpha, \beta \in A$ there exists $\gamma \in A$, such that $\alpha, \beta < \gamma$. If $\alpha = (-m, n)$ and $\beta = (-m', n')$, we set $\max(\alpha, \beta) = (-\max(m, m'), \max(n, n'))$. Let $\rho = (1, 1) \in \mathbf{Z}^2$. The following is straightforward.

LEMMA 2.4. *In the notation above, we have:*
(i) *if $\alpha \leq \beta$, then $\mathcal{R}_\alpha < \mathcal{R}_\beta$;*
(ii) *if $\alpha, \beta \in A$ and $\gamma = \max(\alpha, \beta)$, then $\mathcal{R}_\alpha \vee \mathcal{R}_\beta = \mathcal{R}_\gamma$;*
(iii) *the transformation $T^k : X \to X$ is a polytope exchange on $\mathcal{R}_\alpha$, if $\alpha, \alpha + k\rho \in A$.*

For any X with a partition $\mathcal{P} = \{P_i, i \in I\}$, the space $X_\mathcal{P} = \bigsqcup_{i \in I} P_i$, with the topology of a disjoint union, has a natural (disjoint) partition, $\hat{\mathcal{P}} = \{P_i, i \in I\}$. Let $\pi^\mathcal{P} : X_\mathcal{P} \to X$ be the projection. A polytope exchange $T : X \to X$, on a partition $\mathcal{P}$, with $\mathcal{Q} = T(\mathcal{P})$, induces a homeomorphism, $T_\mathcal{P} : X_\mathcal{P} \to X_\mathcal{Q}$. If $\mathcal{P} < \mathcal{P}'$, then $\mathcal{Q} < \mathcal{Q}'$, and we denote by $\pi_\mathcal{P}^{\mathcal{P}'} : X_{\mathcal{P}'} \to X_\mathcal{P}$, $\pi_\mathcal{Q}^{\mathcal{Q}'} : X_{\mathcal{Q}'} \to X_\mathcal{Q}$ the natural projections. The mappings $\pi_\mathcal{P}^{\mathcal{P}'}, \pi_\mathcal{Q}^{\mathcal{Q}'}, \pi^\mathcal{P}, \pi^\mathcal{Q}, T_\mathcal{P}, T_{\mathcal{P}'}$ satisfy the obvious relations, e.g. $T_\mathcal{P}\pi_\mathcal{P}^{\mathcal{P}'} = \pi_\mathcal{Q}^{\mathcal{Q}'}T_{\mathcal{P}'}$.

Let $T : X \to X$ be a polytope exchange on $\mathcal{R}$, and let partitions $\{\mathcal{R}_\alpha, \alpha \in A\}$ be as above. For $\alpha \in A$ we denote by X_α the space $X_{\mathcal{R}_\alpha}$, and by $\pi^\alpha : X_\alpha \to X$, $\pi_\alpha^\beta : X_\beta \to X_\alpha$ ($\alpha < \beta$) the associated projections. If $\alpha, \alpha + k\rho \in A$, we denote by $T_\alpha^k : X_\alpha \to X_{\alpha+k\rho}$ the homeomorphism corresponding to T^k.

The system of compacta $\{X_\alpha : \alpha \in A\}$ and continuous maps $\{\pi_\alpha^\beta : X_\beta \to X_\alpha, \alpha < \beta\}$ is an *inverse limit spectrum*; let $\hat{X}$ be the *inverse limit space* [**40**]. By construction, $\hat{X}$ is endowed with the family of projections $\{\pi_\alpha : \hat{X} \to X_\alpha, \alpha \in A\}$, satisfying $\pi_\alpha = \pi_\alpha^\beta\pi_\beta$ for $\alpha < \beta$. If Y is any space with a system $\{t_\alpha : Y \mapsto X_\alpha, \alpha \in A\}$ of maps satisfying the requirements above, then there is a unique surjection, $f : Y \to \hat{X}$, such that $t_\alpha = \pi_\alpha f$ for all α.

PROPOSITION 2.5. *Let $T : X \to X$ be a polytope exchange on $\mathcal{R}$, and let the notation be as above. Then there is a unique homeomorphism, $\hat{T} : \hat{X} \to \hat{X}$, such that if $\alpha, \alpha \pm \rho \in A$, then $T_\alpha^{\pm 1}\pi_\alpha = \pi_{\alpha\pm\rho}\hat{T}^{\pm 1}$.*

Proof. A subset $B \subset A$ is *cofinal* if for any $\alpha \in A$ there is $\beta \in B$ such that $\alpha < \beta$. Any subset, $B \subset A$, inherits a partial order. If we denote by $\hat{X}_B$ the corresponding inverse limit space, then there is a canonical map (onto) $\phi_B : \hat{X} \to \hat{X}_B$ which is compatible with the structures involved. If B is cofinal, then ϕ_B is a homeomorphism.

For $k \geq 0$, we denote by $A_k \subset A$ the subset $A_k = \{(-m, n) : m, n \geq k\}$. Then $A = A_0 \supset A_1 \supset \cdots \supset A_k \supset \cdots$, and every A_k is cofinal in A. Denote by $\hat{X}_k$ the inverse limit space corresponding to A_k, let $\pi_k : \hat{X}_k \to X$ be the projections, and let $\phi_k : \hat{X} \to \hat{X}_k$ be the homeomorphisms. If $\alpha \in A_k$, and $|j| < k$, then $\alpha + j\rho \in A_{k-|j|}$. By the preceding discussion and Lemma 2.4, this defines continuous mappings, $T_k^j : \hat{X}_k \to \hat{X}_{k-|j|}$, compatible with all π_r, ϕ_s, hence the homeomorphisms, $\hat{T}^j : \hat{X} \to \hat{X}$. Finally, $\hat{T}^j = (\hat{T})^j$. □

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The homeomorphism $\hat{T} : \hat{X} \to \hat{X}$ depends on $\mathcal{R}$, and we use the notation $\hat{T}_{\mathcal{R}} : \hat{X}_{\mathcal{R}} \to \hat{X}_{\mathcal{R}}$ for this *regularization.* If $\mathcal{R} < \mathcal{S}$, there is a unique projection, $\hat{X}_{\mathcal{S}} \to \hat{X}_{\mathcal{R}}$, compatible with $\hat{T}_{\mathcal{R}}, \hat{T}_{\mathcal{S}}$. Let $\mathcal{R} = \{R_i, i \in I\}$. A sequence of indices, $(s, n) = (i_{-n+1}, \ldots, i_{n-1})$, is a *code* if the set $R_{(s,n)} = \{x : T^k x \in R_{i_k}, |k| < n\}$ has a nonempty interior. Let S_n be the set of n-codes, and let $S = S_{\mathcal{R}}$ be the set of doubly infinite sequences $s = (i_k \in I : -\infty < k < \infty)$, such that for any $n > 0$ we have $(s, n) = (i_{-n+1}, \ldots, i_{n-1}) \in S_n$. Let Σ be the *full shift* on $|I|$ symbols, and let $t : \Sigma \to \Sigma$ be the left shift transformation. Then $S \subset \Sigma$ is a *subshift.* For $s \in S$ set $R_s = \bigcap_{n \geq 1} R_{(s,n)} \subset X$. Then T induces a homeomorphism of R_s onto $R_{t(s)}$.

