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# NOTES AND PROBLEMS

# NONTESTABILITY OF EQUAL WEIGHTS SPATIAL DEPENDENCE

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We show that any invariant test for spatial autocorrelation in a spatial error or spatial lag model with equal weights matrix has power equal to size. This result holds under the assumption of an elliptical distribution. Under Gaussianity, we also show that any test whose power is larger than its size for at least one point in the parameter space must be biased.

## 1. INTRODUCTION

Equal weights matrices have recently attracted considerable attention in the spatial econometric literature; see, for instance, Kelejian and Prucha (2002), Baltagi (2006), Kelejian, Prucha, and Yuzefovich (2006), Baltagi and Liu (2009, 2010), and Smith (2009). Such matrices may arise naturally in applications where, perhaps because no relevant distance between the observational units is observed, all pairs of units are assumed to have the same degree of correlation. Examples are mentioned in the above references.

Regression models with equicorrelation are known to raise some peculiar inferential problems. In particular, in a spatial context, Baltagi and Liu (2009) have pointed out that the Lagrange multiplier (LM) test for spatial lag equal weights dependence is degenerate, in the sense that the associated test statistic is constant on the sample space. Smith (2009) has shown that the Cliff-Ord test for spatial equicorrelation is affected by the same degeneracy. Given such results, it is natural to wonder whether it is at all possible to construct useful tests for equal weights spatial dependence. This paper shows that any invariant test of equal weights spatial dependence must have power equal to size. Under Gaussianity, the same unfavorable property applies to any unbiased test. The main implication of these results is that, in both spatial error and spatial lag models with equal weights, hypotheses concerning the autocorrelation coefficient are fundamentally ill-posed.

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## 2. THE MODELS

Consider the following two specifications of a spatial regression model: the spatial error model (SEM),

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}, \quad \boldsymbol{u} = \rho \boldsymbol{W}\boldsymbol{u} + \boldsymbol{\varepsilon}, \tag{1}$$

and the spatial lag model (SLM),

$$\boldsymbol{y} = \rho \boldsymbol{W}\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}. \tag{2}$$

For both specifications, $\boldsymbol{y}$ is the $n \times 1$ vector of observations on the dependent variable, $\rho$ is a scalar autocorrelation parameter, and $\boldsymbol{W}$ is an $n \times n$ spatial weights matrix. The analysis is conditional upon the $n \times k$ matrix $\boldsymbol{X}$ of regressors, which is assumed to have full rank and, unless otherwise specified, to contain an intercept. We assume that $\rho$ lies in the largest connected interval around the origin where $\boldsymbol{I}_n - \rho \boldsymbol{W}$ is invertible. We also assume that the error vector $\boldsymbol{\varepsilon}$ has a spherically symmetric distribution with $\text{var}(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}_n$, and that its density exists. Spherical symmetry of $\boldsymbol{\varepsilon}$ implies elliptical symmetry of $\boldsymbol{y}$. The class of elliptically symmetric distributions contains, for instance, scale mixtures of normal distributions and multivariate $t$-distributions.

In this paper we are concerned with the case when $\boldsymbol{W}$ is an *equal weights* matrix, i.e., a matrix $\overline{\boldsymbol{W}}$ with entries $(\overline{\boldsymbol{W}})_{ij}$ equal to 0 if $i = j$, and equal to some fixed scalar $w$, which may depend on $n$, if $i \neq j$. Because SEM and SLM are unchanged if $\boldsymbol{W}$ is multiplied by a constant and $\rho$ is suitably rescaled, we set $w = 1$. The largest connected interval of values of $\rho$ around the origin where $\boldsymbol{I}_n - \rho \overline{\boldsymbol{W}}$ is invertible is then $(-1, 1/(n-1))$.

A remark on notation: Throughout, $\boldsymbol{M}_{\boldsymbol{X}}$ denotes the matrix $\boldsymbol{I}_n - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$, and $\boldsymbol{\iota}$ is the $n \times 1$ vector of all ones.

