Math. Proc. Camb. Phil. Soc. (1992), **112**, 233
Printed in Great Britain

Locally free $\mathscr{D}(\mathbb{P}^n)$-modules

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(*Received* 30 *September* 1991; *revised* 3 *February* 1992)

Introduction

The purpose of this paper is to study the structure of locally free modules over the ring of differential operators on projective space. Let $\mathscr{X}$ be a non-singular, complex, algebraic variety. Denote by $\mathscr{D}_{\mathscr{X}}$ the sheaf of rings of differential operators over $\mathscr{X}$ and by $\mathscr{D}(\mathscr{X})$ its ring of global sections. A $\mathscr{D}(\mathscr{X})$-module M is called locally free if the associated sheaf $\mathscr{D}_{\mathscr{X}} \otimes M$ is locally free as a sheaf of $\mathscr{D}_{\mathscr{X}}$-modules. Locally free modules arise naturally in $\mathscr{D}$-module theory as inverse images of determined modules; see [1] for definitions and examples.

In the case of projective space, they can be constructed starting from vector bundles and using the famous localization theorem of Bernstein and Beilinson. This theorem establishes an equivalence between the category of finitely generated $\mathscr{D}(\mathbb{P}^n)$-modules and the category of coherent sheaves of $\mathscr{D}_{\mathbb{P}^n}$-modules. If $\mathscr{L}$ is a locally free sheaf over $\mathscr{O}_{\mathbb{P}^n}$, then $\mathscr{D}_{\mathbb{P}^n} \otimes \mathscr{L}$ is locally free as a sheaf of $\mathscr{D}_{\mathbb{P}^n}$-modules. The localization theorem now makes this sheaf correspond to a locally free $\mathscr{D}(\mathbb{P}^n)$-module. Not surprisingly, the localization theorem is one of the main tools used in this paper.

Most of the paper is concerned with comparing the structure of locally free sheaves of coherent $\mathscr{O}_{\mathbb{P}^n}$-modules with that of finitely generated, locally free $\mathscr{D}(\mathbb{P}^n)$-modules. Section 2 deals with the projective line. Here the two structures are very similar: a Grothendieck-type splitting theorem holds for (most) locally free $\mathscr{D}(\mathbb{P}^1)$-modules. When we turn to projective spaces of higher dimensions the structures of locally free $\mathscr{O}_{\mathbb{P}^n}$-modules and $\mathscr{D}(\mathbb{P}^n)$-modules are very different. In Section 3 we point out some of these differences. Finally Section 4 collects a few miscellaneous results together with some questions and conjectures suggested by this work.

1. *Preliminaries*

In this section we establish the notation and state the results about $\mathscr{D}(\mathbb{P}^n)$-modules that are used in the paper. For details the reader is referred to [**5**]. We shall also make ample use of standard facts about vector bundles over $\mathbb{P}^n$ in this paper. These are not listed here as they can be found in the excellent book [**12**], to which full references are given, whenever necessary. Let A be a commutative, regular $\mathbb{C}$-algebra. The ring of differential operators on A is the subring of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A itself and its $\mathbb{C}$-linear derivations. This ring is denoted by $\mathscr{D}(A)$. An important special case occurs when $A = \mathscr{O}(\mathscr{X})$, the coordinate ring of a non-singular, affine, algebraic variety over $\mathbb{C}$. In this case we write $\mathscr{D}(\mathscr{X})$ instead of $\mathscr{D}(\mathscr{O}(\mathscr{X}))$.

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Let $\mathbb{P}^n$ be projective n-space. The standard open sets of $\mathbb{P}^n$ are denoted by U_i, for $0 \leqslant i \leqslant n$. In homogeneous coordinates $U_i = \{[x_0 : \ldots : x_n] : x_i \neq 0\}$. The structure sheaf of $\mathbb{P}^n$ is denoted by $\mathcal{O}_{\mathbb{P}^n}$, or simply $\mathcal{O}$. In fact, we shall always drop subscripts if there is no danger of confusion. The sheaf of rings of differential operators over $\mathbb{P}^n$, denoted by $\mathscr{D}_{\mathbb{P}^n}$, is defined as follows. For V an open set of $\mathbb{P}^n$, let

$$\Gamma(V, \mathscr{D}_{\mathbb{P}^n}) = \bigcap_{x \in V} \mathscr{D}(\mathcal{O}_x),$$