PROPOSITION 2.6. *Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$, and let the notation be as above. Then:*

(1) *there is a unique continuous semiconjugacy (onto) $\sigma : \hat{X} \to S$;*
(2) *the mapping $\pi \times \sigma : \hat{X} \to X \times S$ is a homeomorphism, and its image is the subset $\{(x, s) : x \in R_s\}$.*

Proof. By construction, $R_s = \bigcap_{n \geq 0} R_{(s,n)} \neq \emptyset$. Set $X_n = \bigsqcup_{s \in S_n} R_{(s,n)}$, and let $\sigma_n^{n+1} : S_{n+1} \to S_n$, $\pi_n^{n+1} : X_{n+1} \to X_n$ be the natural mappings. The set $B = \{(-n, n), n \geq 0\} \subset A$ is cofinal, hence the inverse limit space $\lim_{\leftarrow} X_n$, as $n \to \infty$, is canonically homeomorphic to $\hat{X}$, which defines $\hat{T}$ on $\lim_{\leftarrow} X_n$. The action of $\hat{T}^{\pm 1}$ on $\lim_{\leftarrow} X_n$ is obtained from the commutative diagram:

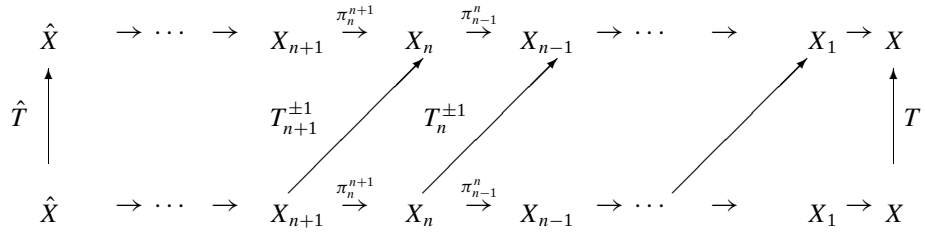


Diagram 1.

The natural projections $\rho_n : X_n \to S_n$, $n \geq 0$, define a mapping of the inverse limit spectra: $\{X_n, \pi_n^{n+1} : X_{n+1} \to X_n, n \geq 1\} \to \{S_n, \rho_n^{n+1} : S_{n+1} \to S_n, n \geq 1\}$. This yields a continuous mapping of the inverse limit spaces, $\sigma : \hat{X} \to S$, and the commutative diagram:

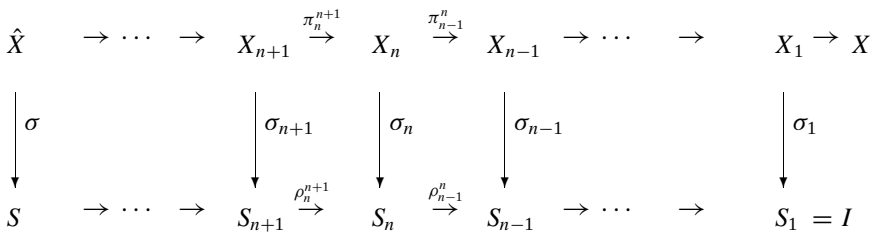


Diagram 2.

The claimed properties are immediate from the diagrams above. □

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2.3. *The entropy of a polytope exchange.* If $\mathcal{P}$ is a finite partition (polyhedral or set-theoretic), $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$. If F is a homeomorphism of a compact metric space, Y, and α is an open cover of Y, we denote by $h(F)$ and $h(F, \alpha)$ the *topological entropy* of F and the *topological entropy* of F *relative to* α, respectively [**39**].

Definition 2.7. Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$, and let the notation be as above. Then $h(\hat{T}, \hat{\mathcal{R}})$ is the *(topological) entropy* of T *relative to* $\mathcal{R}$, and $h(\hat{T})$ is the *(topological) entropy* of T. We will use the notation $h(T, \mathcal{R})$ and $h(T)$ in what follows.

A point, $x \in X$, is *regular* if $T^k x$ is defined for all k. Let X_{reg} be the set of regular points. By definition, an *invariant measure*, μ, on X satisfies $\mu(X_{\text{reg}}) = 1, T_*\mu = \mu$. Let $h_\mu(T)$ be the *metric entropy*. For future reference we list the basic properties of the entropy in the present context. They follow from the standard material [**39**].

PROPOSITION 2.8. *Let $T : X \to X$ be a polytope exchange on $\mathcal{R}$, and let the notation be as above. Then:*

(1) *We have*

$$h(T, \mathcal{R}) = \lim_{n\to\infty} \log |\mathcal{P}_n|/n = h(T^{-1}, \mathcal{R}) = h(t). \tag{2}$$

(2) *Set $\mathcal{R}_n = \mathcal{P}_n$ if $n > 0$, and $\mathcal{R}_n = \mathcal{Q}_{-n}$ if $n < 0$. Then*

$$h(T^n, \mathcal{R}_n) = |n| h(T, \mathcal{R}). \tag{3}$$

(3) *For any partition $\mathcal{P}$ set* $\operatorname{diam}(\mathcal{P}) = \sup \operatorname{diam}(P)$, $P \in \mathcal{P}$ *(with respect to a metric on X). If a sequence $\mathcal{R} < \mathcal{R}^{(1)} < \cdots < \mathcal{R}^{(n)} < \cdots$ of partitions is such that* $\operatorname{diam}(\mathcal{R}^{(n)}) \to 0$, *then $h(T, \mathcal{R}^{(n)}) \to h(T)$ monotonically. If $\mathcal{R}$ is* generating, *i.e.* $\operatorname{diam}(\vee_{n=1}^{\infty} \mathcal{P}_n) = 0$, *then $h(T, \mathcal{R}) = h(T)$.*

(4) *For any invariant measure μ on X, we have*

$$h_\mu(T) \le h(T). \tag{4}$$

Remark. We do not claim that $\sup_\mu h_\mu(T) = h(T)$ (the variational principle), because of the condition $\operatorname{supp} \mu \subset X_{\text{reg}}$ (see [**33**]).

3. *Estimates for entropy*

3.1. *Growth rates for generalized polytope exchanges.* The results of this subsection are valid in any dimension. The main result is Theorem 3.5.

LEMMA 3.1. *Let p_n be a positive sequence, and let* $\limsup_{n\to\infty} p_n/n = h > 0$. *For any number g, $0 \le g < h$, and any integer $m \ge 1$, there are infinitely many n such that*

$$p_{n+m} - p_n \ge mg. \tag{5}$$

Proof. Assume the opposite, i.e. $p_{n+m} - p_n < mg$ for all $n > n_0$. Then for these n and arbitrary $k = 1, 2, \ldots$, we have $p_{n+km} - p_n < kmg$. Therefore

$$\frac{p_{n+km}}{n+km} < \frac{p_n}{n+km} + \frac{km}{n+km} g. \tag{6}$$

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With n fixed and $k \to \infty$, the indices $n_k = n + km$ go to infinity along a subsequence $n_k \equiv \nu \bmod m$, $0 \le \nu \le m-1$. By equation (6), $\limsup_{k\to\infty} p_{n_k}/n_k \le g < h$. Since there are m such subsequences ($\nu = 0, \dots, m-1$), we have $\limsup_{r\to\infty} p_r/r \le g < h$, which contradicts the assumption. □

Definition 3.2. For a positive sequence, a_n, let $h = \limsup_{n\to\infty} n^{-1} \log a_n$. If $h > 0$, we say that the sequence a_n grows exponentially, at the *rate* h. If $h = 0$, then a_n *grows subexponentially*.