## 3. MAIN RESULT

Let $\{p_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \Theta\}$ be a family of distributions with parameter space $\Theta$ and sample space $S$. A possibly randomized test for the null $\boldsymbol{\theta} \in \Theta_0 \subset \Theta$ against the alternative $\boldsymbol{\theta} \in \Theta \backslash \Theta_0$ is fully characterized by a critical function $\phi(\cdot) : S \to [0, 1]$ that associates to each value of $\boldsymbol{y}$ in $S$ a probability of rejecting the null. The power function of a test is the function $\pi(\boldsymbol{\theta}) := \text{E}_{\boldsymbol{\theta}}(\phi(\boldsymbol{y}))$.

Consider testing $\text{H}_0 : \rho = 0$ against $\text{H}_1 : \rho \neq 0$ in the context of a SEM or SLM with equal weights. It is worth pointing out that several tests that are commonly employed for $\rho = 0$ when $\boldsymbol{W} \neq \overline{\boldsymbol{W}}$ are not defined when $\boldsymbol{W} = \overline{\boldsymbol{W}}$, because their test statistics become degenerate in the latter case (in the sense that they do not depend on $\boldsymbol{y}$). This is so not only for the LM test for spatial lag dependence (as observed by Baltagi and Liu, 2009), but also for several other tests, including the Cliff-Ord test for residual correlation, Bera and Yoon's modified LM test for residual correlation, and the likelihood ratio (LR) test.[1]

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For the SEM, the problem of testing $H_0$ against $H_1$ is clearly invariant under the transformations $\boldsymbol{y} \to \gamma\,\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\delta}$, where $\gamma > 0$ and $\boldsymbol{\delta} \in \mathbb{R}^k$, for any $\boldsymbol{W}$. When $\boldsymbol{W} = \overline{\boldsymbol{W}}$, the same invariance also holds for the SLM.[2] We refer to the tests that are invariant under the transformations $\boldsymbol{y} \to \gamma\,\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\delta}$ as *invariant tests*; see, e.g., Lehmann and Romano (2005).

THEOREM 1. *Consider testing $\rho = 0$ in a SEM or SLM with equal weights. The power function of any invariant test and, under Gaussianity, of any unbiased test is equal to the size of the test everywhere.*

The proof is in the Appendix. Theorem 1 consists of two parts. The first part asserts that it is impossible to construct useful invariant tests for the testing problem under analysis. The relationship between invariance requirements and power of tests has been previously studied, for example, in Marsh (2007) and in Martellosio (2010). Even though in the context of Theorem 1 it is entirely natural to restrict attention to the class of invariant tests, powerful tests may exist outside that class. The second part of the theorem implies that, at least under Gaussianity, if a test has power larger than its size in some region of the parameter space then it must be biased. It should be mentioned that the power of an invariant or an unbiased test would be greater than the size if an intercept were not included in the regression, a case that, however, is not very common in practice. Some further remarks follow.

**Remark 1.** Concerning the first part of Theorem 1, it can be easily seen that the problem is caused by the transformations $\boldsymbol{y} \to \gamma\,\boldsymbol{y} + \eta\boldsymbol{\iota}$, $\gamma > 0$, $\eta \in \mathbb{R}$. The class of tests invariant under such transformations is larger than the class of invariant tests as defined above. The power function of the tests in this larger class in general depends on one or more regression slope coefficients, but remains independent of $\rho$.[3]

**Remark 2.** An equivalent formulation of the first part of Theorem 1 is that, in a SEM or SLM with equal weights and homoskedastic "innovations" $\boldsymbol{\varepsilon}$, the parameter $\rho$ cannot be identified by a function of the data that is invariant under the transformations $\boldsymbol{y} \to \gamma\,\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\delta}$. For more on the problem affecting the models considered here, see Kariya (1980b), who employs a generalization of the usual notion of identifiability of a model.

**Remark 3.** Theorem 1 can be seen as a generalization of results available in Arnold (1979) concerning tests about the autocorrelation coefficient in a Gaussian linear model with exchangeable errors.

**Remark 4.** A test whose critical function $\phi(\cdot)$ is equal to the size almost everywhere is usually referred to as a *trivial test*. Not all tests in Theorem 1 are trivial in that sense. As an example, consider a misspecified Cliff-Ord test for $\rho = 0$, i.e., a test based on the statistic $\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{W}\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}$ for some $\boldsymbol{W} \neq \overline{\boldsymbol{W}}$ (with $\boldsymbol{W}$ such that $\boldsymbol{M}_{\boldsymbol{X}}\left(\boldsymbol{W} + \boldsymbol{W}'\right)$ is not a scalar multiple of $\boldsymbol{M}_{\boldsymbol{X}}$, to ensure that the

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test statistic is nondegenerate). Since it is invariant, such a test must have power equal to size everywhere. Yet it is not trivial in the above sense, because, being nonrandomized, its critical function can only take the values 0 or 1.