where $\mathcal{O}_x$ is the stalk of $\mathcal{O}_{\mathbb{P}^n}$ at the point x. Notice that $\Gamma(U_i, \mathscr{D}_{\mathbb{P}^n}) = \mathscr{D}(U_i)$, for $0 \leqslant i \leqslant n$. The ring $\mathscr{D}(\mathbb{P}^n)$, of differential operators on $\mathbb{P}^n$, is the ring of global sections of $\mathscr{D}_{\mathbb{P}^n}$: $\mathscr{D}(\mathbb{P}^n) = \Gamma(\mathbb{P}^n, \mathscr{D}_{\mathbb{P}^n})$. Since $U_0 \cong \mathbb{A}^n(\mathbb{C})$, we can choose coordinates $t_1, \ldots, t_n$ for U_0. Then $\mathscr{D}(U_0)$ is generated by $\{t_j, \partial_j : 1 \leqslant j \leqslant n\}$, where ∂_j is shorthand for the operator $\partial/\partial t_j$. It can then be shown that $\mathscr{D}(U_i)$, for $i \neq 0$, identifies with the $\mathbb{C}$-algebra generated by $\{t_i^{-1} t_j, t_i^{-1}, t_i \partial_j, t_i \pi : 1 \leqslant j \leqslant n\}$, where $\pi = t_1 \partial_1 + \ldots + t_n \partial_n$. Of course, each $\mathscr{D}(U_i)$ is isomorphic to the nth Weyl algebra. Now $\mathscr{D}(\mathbb{P}^n)$ is a subring of $\mathscr{D}(U_0)$ and one can show that, as a $\mathbb{C}$-algebra, it is generated by $\{\partial_i, t_i \partial_j, t_i \pi : 1 \leqslant i, j \leqslant n\}$. We recall the following result from [**5**], §4:

PROPOSITION 1·1. *$\mathscr{D}(\mathbb{P}^n)$ is a right and left noetherian domain. It has only one non-trivial, proper two-sided ideal* $\mathbf{m}$, *and* $\mathscr{D}(\mathbb{P}^n)/\mathbf{m} \cong \mathbb{C}$. *Moreover,* $\mathbf{m}$ *is generated (as a right and left ideal) by* $\{\partial_i, t_i \partial_j, t_i \pi : 1 \leqslant i, j \leqslant n\}$.

A simple consequence of Proposition 1·1, via Goldie's Theorem, is that $\mathscr{D}(\mathbb{P}^n)$ has a quotient division ring, to be denoted by $Q(\mathbb{P}^n)$.

Now let $\mathbf{Mod}(\mathscr{D}(\mathbb{P}^n))$ be the category of finitely generated, left $\mathscr{D}(\mathbb{P}^n)$-modules and $\mathbf{Mod}(\mathscr{D}_{\mathbb{P}^n})$ the category of coherent sheaves of left $\mathscr{D}_{\mathbb{P}^n}$-modules. Then $\mathscr{D}_{\mathbb{P}^n} \otimes_{\mathcal{O}} -$ is a functor from the category $\mathbf{Mod}(\mathcal{O}_{\mathbb{P}^n})$, of coherent sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules, to $\mathbf{Mod}(\mathscr{D}_{\mathbb{P}^n})$. Composing it with the global sections functor $\Gamma(\mathbb{P}^n, -) : \mathbf{Mod}(\mathscr{D}_{\mathbb{P}^n}) \to \mathbf{Mod}(\mathscr{D}(\mathbb{P}^n))$, one obtains a functor from $\mathbf{Mod}(\mathcal{O}_{\mathbb{P}^n})$ to $\mathbf{Mod}(\mathscr{D}(\mathbb{P}^n))$, that is denoted by Ind. A $\mathscr{D}(\mathbb{P}^n)$-module M is *locally free* if the corresponding sheaf $\mathscr{D}_{\mathbb{P}^n} \otimes_{\mathscr{D}(\mathbb{P}^n)} M$ is locally free as a sheaf of $\mathscr{D}_{\mathbb{P}^n}$-modules. Locally free $\mathscr{D}(\mathbb{P}^n)$-modules can be constructed using the functor Ind. Let $\mathrm{Vect}(\mathbb{P}^n)$ be the category of locally free coherent $\mathcal{O}_{\mathbb{P}^n}$-modules. The notation is motivated by the fact that the category of vector bundles of finite rank over $\mathbb{P}^n$ is equivalent to $\mathrm{Vect}(\mathbb{P}^n)$. With this in mind we shall sometimes refer to the elements of $\mathrm{Vect}(\mathbb{P}^n)$ as vector bundles. Now if $\mathscr{F}$ is an object of $\mathrm{Vect}(\mathbb{P}^n)$ then $\mathrm{Ind}(\mathscr{F})$ is a locally free $\mathscr{D}(\mathbb{P}^n)$-module. This follows from the following theorem of Beilinson and Bernstein [**3**]:

THEOREM 1·2. *The mutually inverse functors* $\Gamma(\mathbb{P}^n, -) : \mathbf{Mod}(\mathscr{D}_{\mathbb{P}^n}) \to \mathbf{Mod}(\mathscr{D}(\mathbb{P}^n))$ *and* $\mathscr{D}_{\mathbb{P}^n} \otimes_{\mathscr{D}(\mathbb{P}^n)} - : \mathbf{Mod}(\mathscr{D}(\mathbb{P}^n)) \to \mathbf{Mod}(\mathscr{D}_{\mathbb{P}^n})$ *provide an equivalence of categories.*

An important class of examples is obtained by application of Ind to the rank one bundles. These bundles are usually denoted by $\mathcal{O}(k)$, where k is an integer. Set $D(k) = \mathrm{Ind}(\mathcal{O}(k))$. We recall from [**5**], proposition 3·4 the following result:

THEOREM 1·3. *The only locally free $\mathscr{D}(\mathbb{P}^n)$-modules of rank one (up to isomorphism) are the $D(k)$, for $k \in \mathbb{Z}$. Moreover, $D(k) \cong D(j)$ if and only if $k = j$.*

These modules are also important if one wants to calculate K_0 and G_0 of $\mathscr{D}(\mathbb{P}^n)$. If M is a finitely generated (respectively a finitely generated, projective) $\mathscr{D}(\mathbb{P}^n)$-module, then the class of M in $G_0(\mathscr{D}(\mathbb{P}^n))$ (respectively $K_0(\mathscr{D}(\mathbb{P}^n))$) is denoted by $[M]$. Hodges [**9**] has proved the following result:

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THEOREM 1·4. *The groups $K_0(\mathscr{D}(\mathbb{P}^n))$ and $G_0(\mathscr{D}(\mathbb{P}^n))$ are free with basis $\{[D(0)], [D(1)], \ldots, [D(n)]\}$.*

Still on the subject of locally free D-modules of rank one, one can determine which of these are projective and which are generators, as follows (see [**5**], 4·7 and 4·8):

PROPOSITION 1·5. *Let k be an integer. Then $D(k)$ is projective if and only if $k \leqslant n$ and $D(k)$ is a generator if and only if $k \leqslant 0$. The equation $\mathbf{m}D(k) = D(k)$ holds if and only if $k \geqslant 1$.*

$$D(k) = \sum_{\substack{i_1, \ldots, i_n \geqslant 0 \\ i_1 + \ldots + i_n \leqslant k}} \mathscr{D}(\mathbb{P}^n)\, t_1^{i_1} \ldots t_n^{i_n}, \quad \text{for } k \geqslant 0$$

and

$$D(-k) = \sum_{\substack{i_1, \ldots, i_n \geqslant 0 \\ i_1 + \ldots + i_n \leqslant k}} \mathscr{D}(\mathbb{P}^n)(\pi - 1) \ldots (\pi - k + i_1 + \ldots + i_n)\, \partial_1^{i_1} \ldots \partial_n^{i_n}, \quad \text{for } k \geqslant 0.$$

Let M be a finitely generated, left module over a noetherian ring R. If P is a prime ideal of R then the reduced rank of M at P is denoted by $\rho(M, P)$. We end the section with a useful, although rather technical, result. It is easily proved using the techniques of [**14**], §5.

PROPOSITION 1·6. *Let M be a finitely generated $\mathscr{D}(\mathbb{P}^n)$-module with $\rho(M, 0) \geqslant n + 1$. If $k \in \{1, \ldots, n\}$ then*

(*a*) *there exists a $\mathscr{D}(\mathbb{P}^n)$-module M' such that $M \cong D(k) \oplus M'$, and*

(*b*) *if $M \oplus D(k) \cong N \oplus D(k)$ then $M \cong N$.*

2. $\mathscr{D}(\mathbb{P}^1)$-Modules

In this section we study the structure of locally free $\mathscr{D}(\mathbb{P}^1)$-modules. To simplify the notation we shall often write simply D instead of $\mathscr{D}(\mathbb{P}^1)$. Our results are inspired by Grothendieck's splitting theorem for bundles over $\mathbb{P}^1$ (see [**7**]):

THEOREM 2·1. *Any locally free sheaf of finite rank over $\mathbb{P}^1$ splits uniquely in the form*

$$\mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \oplus \ldots \oplus \mathscr{O}(a_n)$$

where $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ are integers.

However, the analogous result for $\mathscr{D}(\mathbb{P}^1)$-modules is false. The reason is that the category of coherent $\mathscr{O}_{\mathbb{P}^1}$-modules does not have enough projectives, whilst the category of finitely generated $\mathscr{D}(\mathbb{P}^n)$-modules does. We explain this in more detail. Given an integer k consider the following twisted version of the Euler exact sequence ([**12**], p. 6) over $\mathbb{P}^1$:

$$0 \to \mathscr{O}(k-1) \to \mathscr{O}(k)^2 \to \mathscr{O}(k+1) \to 0. \tag{2·2}$$

These sequences are never split. Now apply the exact functor Ind to (2·2). One obtains the sequence

$$0 \to D(k-1) \to D(k)^2 \to D(k+1) \to 0. \tag{2·3}$$

However, this new sequence can be split. For example, when $k \leqslant 0$ the module $D(k+1)$ is projective. Hence, for $k \leqslant 0$, sequence (2·3) splits to give

$$D(k)^2 \cong D(k-1) \oplus D(k+1). \tag{2·4}$$

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Thus in generalizing Grothendieck's Theorem to $\mathscr{D}(\mathbb{P}^1)$-modules one cannot hope to have as strong a uniqueness in the representation of the direct sum as in the original theorem. After some experimentation, we arrived at the following splitting theorem for $\mathscr{D}(\mathbb{P}^1)$-modules.