LEMMA 3.3. *Let a_n, b_n, ℓ_n, d_n $(n \ge 1)$ be positive sequences such that: (i) $a_{m+n} \le a_m a_n$; (ii) for all m, n*

$$a_{m+n} \le b_m a_n + d_m \ell_n. \tag{7}$$

Let $\alpha, \beta, \lambda, \delta$ (≥ 0) be the respective growth rates. Then

$$\alpha \le \max\{\beta, \min(\delta, \lambda)\}. \tag{8}$$

Proof. Suppose $\alpha > \beta$. Set $\phi_n = \log(b_n + 1)$. Then $\beta = \limsup_{n\to\infty}(\phi_n/n)$ and there is $k \ge 1$ such that $\phi_k/k = g < \alpha$. We fix k for the rest of the proof. By Lemma 3.1 there are infinitely many n such that $\log a_{n+k} - \log a_n \ge \phi_k$, i.e. $a_{n+k}/a_n \ge b_k + 1$. By equation (7), $a_n \le a_n(a_{n+k}/a_n - b_k) \le d_k \ell_n$ for infinitely many n. Since, by (i), $\alpha = \inf_n \log a_n/n$, we have $a_n \ge e^{n\alpha}$ for all n, and therefore $\alpha \le \lambda$. This proves the inequality $\alpha \le \max(\beta, \lambda)$. Fix n and divide equation (7) by a_n, obtaining

$$\frac{a_{n+m}}{a_n} \le b_m + \frac{\ell_n}{a_n} d_m.$$

As $m \to \infty$, the growth rate of the sequence on the left-hand side is α, while that of the sequence on the right-hand side is $\max(\beta, \delta)$. Thus $\alpha \le \max(\beta, \delta)$. □

We assume from now on that X is a metric space, and denote the distance by $d(\cdot, \cdot)$. Let $\ell(P)$ be a function on polytopes, $P \subset X$, such that $\ell(P) \ge C \operatorname{diam}(P)$ for connected P, where $C > 0$ is a constant. We say that $\ell(P)$ is a *length-type function*. In §3.3 and §4, $d(\cdot, \cdot)$ is induced by a *Finsler metric*, $|\ |$, and $\ell(P) = |\partial P|$ is the *edge length* of P. For any partition, $\mathcal{P} : X = \bigcup_{i\in I} P_i$, we set $\ell(\mathcal{P}) = \sum_{i\in I} \ell(P_i)$; $D(\mathcal{P}) = \min d(P, Q)$, the minimum over pairs of nonadjacent atoms of $\mathcal{P}$; $b(\mathcal{P}) = \max_{x\in X} |\mathcal{P}_x|$, where $\mathcal{P}_x \subset \mathcal{P}$ consists of atoms containing x.

PROPOSITION 3.4. *Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$, and let $\ell(P)$ be a length-type function. For $n > 0$ set $\ell_n = \ell(\mathcal{P}_n)$, $\bar{b}_n = b(\mathcal{Q}_n)$. Let λ and $\bar{\beta}$ be the exponential growth rates of the sequences ℓ_n and $\bar{b}_n$ respectively.*

If the atoms of $\mathcal{P}_n, n > 0$, are connected, then the growth rate of $|\mathcal{P}_n|$ is, at most, $\max(\bar{\beta}, \lambda)$.

Proof. By the material of §2, for $-m \le k \le n$, T^k yields a one-to-one correspondence between the atoms of $\mathcal{R}_{(-n,m)}$ and $\mathcal{R}_{(-n+k,m+k)}$. In particular, $|\mathcal{Q}_m| = a_m$, hence, for any $m, n \ge 0$,

$$a_{n+m} = |\mathcal{R}_{(-n,m)}| = |\mathcal{P}_n \vee \mathcal{Q}_m| \le a_n a_m. \tag{9}$$

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For any $m, n \geq 0$ we divide the atoms of $\mathcal{P}_n$ into two groups: the 'good' and the 'bad' ones. A good atom, $A \in \mathcal{P}_n$, intersects only the adjacent atoms of $\mathcal{Q}_m$. By equation (9) and preceding remarks, A produces at most $\bar{b}_m$ atoms of $\mathcal{P}_{n+m}$. Hence, the contribution to a_{n+m} by the good atoms of $\mathcal{P}_n$ is, at most, $a_n \bar{b}_m$. The diameter of a bad atom, B, is, at least, $D(\mathcal{Q}_m) = \bar{D}_m$. Since B is connected, $\ell(B) \geq C \operatorname{diam}(B) \geq C\bar{D}_m$. Hence, the number of bad atoms in $\mathcal{P}_n$ is bounded above by $\ell_n / C\bar{D}_m$. A bad atom may intersect all atoms in $\mathcal{Q}_m$. Hence the contribution of bad atoms to $|\mathcal{P}_{n+m}|$ is, at most, $a_m \ell_n / C\bar{D}_m$. Thus

$$a_{n+m} \leq \bar{b}_m a_n + \frac{a_m}{C\bar{D}_m} \ell_n. \tag{10}$$

The claim follows by Lemma 3.3 (with $d_m = a_m/(C\bar{D}_m)$). □

The theorem below is immediate from Proposition 3.4 and §2.

THEOREM 3.5. *Let $T : X \to X$ be a polytope exchange on a partition $\mathcal{R}$, and let $\ell(\cdot)$ be a length-type function. Let λ and $\bar{\beta}$ be the growth rates of the sequences $\ell_n = \ell(\mathcal{P}_n)$ and $\bar{b}_n = b(\mathcal{Q}_n)$ respectively.*

If the atoms of the partitions $\mathcal{P}_n$ are connected, then $h(T, \mathcal{R}) \leq \max(\bar{\beta}, \lambda)$. If, in addition, the partition $\mathcal{R}$ is generating, then $h(T) \leq \max(\bar{\beta}, \lambda)$.

We expect that, in general, the topological entropy of a polytope exchange is smaller than the upper bound of Theorem 3.5. For differentiable maps Yomdin [**41**] gave an example where the topological entropy is strictly smaller than the growth rate of the lengths of curves.

3.2. *Entropy of generalized polygon exchanges.* In this subsection we assume that $\dim X = 2$, and study the *generalized polygon exchanges* (GPEs). We will strengthen the bound on the entropy established in Theorem 3.5. The following is the main result.