**Remark 5.** The result concerning invariant tests holds for any spherical symmetric distribution of $\varepsilon$, and hence is semiparametric in nature. On the other hand, the result concerning unbiased tests requires a fully parametric (Gaussian) setting.

## 4. A GENERALIZATION

An interesting question is whether other weights matrices exist such that the problems described above occur. We now discuss a generalization of Theorem 1 in this direction. For the sake of brevity, we confine attention to the SEM and to invariant tests.

Let $\mathcal{W}$ be the class of matrices $\boldsymbol{W}$ satisfying the following four conditions: (i) $\boldsymbol{W}$ has at least one (real) positive eigenvalue, and the largest of the positive eigenvalues of $\boldsymbol{W}$, to be denoted by $\lambda_{\max}$, has geometric multiplicity one; (ii) any eigenvector of $\boldsymbol{W}$ associated to $\lambda_{\max}$, to be denoted by $\boldsymbol{f}_{\max}$, is also an eigenvector of $\boldsymbol{W}'$; (iii) the smallest eigenvalue of $\boldsymbol{W}+\boldsymbol{W}'$ has multiplicity $n-1$; (iv) the smallest eigenvalue of $\boldsymbol{W}'\boldsymbol{W}$ has multiplicity at least equal to $n-1$. Clearly, due to the severity of these conditions, $\mathcal{W}$ is a small class of matrices. It is easily seen that $\overline{\boldsymbol{W}} \in \mathcal{W}$. Other members of $\mathcal{W}$ are, e.g., the adjacency matrices of any doubly regular tournament (see Gross and Yellen, 2004).

Theorem 1 admits the following generalization, which does not require the assumption that an intercept is included in the regression: In a SEM with $\boldsymbol{W} \in \mathcal{W}$ and with $\boldsymbol{f}_{\max}$ in the column space of $\boldsymbol{X}$, the power function of any invariant test of $\rho = 0$ is equal to the size of the test everywhere.[4] The most important case is when $\boldsymbol{f}_{\max}$ is a vector of identical entries, because then the condition that $\boldsymbol{f}_{\max}$ belongs to the column space of $\boldsymbol{X}$ is satisfied as long as $\boldsymbol{X}$ contains an intercept.

## 5. CONCLUSION

We have shown that an equal weights matrix causes an identification problem, which, under the assumptions made in this note, makes it impossible to sensibly test for spatial autocorrelation. More specifically, several test statistics are degenerate, and the tests that are based on nondegenerate statistics but conform to the criteria of invariance or unbiasedness have power equal to size. These results reinforce the recommendation, made in a different context by Kelejian et al. (2006), that equal weights matrices should not be used in cross-sectional autoregressive models.

### NOTES

1. The statistics associated to the first three tests depend on $\boldsymbol{y}$ only through $\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{W}\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}$ and/or $\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{W}\boldsymbol{y}/\boldsymbol{y}'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}$. When $\boldsymbol{W} = \overline{\boldsymbol{W}}$ such statistics are degenerate because $\boldsymbol{M}_{\boldsymbol{X}}$

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$\overline{\boldsymbol{W}} = \boldsymbol{M}_{\boldsymbol{X}}(\boldsymbol{\iota}'\boldsymbol{\iota} - \boldsymbol{I}_n) = -\boldsymbol{M}_{\boldsymbol{X}}$, and hence cannot define a test. A similar argument can be used to show that the LR test for $\rho = 0$ is degenerate.