THEOREM 2·5. *Let M be a finitely generated, locally free $\mathscr{D}(\mathbb{P}^1)$-module. Assume that one of the two following conditions is satisfied*:

(*P*) *M is projective*;

(*G*) *M is not a generator.*

Then there exist integers k, m, n with $m > 0$ and $n \geqslant 0$, such that

$$M \cong D(k)^m \oplus D(k-1)^n.$$

Moreover, the numbers k, m, n uniquely determine M up to isomorphism.

In the statement of the theorem we use the convention that $D(k)^0 = 0$. The proof of the theorem depends on whether (*P*) or (*G*) is assumed. However, the proof of the uniqueness is the same for both and will be done first.

2·6. *Proof of uniqueness in Theorem* 2·5. For $i = 1, 2$ write $M \cong D(k_i)^{m_i} \oplus D(k_i-1)^{n_i}$, where $(k_i, m_i, n_i) \in \mathbb{Z} \times (\mathbb{N}\backslash\{0\}) \times \mathbb{N}$. This gives a relation in $G_0(\mathscr{D}(\mathbb{P}^1))$. Recall from Theorem 1·4 that $\{[D], [D(1)]\}$ form a free basis for $G_0(\mathscr{D}(\mathbb{P}^1))$. This basis allows us to identify elements of $G_0(\mathscr{D}(\mathbb{P}^1))$ with pairs of integers. Using induction on sequence (2·3) the class of $D(k)$ in $G_0(\mathscr{D}(\mathbb{P}^1))$ can be written in the form $(1-k, k)$. Hence the class of M is $(m_i(1-k_i)+n_i(2-k_i), m_i k_i + n_i(k_i-1))$, for $i = 1, 2$. But these two pairs must be equal. Equating second coordinates and using that $m_1+n_1 = m_2+n_2$, one has

$$(m_1+n_1)(k_1-k_2)+(n_2-n_1) = 0.$$

Thus
$$|k_1-k_2| = |n_1-n_2|/(m_1+n_1) \leqslant 1.$$

If the above expression is equal to 1 then $|n_1-n_2| = m_1+n_1$. But this implies that $m_1 \leqslant 0$ or $m_2 \leqslant 0$, which is not allowed. Hence the inequality must be strict and so $k_1 = k_2$ and $n_1 = n_2$, as required.

We now turn to the proof of the existence of the decomposition proposed in Theorem 2·5. That locally free modules of rank one are of the prescribed form follows from Proposition 1·3. Hence it can be assumed, in proving the theorem, that the module has rank at least two. We deal first with the case of a projective module. A variation of Bass's Cancellation Theorem is needed.

LEMMA 2·7. *Let M, M' and N be finitely generated, projective $\mathscr{D}(\mathbb{P}^1)$-modules such that (a) $\rho(M, 0) \geqslant 2\rho(N, 0)$ and (b) $\rho(M, \mathbf{m}) \geqslant \rho(N, \mathbf{m})$. If $M \oplus N \cong M' \oplus N$ then $M \cong M'$.*

Proof. If $\rho(N, \mathbf{m}) = 0$, this is [**14**], corollary 5·11. If $\rho(N, \mathbf{m}) \geqslant 1$, then it follows from [**4**], corollary 8·4.

2·8 *Proof of Theorem* 2·5 *for projective modules.* Let M be a locally free, finitely generated, projective $\mathscr{D}(\mathbb{P}^1)$-module with $\rho(M, 0) \geqslant 2$. Suppose first that $\rho(M, \mathbf{m}) = 0$. Let $M' = D(1)^* \oplus_D M$. Then M' is a finitely generated, projective $\mathrm{End}_D D(1)$-module of the same rank as M. Since $\mathrm{End}_D D(1) \cong \mathscr{D}_{-1}(\mathbb{P}^1)$, by [**5**], proposition 3·8, and $K_0(\mathscr{D}_{-1}(\mathbb{P}^1)) \cong \mathbb{Z}$, by [**6**], theorem 4·6, then M' is stably free. But $\mathscr{D}_{-1}(\mathbb{P}^1)$ is a simple,

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noetherian domain of Krull dimension one. Hence all finitely generated, stably free, projective $\mathscr{D}_{-1}(\mathbb{P}^1)$-modules of rank at least two are free. Thus M' is a free $\mathscr{D}_{-1}(\mathbb{P}^1)$-module and one has

$$M \cong D(1) \otimes_{\mathscr{D}_{-1}(\mathbb{P}^1)} M' \cong D(1)^{\rho(M,0)},$$

which is of the required form.