THEOREM 3.6. *Let $T : X \to X$ be a GPE on a partition $\mathcal{R}$, and let the notation be as in Proposition 3.4. Assume that the atoms of the partitions $\mathcal{P}_n$ are homeomorphic to the disc. Let $\ell(\cdot)$ be any length-type function, and let λ be the growth rate of the sequence $\ell(\mathcal{P}_n)$. Then $h(T, \mathcal{R}) \leq \lambda$.*

We view $\mathcal{P}_n = \mathcal{R}_{(-n,0)}$, $n \geq 1$, as graphs on X, with faces, edges and vertices. We make a standing assumption that every vertex is the terminus of at least three edges (no 'false vertices'). We denote by a_n, e_n and v_n the numbers of faces, edges and vertices of the graph $\mathcal{P}_n$. The respective growth rates, as n goes to infinity, are α, ε and ν, and let β be the growth rate of the sequence $b_n = b(\mathcal{P}_n)$ (*vertex complexity* of $\mathcal{P}_n$). In a similar way, we interpret $\mathcal{Q}_n = \mathcal{R}_{(0,n)}$, $n \geq 1$, as graphs, and use notation $\bar{a}_n$, $\bar{e}_n$, $\bar{v}_n$, $\bar{b}_n$, and $\bar{\alpha}, \bar{\varepsilon}, \bar{\nu}, \bar{\beta}$ for the corresponding quantities.

The centerpiece in the proof of Theorem 3.6 is Proposition 3.8, which gives efficient bounds on the growth rate of the vertex complexity of the sequence of graphs generated by a GPE. We will need the following elementary lemma.

LEMMA 3.7. *Let $b_k, k \geq 1$, be a positive sequence, such that for m, n sufficiently large*

$$b_{m+n} \leq e^{\varepsilon m} + e^{\mu n}, \tag{11}$$

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with $\varepsilon, \mu > 0$. *Then the exponential growth rate,* β, *of the sequence* b_k *satisfies* $\beta \leq \varepsilon\mu/(\varepsilon+\mu)$.

Proof. The function $f(x, y) = e^{\varepsilon x} + e^{\mu y}$ assumes its unique minimum on any line, $x + y = r$, at

$$(x_r, y_r) = \left(\frac{\mu r}{\varepsilon+\mu} + \frac{\log\mu - \log\varepsilon}{\varepsilon+\mu}, \frac{\varepsilon r}{\varepsilon+\mu} + \frac{\log\varepsilon - \log\mu}{\varepsilon+\mu}\right),$$

and

$$\min_{x+y=r} f(x, y) = f(x_r, y_r) = [(\mu/\varepsilon)^{\varepsilon/(\varepsilon+\mu)} + (\varepsilon/\mu)^{\mu/(\varepsilon+\mu)}]e^{\varepsilon\mu r/(\varepsilon+\mu)}.$$

Let r be a positive integer, let m_r be an integer such that $|m_r - x_r| \leq 1/2$, and set $n_r = r - m_r$. Then $|y_r - n_r| \leq 1/2$, and $m_r + n_r = r$. Substituting $m = m_r, n = n_r$ into equation (11), we obtain

$$b_r \leq \max(e^{\varepsilon/2}, e^{\mu/2})[(\mu/\varepsilon)^{\varepsilon/(\varepsilon+\mu)} + (\varepsilon/\mu)^{\varepsilon/(\varepsilon+\mu)}]e^{\varepsilon\mu r/(\varepsilon+\mu)},$$

which implies the claim. □

PROPOSITION 3.8. *Let* $T : X \to X$ *be a GPE on a partition* $\mathcal{R}$, *and let* $\mathcal{P}_n = \mathcal{R}_{(-n,0)}$, $n \geq 1$, *be the corresponding sequence of graphs on* X. *Let* a_n, e_n, v_n, *and* b_n *be as above, and let* ε *and* β *be the growth rates of the sequences* e_n *and* b_n. *Set* $\mu = \log M$, *where* $M = \min\{a_1, b_2\}$. *Then*

$$\beta \leq \frac{\epsilon\mu}{\epsilon+\mu}.$$

Proof. If $\mathcal{P}$ is any graph on X, and $x \in X$, we denote by $b(x) \geq 0$ the number of edges emanating from x. Thus $b(x) = 0$ (2), if x is an interior point of a face (edge) of $\mathcal{P}$, and $b(x) \geq 3$ if x is a vertex of $\mathcal{P}$. Note that $b(x) \neq 1$. Set $b(\mathcal{P}) = \max_{x\in\mathcal{P}} b(x)$. We denote by $b_n(x)$ the number $b(x)$ with respect to $\mathcal{P}_n$. Using the recursion $\mathcal{P}_{n+1} = \mathcal{R} \vee T^{-1}\mathcal{P}_n$, $\mathcal{P}_1 = \mathcal{R}$, we divide the vertices, x, of $\mathcal{P}_{n+1}$ into two categories:

(i) 'new vertices': x is not a vertex of $T^{-1}\mathcal{P}_n$. Thus $x = T^{-1}x'$, where $b_n(x') \leq 2$, and $b_{n+1}(x) = b_1(x) + b_n(x') \leq b_1 + 2$;

(ii) 'old vertices': x is a vertex of $T^{-1}\mathcal{P}_n$. Then $x = T^{-1}x'$, where x' is a vertex of $\mathcal{P}_n$. These vertices x' in $\mathcal{P}_n$ are the 'parents' of x. Since T^{-1} is a homeomorphism on the atoms of $\mathcal{R} = \mathcal{P}_1$, the maximal number of parents, an old vertex can have is, at most, a_1. On the other hand, $\mathcal{P}_2$ determines how T^{-1} puts the atoms together. Thus the number of parents is at most $\min\{a_1, b_2\} = M$.

Let $m, n \geq 1$. Let x be a vertex of $\mathcal{P}_{n+m}$. If x is an old vertex, we trace its ancestors in the 'family tree' all the way back to the generation m.

(I) For $0 < k < n$, we restrict our attention to the ancestors of x that were new vertices at the generation $m + k$. The number of those ancestors is, at most, M^{n-k}. Since an ancestor, say x', is a new vertex, it has by (i) at most $b_1 + 2$ edges. Between the generations $m+k$ and $m+n$, x' can acquire, through mergings with the vertices of $\mathcal{R}$ (or by intersection with edges of $\mathcal{R}$), at most $(n-k)b_1$ more edges (not considering mergings with other vertices). Thus the total number of edges of x these ancestors

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supply is at most

$$I = \sum_{k=1}^{n-1} (b_1 + 2 + (n-k)b_1) M^{n-k} < \frac{(n+1)b_1 + 2}{M-1} M^n.$$

(II) The ancestors of x that already existed at the generation m had at most e_m edges altogether. Each ancestor acquires between the generations m and $m+n$ at most nb_1 edges. Since there were, at most, M^n of these ancestors, their total contribution to the edges of x is, at most,

$$II = e_m + nb_1 M^n.$$

We finally obtain the estimate

$$b_{m+n} \leq I + II \leq e_m + \frac{(2n+1)b_1 + 2}{M-1} e^{n\mu}. \tag{12}$$

By equation (12), for any $\epsilon_1 > \epsilon \geq 0$, $\mu_1 > \mu$, there exists $t = t(\epsilon_1, \mu_1)$ such that for $m, n > t$, we have $b_{m+n} < e^{\epsilon_1 m} + e^{\mu_1 n}$. By Lemma 3.7, $\beta \leq \epsilon_1 \mu_1 / (\epsilon_1 + \mu_1)$. □