2. To see this, assume that the columns of $\boldsymbol{X}$ are orthogonal, one of them being a vector of identical entries. This can be done without loss of generality because any test invariant under the transformations $\boldsymbol{y} \to \gamma\,\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\delta}$ depends on $\boldsymbol{X}$ only through the column space of $\boldsymbol{X}$. With such a choice of $\boldsymbol{X}$, $\mathrm{E}(\boldsymbol{y}) = (\boldsymbol{I}_n - \rho\boldsymbol{W})^{-1}\boldsymbol{X} = \boldsymbol{\Delta}_\rho\boldsymbol{X}$, where $\boldsymbol{\Delta}_\rho$ is a diagonal matrix containing the eigenvalues of $(\boldsymbol{I}_n - \rho\boldsymbol{W})^{-1}$. It follows that the distribution of $\gamma\,\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\delta}$ is the same as the distribution of $\boldsymbol{y}$, provided that the parameters $\beta, \sigma^2, \rho$ are replaced by $\gamma\,\boldsymbol{\Delta}_\rho\beta + \delta, \gamma^2\sigma^2, \rho$.

3. This can be shown by a straightforward extension of the proof of Theorem 1. Partition $\boldsymbol{X}$ as $(\boldsymbol{\iota}, \widetilde{\boldsymbol{X}})$ and, accordingly, $\boldsymbol{\beta}$ as $(\alpha, \widetilde{\boldsymbol{\beta}}')'$, and let $\boldsymbol{C}$ be an $(n-1)\times n$ matrix such that $\boldsymbol{C}\boldsymbol{C}' = \boldsymbol{I}_{n-1}$ and $\boldsymbol{C}'\boldsymbol{C} = \boldsymbol{M}_{\boldsymbol{\iota}}$. Then $\boldsymbol{C}\boldsymbol{y}$ has an elliptical distribution with $\mathrm{var}(\boldsymbol{C}\boldsymbol{y}) = h_\rho^2\sigma^2\boldsymbol{I}_{n-1}$, and with $\mathrm{E}(\boldsymbol{C}\boldsymbol{y}) = \boldsymbol{C}\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{\beta}}$ for the SEM and $\mathrm{E}(\boldsymbol{C}\boldsymbol{y}) = h_\rho\boldsymbol{C}\widetilde{\boldsymbol{X}}\widetilde{\boldsymbol{\beta}}$ for the SLM. It follows that the distribution of $\boldsymbol{v}$ does not depend on $\rho$.

4. If $\boldsymbol{W} \in \mathcal{W}$, then $\boldsymbol{W} + \boldsymbol{W}'$ has spectral projectors $\boldsymbol{F} := \boldsymbol{f}_{\max}\boldsymbol{f}'_{\max}/\boldsymbol{f}'_{\max}\boldsymbol{f}_{\max}$ and $\boldsymbol{I}_n - \boldsymbol{F}$, and $\boldsymbol{W}'\boldsymbol{W}$ either has the spectral projectors $\boldsymbol{F}$ and $\boldsymbol{I}_n - \boldsymbol{F}$ or is a scalar multiple of $\boldsymbol{I}_n$. It follows that the variance matrix $\sigma^2(\boldsymbol{I}_n - \rho(\boldsymbol{W} + \boldsymbol{W}') + \rho^2\boldsymbol{W}'\boldsymbol{W})^{-1}$ has spectral projectors $\boldsymbol{F}$ and $\boldsymbol{I}_n - \boldsymbol{F}$, and hence can be written as $\sigma^2(a(\rho)\boldsymbol{I}_n + b(\rho)\boldsymbol{F})$, for some functions $a(\rho)$ and $b(\rho)$. Thus the stated generalization can be proved by a straightforward extension of the first part of the proof of Theorem 1.

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# APPENDIX: Proof

**Proof of Theorem 1.** Consider first invariant tests. A maximal invariant under the transformations $\boldsymbol{y} \to \gamma\,\boldsymbol{y} + \boldsymbol{X}\delta$ is $\boldsymbol{v} := \boldsymbol{C}\boldsymbol{y}/\|\boldsymbol{C}\boldsymbol{y}\|$, where $\boldsymbol{C}$ is an $(n-k)\times n$ matrix such