We are left to deal with the case when $\rho(M, \mathbf{m}) \geqslant 1$. From Theorem 1·4, $K_0(D)$ is generated by $[D]$ and $[D(1)]$. Hence there are integers p and q such that the following relation holds in $K_0(D)$:

$$[M] = p[D] + q[D(1)]. \tag{2·9}$$

Moreover $$p+q = \rho(M,0) \geqslant 2, \quad p = \rho(M, \mathbf{m}) \geqslant 1.$$

If $q \geqslant 0$ there exists an integer $s > 0$ such that

$$M \oplus D^s \cong D^{s+p} \oplus D(1)^q.$$

By Lemma 2·7 it follows that $M \cong D^p \oplus D(1)^q$. Thus we can assume that $q < 0$. Consider the following inductive statement:

$H(k)$: there exist integers $0 < r_k < r_{k-1}$ such that $M \oplus D(1-k)^{r_k} \cong D(-k)^{r_{k-1}}$.

We show that $H(0)$ is true. From (2·9),

$$M \oplus D(1)^{-q} \oplus D^s \cong D^{s+p}$$

for some integer $s \geqslant 0$. Using Lemma 2·7, it is possible to cancel D^s to get $M \oplus D(1)^{-q} \cong D^p$. Since $p > -q > 0$, this proves $H(0)$.

Now we show that if $H(k)$ holds and $2r_k > r_{k-1}$ then $H(k+1)$ also holds. Adding $D(-k-1)^{r_k}$ to both sides of $H(k)$ and using (2·4) one obtains

$$M \oplus D(-k)^{2r_k} \cong D(-k)^{r_{k-1}} \oplus D(-k-1)^{r_k}. \tag{2·10}$$

If $2r_k > r_{k-1}$, Lemma 2·7 can be used and one has

$$M \oplus D(-k)^{2r_k - r_{k-1}} \cong D(-(k+1))^{r_k}.$$

Since $r_k < r_{k-1}$ then $0 < 2r_k - r_{k-1} < r_k$, which is the statement of $H(k+1)$. However, the r_k are all non-negative and $r_{-1} > r_0 > r_1 > \ldots$. Hence, for some k, $H(k)$ holds but $2r_k \leqslant r_{k-1}$. Once again (2·10) holds and Lemma 2·7 can be applied. But now $2r_k \leqslant r_{k-1}$. Thus

$$M \cong D(-k)^{r_{k-1} - 2r_k} \oplus D(-k-1)^{r_k},$$

which is the desired result.

When M satisfies condition (G) the proof of Theorem 2·5 utilises a reduction to the projective case. This is achieved with the help of the following technical lemmas.

LEMMA 2·11. *Let M be a finitely generated, torsion-free, left $\mathscr{D}(\mathbb{P}^1)$-module that is not a generator. Then $M = \mathbf{m}M^{**}$.*

Proof. From Proposition 1·6 there exists an integer $p \geqslant 0$ for which M splits as a direct sum of $D(1)^p$ and a left ideal of D. Thus one can assume that M itself is a left ideal of D. Now [**10**], lemma 1·5 shows that $\mathbf{m}M^{**} \subseteq M$. Together with $\mathbf{m}^2 = \mathbf{m}$, this implies that $\mathbf{m}M^{**} \subseteq \mathbf{m}M \subseteq \mathbf{m}M^{**}$. So $\mathbf{m}M = \mathbf{m}M^{**}$. From [**15**], theorem 2·6 we know that $\mathscr{D}(\mathbb{P}^1)$ has global dimension two. Hence, by [**2**], it follows that M^* is

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projective. Thus $M(\mathrm{End}\,(M^*)) = M(M^*M^{**})$. Since M is not a generator, we have $MM^* = \mathbf{m}$. So $MM^*M^{**} = \mathbf{m}M$. Finally, because $M \subseteq M(\mathrm{End}\,(M^*))$, we get $M \subseteq \mathbf{m}M$. The other inclusion is obvious, and the proof is complete.

LEMMA 2·12. *The right $\mathscr{D}(\mathbb{P}^1)$-module $D(k)^*$ is generated by $\partial^{k-1}t^{k-i}$, for $2 \leqslant i \leqslant k$.*

Proof. Put $p_i = \partial^{k-1}t^{k-i}$ and let V be the right D-module generated by p_i, for $2 \leqslant i \leqslant k$. We show first that the p_i belong to $D(k)^*$ for $2 \leqslant i \leqslant k$. Recall that $D(k)^* = \{q \in Q(\mathbb{P}^1) : D(k)\,q \subseteq D\}$. Since $D(k)$ is generated by 1 and t^k, this is equivalent to proving that p_i and $t^k p_i$ belong to D, for $2 \leqslant i \leqslant k$. But

$$p_i = \partial^{k-1}t^{k-i} = \partial^{i-1}(t\partial + k - i)\dots(t\partial + 1) \in D.$$

On the other hand,

$$t^k p_i = t(t\partial)\dots(t\partial - k + 2)\,t^{k-i} = (t^2\partial)^{k-i+1}(t\partial - 1)\dots(t\partial - i + 2) \in D,$$

whenever $2 \leqslant i \leqslant k$. The proof that $D(k)^*$ is contained in V is far more technical. In it we use the natural grading of $\Gamma(U_0 \cap U_1, \mathscr{D}_{\mathbb{P}^1}) = \mathbb{C}[t, t^{-1}, \partial]$: that means giving t degree 1 and ∂ degree -1. Clearly this grading descends to $\mathbb{C}[t\partial]$-modules generated by homogeneous elements. Let

$$E = \Gamma(\mathbb{P}^1, \mathscr{O}(-k) \otimes_{\mathscr{O}} \mathscr{D}_{\mathbb{P}^1}) = \Gamma(U_0, \mathscr{D}_{\mathbb{P}^1}) \cap t^{-k}\Gamma(U_1, \mathscr{D}_{\mathbb{P}^1}).$$