LEMMA 3.9. *Let $\mathcal{P}_n, n \geq 1$, be a sequence of graphs on a compact surface X, and let a_n, e_n, v_n be the numbers of faces, edges and vertices of $\mathcal{P}_n$. Let α, ε, ν be the corresponding growth rates. Assume that the faces of $\mathcal{P}_n$ are topological discs. Then $\nu \leq \alpha$, $\varepsilon = \alpha$.*

Proof. Associating to every vertex of $\mathcal{P}_n$ the oriented edges 'emanating from' it, we obtain $v_n \leq 2e_n/3$. Let $\chi = \chi(X)$ be the Euler characteristic. The identity $a_n + v_n = e_n + \chi$ and the inequality above imply $v_n \leq 2a_n - 2\chi$, $e_n \leq 3a_n - 3\chi$. Thus $\alpha, \nu \leq \epsilon$ and $\epsilon, \nu \leq \alpha$. □

Proof of Theorem 3.6. We use the preceding notation and terminology. Since T^n maps the atoms of $\mathcal{P}_n$ homeomorphically onto the atoms of $\mathcal{Q}_n$, $n \geq 1$, we conclude that the atoms of $\mathcal{Q}_n$ are topological discs and, moreover, $\bar{\alpha} = \alpha$. By Proposition 3.8 and Lemma 3.9, $\bar{\beta} < \alpha$ if $\alpha > 0$ and $\bar{\beta} = 0$ if $\alpha = 0$. Since $h(T, \mathcal{R}) = \alpha$, Theorem 3.5 implies the claim. □

Our techniques yield useful relations between various growth rates associated with the sequence of graphs generated by a GPE. In particular, we get an efficient bound on the asymptotic vertex complexity of these graphs. We formulate these relations below.

COROLLARY 3.10. *Let $T : X \to X$ be a GPE on a partition $\mathcal{R}$, satisfying the assumptions of Theorem 3.6, and let the notation be as above. Let $\ell(P)$ be a length-type function on X, and let $\lambda, \bar{\lambda}$ be the growth rates of $\ell(\mathcal{P}_n), \ell(\mathcal{Q}_n)$. Set $h = h(T, \mathcal{R})$. Then the following assertions hold:*

(i) *if $h = 0$, then $\epsilon = \nu = \beta = 0$;*

(ii) *if $h > 0$, then $\epsilon = \bar{\epsilon} = h$, $\beta \leq h/(1 + h/\mu)$, $h^2/(h+\mu) \leq \nu, \bar{\nu} \leq h$, and*

$$\max(\beta, \bar{\beta}) < h \leq \min(\lambda, \bar{\lambda}). \tag{13}$$

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Proof. The case $h = 0$ is obvious, hence we assume $h > 0$. For any $\beta' > \beta$, and large enough n, every vertex in the graph $\mathcal{P}_n$ has at most $e^{n\beta'}$ edges. For any $0 < h' < h$, by Lemma 3.9, $e_n \geq e^{nh'}$ for large enough n. Therefore the number of vertices in $\mathcal{P}_n$ satisfies $v_n \geq 2e_n/b_n \geq 2e^{n(h'-\beta')}$. Thus $\nu \geq h - \beta$. Substituting the upper bound on β from Proposition 3.8, we obtain the first line of relations in (ii). Since $\epsilon = \bar{\epsilon} = h$, Proposition 3.8 implies that $\beta, \bar{\beta} < h$. As $h = \alpha = \bar{\alpha}$, we have $h \leq \lambda, \bar{\lambda}$, by Theorem 3.6. □

3.3. *Growth rate of lengths of curves, Lyapunov exponents and entropy.* Let $T : X \to X$ be a GPE on a partition $\mathcal{R}$, satisfying the assumptions of Theorem 3.6. If $\mathcal{R}$ does not generate, then, in general, $h(T, \mathcal{R}) < h(T)$, and Theorem 3.6 does not yield an estimate on the entropy of T. Theorem 3.11 below rectifies the situation. It is a direct analog of Theorem 1.1 in the present context.

Let X be a compact Finsler surface (cf. [**37**]), and let $T : X \to X$ be a GPE on a partition $\mathcal{R}$. For a (piecewise) C^1 curve, γ, in X (i.e. γ consists of a finite number of compact C^1 curves), we denote by $|\gamma|$ its length, and set $\gamma_k = \gamma \setminus (\gamma \cap \partial\mathcal{P}_k)$. Since T^k is differentiable on the interiors of the atoms of $\mathcal{P}_k$, the curve $T^k(\gamma_k)$ is piecewise C^1, and we define the *growth rate of the lengths of curves* by

$$\lambda(T) = \sup_{\gamma} \limsup_{n\to\infty} \frac{1}{n} \log\left(\sum_{k=0}^{n} |T^k(\gamma_k)| + \ell_k\right),$$

where the supremum is over all curves γ in X as above, and $\ell_k = |\partial\mathcal{P}_k|$. If T is a diffeomorphism, the definition above agrees with equation (1) in §1.

If $\mathcal{R}$ is a defining partition of a GPE, $T : X \to X$, we say that $\mathcal{R}^{(n)}$, $n \geq 1$, is a *refining sequence of partitions* if $\mathcal{R} < \mathcal{R}^{(1)} < \mathcal{R}^{(2)} < \cdots$ and $\operatorname{diam} \mathcal{R}_\infty^{(n)} \searrow 0$ as $n \to \infty$. Here we use the notation $\mathcal{R}_m^{(n)}$ for the mth join corresponding to $\mathcal{R}^{(n)}$, and set $\mathcal{R}_\infty^{(n)} = \vee_m \mathcal{R}_m^{(n)}$.

THEOREM 3.11. *Let $T : X \to X$ be a GPE on $\mathcal{R}$, and let $\lambda(T) \geq 0$ be the growth rate of curve lengths. Suppose there exists a refining sequence of partitions, $\mathcal{R}^{(n)}$, $n \geq 1$, such that the atoms of $\mathcal{R}_m^{(n)}$ are topological discs. Then $h(T) \leq \lambda(T)$.*

Proof. Setting $\ell(P) = |\partial P|$, we define a length-type function on X. Let λ_n be the growth rate of $\ell(\mathcal{P}_m^{(n)})$, as $m \to \infty$. Then, by Theorem 3.6, $h(T, \mathcal{R}^{(n)}) \leq \lambda_n$ for all n. Since $\lambda_n \leq \lambda(T)$, we have $h(T, \mathcal{R}^{(n)}) \leq \lambda(T)$. By Proposition 2.8, $h(T, \mathcal{R}^{(n)}) \to h(T)$, as $n \to \infty$. □

Since the maps T^k are smooth on the interiors of the atoms in $\mathcal{P}_n$, $0 \leq k \leq n$, the differentials DT_x^k, $k \geq 1$, are defined for every $x \in X_{\text{reg}}$. We define on X_{reg} the function $L_k(x) = \log(\|DT_x^{-k}\| + \|DT_x^k\|)$, where $\|\cdot\|$ denotes the matrix norm. We define the *Lyapunov exponent* by

$$L(x) = \limsup_{n\to\infty} L_n(x)/n = \limsup_{n\to\infty} \log(\|DT_x^{-n}\| + \|DT_x^n\|)^{1/n}.$$

COROLLARY 3.12. *Let $T : X \to X$ be a GPE on $\mathcal{R}$, and let $L = \sup_{x\in X_{\text{reg}}} L(x)$. Suppose the following conditions are satisfied:*

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(i) *there exists a refining sequence of partitions, $\mathcal{R}^{(n)}$, $n \geq 1$, such that the atoms of $\mathcal{R}^{(n)}_m$ are topological discs;*
(ii) *for any $L' > L$ there is an integer $N = N(L')$ such that $L_n(x)/n \leq L'$ for $n \geq N$ and all $x \in X_{\text{reg}}$.*

Then $h_\mu(T) \leq h(T) \leq L$, for any invariant measure μ on X.