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that $\boldsymbol{C}\boldsymbol{C}' = \boldsymbol{I}_{n-k}$ and $\boldsymbol{C}'\boldsymbol{C} = \boldsymbol{M}_{\boldsymbol{X}}$, and $\|\cdot\|$ denotes the euclidean norm. Let $\boldsymbol{\Sigma}(\rho) := \sigma^2 \left(\boldsymbol{I}_n - \rho\overline{\boldsymbol{W}}\right)^{-2}$, the variance matrix of the SEM and SLM with $\boldsymbol{W} = \overline{\boldsymbol{W}}$. In the case of the SEM, the density of $\boldsymbol{v}$ with respect to the normalized Haar measure on the sphere $\left\{\boldsymbol{s} \in \mathbb{R}^{n-k} : \|\boldsymbol{s}\| = 1\right\}$ is

$$f(\boldsymbol{v}; \rho) = \left|\boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{C}'\right|^{-1/2} \left[\boldsymbol{v}' \left(\boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{C}'\right)^{-1} \boldsymbol{v}\right]^{-(n-k)/2}, \tag{A.1}$$

(see Kariya, 1980a, eqn. (3.7)). Write

$$\mathrm{E}(\boldsymbol{C}\boldsymbol{y}) = \boldsymbol{C}\left(\boldsymbol{I}_n - \rho\overline{\boldsymbol{W}}\right)^{-1} \boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{C}\left(h_\rho \boldsymbol{I}_n + \rho l_\rho \boldsymbol{\iota}\boldsymbol{\iota}'\right) \boldsymbol{X}\boldsymbol{\beta},$$

where $h_\rho := 1/(1+\rho)$ and $l_\rho := h_\rho/(1+\rho-n\rho)$. Because the regression is assumed to contain an intercept, $\boldsymbol{C}\boldsymbol{\iota} = \boldsymbol{0}$ and hence $\mathrm{E}(\boldsymbol{C}\boldsymbol{y}) = \boldsymbol{0}$. Thus, expression (A.1) also holds for the SLM. Next write

$$\boldsymbol{\Sigma}(\rho) = \sigma^2(h_\rho^2 \boldsymbol{I}_n + b_\rho \boldsymbol{\iota}\boldsymbol{\iota}'),$$

with $b_\rho := \rho(2+2\rho-n\rho)l_\rho^2$, so that $\boldsymbol{C}\boldsymbol{\Sigma}(\rho)\boldsymbol{C}' = h_\rho^2\sigma^2\boldsymbol{I}_{n-k}$. Plugging this expression into (A.1) shows that $f(\boldsymbol{v}; \rho)$ does not depend on $\rho$. Since the critical function $\phi(\boldsymbol{y})$ of any invariant test must depend on $\boldsymbol{y}$ only through $\boldsymbol{v}$, it follows that $\mathrm{E}(\phi(\boldsymbol{y}))$ is independent of $\rho$, which in turn implies that $\mathrm{E}(\phi(\boldsymbol{y}))$ equals the size of $\phi(\boldsymbol{y})$. Let us now move to unbiased tests. The matrix $\overline{\boldsymbol{W}}$ has only two eigenspaces: the line spanned by $\boldsymbol{\iota}$, associated to the eigenvalue $n-1$, and its orthogonal complement, associated to the eigenvalue $-1$. Thus, letting $\boldsymbol{H}$ be an $n \times n$ orthonormal matrix whose first column has identical entries, $\boldsymbol{\Sigma}(\rho)$ can be decomposed as $\boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}'$, where $\boldsymbol{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\boldsymbol{\Sigma}(\rho)$, that is, $(\boldsymbol{\Lambda})_{1,1} = (1+\rho-n\rho)^{-2}$, and $(\boldsymbol{\Lambda})_{2,2} = \ldots = (\boldsymbol{\Lambda})_{n,n} = (1+\rho)^{-2}$. Consider now the vector $\boldsymbol{z} := \boldsymbol{H}'\boldsymbol{y}$. For the SEM, we have

$$\boldsymbol{z} \sim \mathrm{N}\left(\boldsymbol{H}'\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Lambda}(\rho)\right).$$