But $D(k)\,E \subseteq \mathscr{D}(\mathbb{P}^1)$, and so $E \subseteq D(k)^*$. On the other hand, if q is an element of the quotient ring of $\mathscr{D}(\mathbb{P}^1)$ satisfying $D(k)\,q \subseteq D$ then $q \in E$, by Theorem 1·2. Thus $E = D(k)^*$. But E is graded, say $E = \oplus\{E_i : -\infty < i < \infty\}$. To finish the proof it is enough to show that each homogeneous component E_i is in V. There are three cases to be considered.

Case 1. $i \geqslant 0$. In this case E_i has the form $t^i\mathbb{C}[t\partial] \cap t^{-k}(t^2\partial)^{i+k}\mathbb{C}[t\partial]$. Thus, as one easily checks, it is generated by $t^i(t\partial + i + k - 1)\dots t\partial$ as a right $\mathbb{C}[t\partial]$-module. Moreover, by induction, E_i is contained in $(t\partial + k - 1)\dots t\partial D$, for $i \geqslant 0$. But

$$(t\partial + k - 1)\dots t\partial = \partial\partial^{k-2}t^{k-2}t^2\partial \in V.$$

Hence $E_i \subseteq V$, as required.

Case 2. $-k+1 \leqslant i \leqslant -1$. Let $j = -i$. Thus $1 \leqslant j \leqslant k-1$ and

$$E_i = \partial^j\mathbb{C}[t\partial] \cap t^{-k}(t^2\partial)^{k-j}\mathbb{C}[t\partial].$$

Let $x \in E_i$. Then there exist $f, h \in \mathbb{C}[t\partial]$ such that $x = \partial^j f = t^{-k}(t^2\partial)^{k-j}h$. We have

$$t^j\partial^j f = (t\partial + k - j - 1)\dots t\partial h$$

which can be rephrased as follows:

$$(t\partial - j + 1)\dots t\partial f = (t\partial + k - j - 1)\dots t\partial h.$$

It follows that $x \in E_i$ if and only if $(t\partial + k - j - 1)\dots(t\partial + 1)$ divides f, as polynomials in $\mathbb{C}[t\partial]$. Hence E_i is the cyclic $\mathbb{C}[t\partial]$-module generated by $\partial^j(t\partial + k - j - 1)\dots(t\partial + 1)$. But this last element equals $\partial^{k-1}t^{k-j-1}$ and so belongs to V.

Case 3. $i \leqslant -k$. In this case $E_i = \partial^{-i}\mathbb{C}[t\partial] \cap t^i\mathbb{C}[t\partial]$. But $\partial^{k-1} \in V$, and thus $\partial^{-i} \in V$ for $-i \leqslant k$, and the proof is complete.

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LEMMA 2·13. *If $k \geqslant 1$, then $D(k)^{**} \cong D(-k+2)$.*

Proof. Using [5], theorem 4·7, the formula in Lemma 2·12 becomes

$$D(k)^* \cong \partial^{k-1} D(-k+2)^*.$$

Since $k \geqslant 1$, it follows from Proposition 1·5 that $D(-k+2)$ is projective, hence reflexive. Thus

$$D(k)^{**} \cong D(-k+2)^{**}\partial^{1-k} \cong D(-k+2),$$

as claimed.

2·14 *Proof of Theorem 2·5 when M is not a generator.* We may assume that M is not projective and has rank at least two. As observed in the proof of Lemma 2·11, the global dimension of $\mathscr{D}(\mathbb{P}^1)$ is two. Since M^{**} is reflexive, it follows that it is projective and, in fact, by virtue of Lemma 2·11, a progenerator. From Theorem 2·5, applied to the projective case, the module M^{**} can be written in the form $D(k)^n \oplus D(k-1)^m$, where k, m, n are integers and $k \leqslant 1$. By Lemmas 2·11 and 2·13, one has $\mathbf{m}D(k) \cong D(-k+2)$. Therefore

$$\mathbf{m}M^{**} \cong D(-k+2)^n \oplus D(-k+3)^m.$$

Setting $r = -k+3$ and using Lemma 2·11, again, we have

$$M \cong D(r-1)^n \oplus D(r)^m$$

as required.

One should observe that Theorem 2·5 cannot possibly hold for locally free D-modules which are generators but not projective. This follows from the fact that $D(k)^m \oplus D(k-1)^n$ cannot be a non-projective generator. We end with a conjecture.

Conjecture 2·15. Every finitely generated, locally free $\mathscr{D}(\mathbb{P}^1)$-module splits as a direct sum of locally free modules of rank one.