Proof. By assumption (ii), for any $L' > L$ there is a constant $C = C(L')$ such that $\|DT_x^{-k}\| + \|DT_x^k\| \leq Ce^{kL'}$ for all k and all regular points x. Let $\mathcal{P}_n, n > 1$, be the sequence of joins generated by T from $\mathcal{R}$. By continuity, for any $P \in \mathcal{P}_n$, and all points $x \in P$, we have $\|DT_x^{-k}\| \leq Ce^{kL'}$, for $0 \leq k \leq n-1$. We set $\partial\mathcal{P}_n = B_1 \cup B_2 \cup \cdots \cup B_n$, where $B_1 = \partial\mathcal{R} = \partial\mathcal{P}_1$, $B_2 = \partial\mathcal{P}_2 \setminus B_1$, and $B_k = \partial\mathcal{P}_k \setminus \bigcup_{j=1}^{k-1} B_j$, for $k = 2, \ldots, n$. Then $B_k \subset T^{-k}(\partial\mathcal{R})$, therefore $|B_k| \leq Ce^{kL'}|\partial\mathcal{R}|$, which gives the estimate $|\partial\mathcal{P}_n| \leq C'e^{nL'}$, for some constant C'. Hence the growth rate of the sequence $\ell_n = |\partial\mathcal{P}_n|$ is bounded above by L. For a regular curve γ in X set, as above, $\gamma_k = \gamma \setminus (\gamma \cap \partial\mathcal{P}_k)$, $k \geq 1$. Then $|T^k(\gamma_k)| \leq C_1 e^{kL'}|\gamma|$, for some constant C_1, therefore $\limsup_{n\to\infty}(\log \sum_{k=0}^{n-1} |T^k(\gamma_k)|)/n \leq L'$. Thus $\lambda(T) \leq L$, and Theorem 3.11 and Proposition 2.8 imply the claim. □

4. *Applications and examples*

We apply the preceding material to a variety of geometrical transformations with singularities.

4.1. *Piecewise isometries and related transformations.*

Although the notion of a piecewise isometry is quite general, we restrict our exposition to the special case of Finsler surfaces.

Definition 4.1.
(i) Let X be a compact Finsler surface (possibly, with a boundary). A GPE, $T : X \to X$, on a partition $\mathcal{R}$, is a *piecewise isometry* if the restriction on every atom, $T|_P$, $P \in \mathcal{R}$, is an isometry.
(ii) Let X, Y be Finsler surfaces. A diffeomorphism, $\phi : X \to Y$, is *(Finsler) affine* if it sends geodesics into geodesics.
(iii) Let X be a compact Finsler surface, and let $T : X \to X$ be a GPE on a partition $\mathcal{R} = \bigcup_{i\in I} P_i$. If P_i are Finsler polygons, and $T|_{P_i} : P_i \to X$ are Finsler affine, we say that T is a *(Finsler) affine GPE*.

The boundary of a *Finsler polygon*, $P \subset X$, consists of *Finsler geodesic* segments.

THEOREM 4.2. *Let X be a Finsler surface, and let a GPE, $T : X \to X$, be a piecewise isometry on a partition $\mathcal{R}$, whose atoms are Finsler polygons. Then $h(T) = 0$.*

Proof. Refining $\mathcal{R}$, if necessary, we can assume that the atoms of $\mathcal{R}$ are *geodesically convex*. Recall that $P \subset X$ is geodesically convex if for any $x, y \in P$ there is a unique shortest geodesic, $\gamma_{(x,y)} \subset P$, joining them. The geodesic convexity is preserved by isometries, and by taking finite intersections. Thus, the atoms of the joins $\mathcal{P}_n, n \geq 0$, are geodesically convex, hence homeomorphic to the disc. Since T is a piecewise isometry, the length of curves grows linearly. Thus $h(T) = 0$, by Theorem 3.11. □

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Theorem 4.2 applies to any piecewise isometry of a Riemann surface whose singular set is a union of geodesic segments. If X is a *flat surface*, whose singular points are contained in the set of vertices of $\mathcal{R}$, then T is a Euclidean polygon exchange. By Theorem 4.2, these transformations have entropy zero. When X is the unit square (cf. Example 2 in §2.1), and the atoms of $\mathcal{R}$ are rectangles (necessarily with vertical and horizontal sides), then T is a rectangle exchange.

THEOREM 4.3. *Let X be a compact Finsler surface (possibly with a boundary), such that for any $x, y \in X$ there is a unique geodesic joining them. Let $T : X \to X$ be a Finsler affine GPE, and set $\lambda = \max_x \|DT_x\|$. Then $h(T) \leq \log \lambda$.*

Proof. By the uniqueness of geodesic segments, T preserves geodesic convexity. There is a refining sequence of partitions, $\mathcal{R} < \mathcal{R}^{(1)} < \mathcal{R}^{(2)} < \cdots$, such that the atoms of $\mathcal{R}^{(n)}$ are geodesically convex, and $\operatorname{diam} \mathcal{R}^{(n)} \searrow 0$ as $n \to \infty$. Since the atoms of the joins $\mathcal{R}_m^{(n)}$ are convex, they are topological discs. Thus T satisfies the assumptions of Corollary 3.12, with $L \leq \log \lambda$. □

Let, for example, X be the unit square, and let the atoms of $\mathcal{R}$ be rectangles. We can speak of affine rectangle exchanges, $T : X \to X$. These are easier to 'construct' than the isometric rectangle exchanges. For instance, the classical 'baker transformation' is an affine rectangle exchange on two rectangles. Theorem 4.3 yields the estimate $h(T) \leq \log 2$ for the baker transformation.

4.2. *Polygonal billiards.* The general billiard dynamics is an extension of the concept of the geodesic flow. We refer the reader to [**7, 22, 23**] for more information, and consider the case of a polygon in $\mathbf{R}^2$, or, more generally, a *polyhedral surface* [**11**]. A polyhedral surface, S, is tiled by a finite number of polygons, P_i, which determine the billiard on S. Thus, the billiard in a polygon is a crucial special case. To shorten the exposition, we will state our results for the billiards on general polyhedral surfaces, and prove them for polygons, leaving the general case to the reader, as an exercise. We will use the name 'polygonal billiards' for our dynamical systems.