Partition $\boldsymbol{X}$ as $(\boldsymbol{\iota}, \widetilde{\boldsymbol{X}})$. Let $\boldsymbol{U}_1$ be the first row of $\boldsymbol{H}'\widetilde{\boldsymbol{X}}$ and $\boldsymbol{U}_2$ the remaining $(n-1) \times (k-1)$ matrix. Partitioning $\boldsymbol{\beta}$ accordingly as $\left(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2'\right)'$ where $\boldsymbol{\beta}_2$ is a $(k-1) \times 1$ vector, and $\boldsymbol{z}$ as $\left(z_1, \boldsymbol{z}_2'\right)'$, where $\boldsymbol{z}_2$ is $(n-1) \times 1$, we obtain

$$\boldsymbol{z}_2 \sim \mathrm{N}\left(\boldsymbol{U}_2\boldsymbol{\beta}_2, \frac{\sigma^2}{(1+\rho)^2}\boldsymbol{I}_{n-1}\right). \tag{A.2}$$

Similarly, for the SLM,

$$\boldsymbol{z} \sim \mathrm{N}\left(\boldsymbol{H}'(\boldsymbol{I}_n - \rho\overline{\boldsymbol{W}})^{-1}\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Lambda}(\rho)\right) = \mathrm{N}\left(\boldsymbol{\Lambda}^{-1/2}\boldsymbol{H}'\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Lambda}(\rho)\right),$$

where we have used the fact that $(\boldsymbol{I}_n - \rho\overline{\boldsymbol{W}})^{-1} = \boldsymbol{H}\boldsymbol{\Lambda}^{-1/2}\boldsymbol{H}'$, and hence

$$\boldsymbol{z}_2 \sim \mathrm{N}\left(\frac{1}{1+\rho}\boldsymbol{U}_2\boldsymbol{\beta}_2, \frac{\sigma^2}{(1+\rho)^2}\boldsymbol{I}_{n-1}\right). \tag{A.3}$$

By the factorization theorem, the statistics $\boldsymbol{U}_2'\boldsymbol{z}_2$ and $\boldsymbol{z}_2'\boldsymbol{z}_2$ are complete sufficient both in the case of (A.2) (for the parameters $\boldsymbol{\beta}_2$ and $(1+\rho)^{-2}\sigma^2$) and in the case of (A.3) (for the

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parameters $(1+\rho)^{-1}\boldsymbol{\beta}_2$ and $(1+\rho)^{-2}\sigma^2$). Thus, the statistics $z_1$, $\boldsymbol{U}_2'\boldsymbol{z}_2$, and $\boldsymbol{z}_2'\boldsymbol{z}_2$ are complete sufficient in both the SEM and the SLM. The proof can now be completed by using results in Arnold (1979). First, by Lemma 2 in Arnold (1979), the statistic $(\hat{\boldsymbol{\beta}}', s^2)'$, where $\hat{\boldsymbol{\beta}} := \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{y}$ and $s^2 := (n-k)^{-1}(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})'(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}})$, is a 1-to-1 function of $(z_1, z_2'\boldsymbol{U}_2, \boldsymbol{z}_2'z_2)'$, and hence is complete sufficient for both the SEM and the SLM. Next, consider a size-$\alpha$ unbiased test for $\rho = 0$ (in either the SEM or the SLM), and let $\phi$ be its critical function. If $\phi$ depends on $y$ only through $\hat{\boldsymbol{\beta}}$ and $s^2$, then $\phi = \alpha$ almost everywhere, by Theorem 4 of Arnold, and consequently $\pi(\boldsymbol{\theta}) = \alpha$, for all $\boldsymbol{\theta} := (\boldsymbol{\beta}, \sigma^2, \rho)'$ in the parameter space $\Theta$. If, on the other hand, $\phi$ does not depend on $\boldsymbol{y}$ only through $\hat{\boldsymbol{\beta}}$ and $s^2$, consider the critical function $\psi = \mathrm{E}(\phi \mid \hat{\boldsymbol{\beta}}, s^2)$. Note that $\psi$ has the same power and the same size as $\phi$, because $\mathrm{E}_{\boldsymbol{\theta}}(\psi) = \mathrm{E}_{\boldsymbol{\theta}}(\phi)$ for all $\boldsymbol{\theta} \in \Theta$. In addition, $\psi$ depends on $\boldsymbol{y}$ only through $\hat{\boldsymbol{\beta}}$ and $s^2$. Thus, again by Theorem 4 of Arnold, $\psi$ is a randomized test with $\psi = \alpha$ almost everywhere. It follows that $\phi$ is a (randomized or nonrandomized) test with power equal to $\alpha$ everywhere. ■

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