3. $\mathscr{D}(\mathbb{P}^n)$-modules for $n \geqslant 2$

In the previous section it was proved that every finitely generated, locally free, projective $\mathscr{D}(\mathbb{P}^1)$-module splits as a direct sum of locally free modules of rank one. Call a locally free $\mathscr{D}(\mathbb{P}^n)$-module extended if it is of the form Ind $(\mathscr{E})$, for some bundle $\mathscr{E}$ over $\mathbb{P}^n$. In particular, it follows from Theorem 2·5 that every locally free, projective $\mathscr{D}(\mathbb{P}^1)$-module is extended. In this section we show that both the splitting off result and the fact that modules are extended fail when $n \geqslant 2$. We briefly review some facts about Chern classes that are used below. For details see [8], appendix A. The Chow ring of $\mathbb{P}^n$ is the ring $A = A(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$. The grading of the polynomial ring $\mathbb{Z}[h]$ induces a grading on A. The total Chern class is a map c_t from $K_0(\mathbb{P}^n)$ to A. It satisfies the following condition:

(3·1) $c_t(\mathscr{O}(k)) = 1 + kh$;

(3·2) if $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$ is an exact sequence of locally free sheaves over $\mathbb{P}^n$ then $c_t(\mathscr{E}_2) = c_t(\mathscr{E}_1)\, c_t(\mathscr{E}_3)$.

In fact, (3·1) and (3·2) completely determine the map c_t. Let $\mathscr{E}$ be a bundle over $\mathbb{P}^n$ of rank r. Decomposing $c_t(\mathscr{E})$ into its homogeneous components one arrives at the equation

$$c_t(\mathscr{E}) = 1 + c_1(\mathscr{E})\, h + \ldots + c_r(\mathscr{E})\, h^r,$$

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where $c_i(\mathscr{E})$ are integers, for $1 \leqslant i \leqslant r$. The numbers $c_i(\mathscr{E})$ are called the *Chern classes* of $\mathscr{E}$. Notice that since $\mathscr{E}$ has rank r, the Chern classes $c_i(\mathscr{E})$ are zero for $r+1 \leqslant i \leqslant n$. Since $K_0(\mathscr{D}(\mathbb{P}^n)) \cong K_0(\mathbb{P}^n)$, by Theorem 1·4, the total Chern class can be extended to a map $K_0(\mathscr{D}(\mathbb{P}^n)) \to A(\mathbb{P}^n)$ satisfying the following conditions:

(3·3) $c_t(D(k)) = 1 + kh$;

(3·4) if $0 \to P_1 \to P_2 \to P_3 \to 0$ is an exact sequence of finitely generated, projective $\mathscr{D}(\mathbb{P}^n)$-modules then $c_t(P_2) = c_t(P_1)\, c_t(P_3)$.

Note also that if $\mathscr{E}$ is a locally free sheaf of finite rank over $\mathbb{P}^n$, then $c_t(\mathrm{Ind}\,(\mathscr{E})) = c_t(\mathscr{E})$. We use Chern classes to show that there are non-extended, locally free, projective $\mathscr{D}(\mathbb{P}^n)$-modules of finite rank. First, we prove a lemma.

LEMMA 3·5. *If $n \geqslant 2$, then $D(1)$ can be generated by the n elements $t_1, \ldots, t_{n-2}$, $t_{n-1} + \partial_n, t_n - \partial_{n-1}$.*

Proof. Let $D = \mathscr{D}(\mathbb{P}^n)$ and let V be the left D-module generated by the elements $t_1, \ldots, t_{n-2}, t_{n-1} + \partial_n, t_n - \partial_{n-1}$. Put $\pi = t_1 \partial_1 + \ldots + t_n \partial_n$ and $\nu = t_{n-1} \partial_{n-1} + t_n \partial_n$. If $R = \mathbb{C}[\pi, \nu]$, then

$$R(\nu+2) = R(\partial_{n-1}(t_{n-1} + \partial_n) + \partial_n(t_n - \partial_{n-1})) \subseteq V.$$

Also $$R\nu(\pi-1) = R(t_n\, \pi(t_{n-1} + \partial_n) - t_{n-1}\, \pi(t_n - \partial_{n-1})) \subseteq V.$$

Thus $$R(\pi-1) = [R(\nu+2) + R\nu](\pi-1) = R(\nu+2)(\pi-1) + R\nu(\pi-1) \subseteq V.$$

But $\pi - 1 - (\nu+2) = t_1 \partial_1 + \ldots + t_{n-2} \partial_{n-2} - 3$ belongs to V, and so does

$$\partial_1 t_1 + \ldots + \partial_{n-2} t_{n-2} = t_1 \partial_1 + \ldots + t_{n-2} \partial_{n-2} + (n-2).$$

Thus V contains 1, and so equals $D(1)$, as required.