The polygonal *billiard flow*, T^t, lives on the unit tangent bundle of S. The edges of S define a natural cross-section, X, and the first return map, $T : X \to X$, retains most of the information. We say that T is the *polygonal billiard map*, and X is its *phase space*. In the standard (arclength, angle) coordinates on X we have $T(s, \theta) = (s_1, \theta_1)$, where $0 \leq s \leq 1$ (normalized 'perimeter' of S), and $0 \leq \theta \leq \pi$. The Liouville volume element yields the standard invariant measure ω on X which is defined by $d\omega = \sin\theta \, ds \, d\theta$.

The edges of S form the boundary, $\partial X = \{(s, 0), (s, \pi)\}, 0 \leq s \leq 1$. To describe its singularities, it is useful to think of elements of X as light rays in S. The light ray corresponding to $x = (s, \theta)$ emanates from s in direction θ. A billiard trajectory is a sequence of light rays in S, transforming by the laws of geometric optics. Light rays ending (starting) at vertices of S are singular for T (T^{-1}). Let e, A be an edge and a vertex of S respectively. We denote by $L(e, A)$ and $L(A, e)$ the set of light rays emanating from e and reaching A and emanating from A and reaching e, respectively. The union of real analytic curves $L(e, A)$, $L(A, e) \subset X$ (some may be empty) is the singular set.

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THEOREM 4.4. *Let S be a compact polyhedral surface, and let $T : X \to X$ be the billiard map. Then there is a (generalized) polygonal partition, $\mathcal{R}$, whose boundary is the union of curves $L(e, A)$, $L(A, e)$. The billiard map T is a GPE on $\mathcal{R}$, and the entropy of T is zero.*

Proof. The time-reversal involution, $\sigma : X \to X$, reverses the direction of light rays, and satisfies $\sigma T \sigma = T^{-1}$. By definition, $\sigma(L(e, A)) = L(A, e)$. Without loss of generality we can assume that S is a polygon, and, for simplicity of exposition, that S is convex. For any pair, $e \neq f$, of edges of S, let $P(e, f) \subset X$ consist of light rays that emanate from e and hit f. If e, f are disjoint, then $P(e, f)$ is a (generalized) quadrilateral. Its boundary curves are $L(e, C), L(e, D), L(A, f), L(B, f) \subset \text{int}(X)$, where A, B and C, D are the endpoints of e and f respectively. If e and f have a common vertex, A, then $P(e, f)$ degenerates into a trilateral, with one 'horizontal side' ($\theta = 0$ or $\theta = \pi$), and one 'vertical side' ($s = s(A)$). Note that $\sigma P(e, f) = P(f, e)$.

Let $e_i, 1 \leq i \leq p$, be the edges of S. Then the regions $P_{ij} = P(e_i, e_j), i \neq j$, form a (generalized) polygonal partition, $\mathcal{R}$, of X, and T is a GPE on $\mathcal{R}$. Note that $\sigma(\mathcal{R}) = \mathcal{R}$, hence σ conjugates the GPEs T and T^{-1}.

By 'inserting false vertices' on the sides of S, we can treat S as a q-gon, with q arbitrarily large. Let $\mathcal{R}(q)$ be the corresponding partition of X. Then $|\mathcal{R}(q)| = q(q-1)$, and, as $q \to \infty$, the size of atoms of $\mathcal{R}(q)$ goes to zero. We will use the Finsler metric, $\| \ \| = \sin\theta |ds| + |d\theta|$. Let $l(s, s_1)$ be the Euclidean distance between $s, s_1 \in S$. The general formula for the differential of the billiard map [**16**] yields

$$DT = \begin{pmatrix} -\dfrac{\sin\theta}{\sin\theta_1} & \dfrac{h(s, s_1)}{\sin\theta_1} \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Let $v \in V_{(s,\theta)}$ be a tangent vector to X at (s, θ). Then $DT(v) \in V_{(s_1,\theta_1)}$, and

$$\|DT(v)\| \leq \|v\| + h(s, s_1)|d\theta(v)| \leq \|v\| + \text{diam}(S)|d\theta(v)|.$$

Iterating T, and using the fact that, by equation (14), $|d\theta(DT(v))| = |d\theta(v)|$, we obtain

$$\|DT^n(v)\| \leq \|v\| + |n|\,\text{diam}(S)|d\theta(v)| \leq C(1 + |n|)\|v\|,$$

where $C = 2\max(1, \text{diam}(S))$. Thus, for any C^1 curve, $\gamma \subset X$, we have $|T^n(\gamma)| \leq C|n|^2|\gamma|$, hence $\lambda(T) = 0$. In the coordinates $(s, t = -\cot\theta)$ on X, the curves $L(e, A)$ are linear segments, and T is piecewise projective [**20**]. Refining $\mathcal{R}$, if necessary, we can assume that the atoms are convex polygons in the (s, t)-plane. Thus the atoms of the joins $\mathcal{P}_n$ are convex, hence homeomorphic to the disc. Since $\mathcal{R}$ has a refining sequence of partitions, $\mathcal{R} < \mathcal{R}^{(1)} < \mathcal{R}^{(2)} < \cdots$, with geodesically convex atoms, and $\text{diam}\,\mathcal{R}^{(n)} \searrow 0$, Theorem 3.11 implies the claim. □

The canonical partition $\mathcal{R}$ of $X(S)$, for a polyhedral surface S, has an obvious geometric meaning and so do the graphs $\mathcal{P}_n, \mathcal{Q}_n$ on X defined by iterating T, T^{-1}. By the *complexity, $c(\Gamma)$, of a graph* (see §3.3), we mean the total number of faces, edges, and vertices of Γ, i.e. $c(\Gamma) = a(\Gamma) + e(\Gamma) + v(\Gamma)$. The results of §3 and Theorem 4.4 imply the assertions below.

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COROLLARY 4.5. *Let $\mathcal{P}_n, \mathcal{Q}_n$ be the sequences of graphs defined by iterating the billiard map on a polyhedral surface S. Then the complexities $c(\mathcal{P}_n), c(\mathcal{Q}_n)$ grow subexponentially.*

COROLLARY 4.6. *Let S be a polyhedral surface, and let T^t be the billiard flow of S. For any measure μ, invariant under T^t, we have $h_\mu(T^t) = 0$.*

4.3. *Billiards in rational polyhedra.* We study the billiard dynamics in polyhedra, $P \subset \mathbf{R}^3$. Let $L_i \subset \mathbf{R}^3$ be the hyperplanes containing the faces, $F_i, 1 \le i \le p$, of P, and let $s_i \in O(3)$ be the linear part of the Euclidean reflection about L_i. The group $G = G_P \subset O(3)$, generated by $s_i, 1 \le i \le p$, is an important characteristic of P.

Definition 4.7. A polyhedron P is *rational* if the group G_P is finite.

The analog of the definition above for a polygon, $P \subset \mathbf{R}^2$, means that the angles of P are commensurable with π [**24**]. In $\mathbf{R}^3$ the rationality condition is more restrictive, but there are interesting classes of rational polyhedra, e.g. the polyhedra whose faces are parallel to the coordinate planes.

Let P be a rational polyhedron. The finite group G naturally acts on the unit sphere, $S^2 \subset \mathbf{R}^3$, and let $\Omega \subset S^2$ be a fundamental (spherical) polygon. The phase space $V = P \times S^2$ of the billiard flow (the unit tangent bundle of P) decomposes as a disjoint union of 'invariant submanifolds', $V_\omega = \{(q, v) \in V : v \in G\omega \subset S^2\}$, $\omega \in \Omega$. Thus $V = \bigcup_{\omega\in\Omega} V_\omega$, where the V_ω are invariant under the billiard flow T^t.

Definition 4.8. The restriction $T^t|_{V_\omega} = T^t_\omega : V_\omega \to V_\omega$ is the *billiard flow in direction ω.*

The topology of V_ω depends only on the type of $\omega \in \Omega$. In particular, for $\omega \in \mathrm{int}(\Omega)$, the invariant sets V_ω are isomorphic to an orbifold $Q = Q(P)$, partitioned by $|G|$ copies of P. The flat structure on Q yields the family $T^t_\omega : Q \to Q, \omega \in S^2$, of the directional geodesic flows. We have $T^t_{g\omega} = T^t_\omega$, and the isomorphism $V_\omega = Q, \omega \in \mathrm{int}(\Omega)$, identifies the flows T^t_ω on Q and $T^t|_{V_\omega}$ (hence the identical notation). The 2-skeleton, $S \subset Q$, is a polyhedral surface tiled by $p|G|/2$ polygons congruent to the faces of P, where p is the number of faces of P. The surface S is transversal to the directional flows $T^t_\omega, \omega \in \mathrm{int}(\Omega)$, and the first return maps, $T_\omega : S \to S$, are the *directional billiard maps*.

The material above is the three-dimensional version of a well-known construction of the directional flows for rational polygonal billiards [**12, 13**].

THEOREM 4.9. *Let $P \subset \mathbf{R}^3$ be a rational polyhedron, let $T^t_\omega, \omega \in \mathbf{S}^2$, be the directional billiard flows, and let $T_\omega : S \to S$ be the directional billiard maps. Then T_ω are affine GPEs, and $h(T_\omega) = 0$ for all ω.*

Proof. We will use the three-dimensional version of a well-known construction for rational polygons [**11**]. This construction allows us to represent Q by a polyhedron, $\tilde{Q}$. The polyhedron $\tilde{Q}$ is not unique and, in general, it is not embedded into $\mathbf{R}^3$. Instead, $\tilde{Q}$ lies in a branched covering of $\mathbf{R}^3$. For simplicity of exposition we assume that $\tilde{Q} \subset \mathbf{R}^3$. The boundary faces of $\tilde{Q}$ occur in parallel pairs $\tilde{F}_i, \tilde{F}'_i$ $(i = 1, \ldots, q)$. For each pair $\tilde{F}_i, \tilde{F}'_i$ of parallel faces of $\tilde{Q}$ let $t_i \in \mathbf{R}^3$ be the vector such that $\tilde{F}_i = \tilde{F}'_i + t_i$. Identifying $\tilde{F}_i$ and $\tilde{F}'_i$ as above, we obtain Q.

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In this representation, the directional flow T_ω^t on Q becomes the constant flow $\tilde{T}_\omega^t$ on $\tilde{Q}$, with velocity ω. Let $\tilde{F}_i, \tilde{F}_i'$ be a pair of parallel boundary faces of $\tilde{Q}$, and assume that the vector ω is directed outward on $\tilde{F}_i'$. Then upon reaching $\tilde{F}_i'$, the flowpoint is transferred to $\tilde{F}_i$ by the parallel translation t_i.

Set $\tilde{S} \subset \tilde{Q}$ to be the union of the faces $\tilde{F}_i, 1 \leq i \leq q$, and let $\tilde{T}_\omega : \tilde{S} \to \tilde{S}$ be the first return map. Note that the cross-section $\tilde{S}$ depends on the choice of the polyhedron $\tilde{Q}$, and that, in general, $\tilde{S}$ is a proper subset of S. In fact, $\tilde{S}$ is formed by a subset of the set of faces (i.e. polygons) tiling S, and $\tilde{T}_\omega$ is the induced map. It suffices to show that $h(\tilde{T}_\omega) = 0$.

It is elementary to see that $\tilde{T}_\omega$ is an affine polygon exchange. The affine transformations, A_{ij}, forming $\tilde{T}_\omega$ correspond to the q^2 pairs $(\tilde{F}_i, \tilde{F}_j), 1 \leq i, j \leq q$. The linear part of A_{ij} is the projection of the plane, L_i, containing $\tilde{F}_i$ onto the plane L_j, parallel to ω. The polygon, $P_{ij} = \tilde{F}_i \cap \tilde{T}_\omega^{-1}(\tilde{F}_j)$, supporting A_{ij} consists of points in $\tilde{F}_i$ whose flow lines return to $\tilde{S}$ at $\tilde{F}_j$.

Let p_ω be the projection onto L_ω, the plane orthogonal to ω, and let $R_\omega = p_\omega(\tilde{Q})$. Then R_ω is a cross-section for $\tilde{T}_\omega^t$, and let τ_ω be the induced map. Since ω is transversal to the faces $\tilde{F}_i$, the orthogonal projection of $\tilde{S}$ on L_i, along ω, conjugates $\tilde{T}_\omega$ with τ_ω, where τ_ω is an affine polygon exchange, thus $h(\tilde{T}_\omega) = h(\tau_\omega)$.

In fact, the affine transformations forming τ_ω are parallel translations by the vectors $v_i = v_i(\omega)$, the orthogonal projections of the vectors t_i on L_ω. Thus τ_ω is a piecewise Euclidean isometry, and $h(\tau_\omega) = 0$, by Theorem 4.2. □

COROLLARY 4.10. *Let P be a rational polyhedron, and let μ be a measure on $V = V(P)$, invariant under the billiard flow T^t. Then $h_\mu(T^t) = 0$. In particular, the entropy of the billiard flow with respect to the Lebesgue measure is zero.*

Proof. By Proposition 2.8, the metric entropy of the directional billiard maps T_ω, with respect to any invariant measure, vanishes. Therefore, $h_\nu(T_\omega^t) = 0$ for all ω, where ν is any flow-invariant measure on V_ω. Decomposing the Lebesgue measure μ into invariant measures, μ_ω on V_ω, we obtain the claim. □

Concluding remarks. The family, $\tau_\omega, \omega \in \Omega$, of piecewise isometries, introduced in the proof of Theorem 4.9, is the analog, in two dimensions, of the one-parameter family of interval exchange maps associated with a rational polygon [**11, 13, 24, 38**]. The isomorphism of τ_ω and the directional billiard map T_ω implies, by Theorem 4.2 and the results of §3, that the complexity of graphs, generated by iterating T_ω, grows subexponentially (compare with Corollary 4.5). Moreover, we can show that the growth rate is, at most, polynomial. The analogous questions of polynomial growth, for billiards in general polyhedra, and even polygons, remain open [**13**].

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