THEOREM 3·6. *If $n \geqslant 3$ there exist finitely generated locally free projective $\mathscr{D}(\mathbb{P}^n)$-modules of rank $n-1$ which are not extended.*

Proof. Let $D = \mathscr{D}(\mathbb{P}^n)$. By Lemma 3·5, there exists a surjective map $\phi : D^n \to D(1)$. Let $K = \mathrm{Ker}\,(\phi)$. Since $D(1)$ is projective, one has $D^n \cong K \oplus D(1)$. Hence K is projective of rank $n-1 \geqslant 2$. Thus K is locally free by [**13**], theorem 3·6. Taking total Chern classes one has $c_t(K)(1+h) = 1$. Hence $c_t(K) = 1 - h + h^2 - \ldots + (-1)^n h^n$. Now if K is extended, then $K = \mathrm{Ind}\,(\mathscr{E})$, where $\mathscr{E}$ is a locally free sheaf of finite rank over $\mathbb{P}^n$. Since K has rank $n-1$, so does $\mathscr{E}$. Hence $c_n(\mathscr{E}) = 0$. But

$$c_n(\mathscr{E}) = c_n(\mathrm{Ind}\,(\mathscr{E})) = c_n(K) = (-1)^n,$$

a contradiction. Thus K cannot be extended.

Let $\mathscr{T}$ and Ω denote the tangent sheaf and the sheaf of differentials over $\mathbb{P}^n$, respectively. As usual, set $\mathscr{T}(k) = \mathscr{T} \otimes_{\mathscr{O}} \mathscr{O}(k)$ and $\Omega(k) = \Omega \otimes_{\mathscr{O}} \mathscr{O}(k)$. We are going to prove that neither $\mathrm{Ind}\,(\mathscr{T}(k))$ nor $\mathrm{Ind}\,(\Omega(K))$ can be split as direct sums of locally free $\mathscr{D}(\mathbb{P}^n)$-modules of rank one.

LEMMA 3·7. $c_n(\mathrm{Ind}\,(\Omega(k)) = 0$ *if and only if $k = 2$ and n is odd.*

Proof. Consider the sequence

$$0 \to \Omega \to \mathscr{O}(-1)^{n+1} \to \mathscr{O} \to 0;$$

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(see [12], p. 7). Taking Chern classes, one obtains

$$c_t(\Omega) = \sum_0^n \binom{n+1}{j}(-h)^j. \tag{3·8}$$

Using the formula for the Chern class of a tensor product by a line bundle (see [12], p. 16), we arrive at the following equation:

$$c_n(\Omega(k)) = \sum_0^n \binom{n+1}{j}(-1)^j k^{n-j} = k^{-1}((k-1)^{n+1}-(-1)^{n+1}) \quad \text{for } k \neq 0.$$

It is now clear that $c_n(\Omega(k)) = 0$ if and only if $k = 2$ and n is odd. The lemma now follows since $c_n(\Omega(k)) = c_n(\mathrm{Ind}\,(\Omega(k))$.

THEOREM 3·9. *Let k and n be integers, where $n \geqslant 2$. The modules* $\mathrm{Ind}\,(\Omega(k))$ *and* $\mathrm{Ind}\,(\mathscr{T}(k))$ *do not split as a direct sum of locally free $\mathscr{D}(\mathbb{P}^n)$-modules of rank one.*

In the proof of the theorem we use the fact that taking the dual of a bundle defines an automorphism of the abelian group $K_0(\mathbb{P}^n)$. Recall that if $\mathscr{E}$ is a locally free sheaf of finite rank over $\mathbb{P}^n$, the dual of $\mathscr{E}$ is the locally free sheaf $\mathscr{E}^*$ defined by $\mathscr{H}om\,(\mathscr{E}, \mathscr{O})$. We also use the fact that $K_0(\mathbb{P}^n)$ has a natural ring structure, the product being defined by $[\mathscr{E}_1][\mathscr{E}_2] = [\mathscr{E}_1 \otimes_{\mathscr{O}} \mathscr{E}_2]$ (see [8], p. 435).

Proof. Consider $\mathrm{Ind}\,(\Omega(k))$ first. Assume, for a contradiction, that

$$\mathrm{Ind}\,(\Omega(k)) \cong \oplus_{j=1}^n D(a_j).$$

By Theorem 1·4, this gives rise to the following formula in $K_0(\mathbb{P}^n)$:

$$[\Omega(k)] = \sum_1^n [\mathscr{O}(a_j)].$$

Multiplying both sides by $[\mathscr{O}(-k)]$, we can assume that $k = 0$. Now for each a_i with $1 \leqslant i \leqslant n$, we have

$$[\Omega(-a_i)] = \sum_{j \neq i} [\mathscr{O}(a_j - a_i)] + [\mathscr{O}].$$

Talking total Chern classes we obtain

$$c_t(\Omega(-a_i)) = \prod_{j \neq i} c_t(\mathscr{O}(a_j - a_i)).$$

Since the product on the right has degree $n-1$ in h we obtain that $c_n(\Omega(-a_i)) = 0$. Using Lemma 3·7, one concludes that $a_1 = \ldots = a_n = -2$ and that n is odd. Hence $[\Omega] = [\mathscr{O}(-2)]^n$ in $K_0(\mathbb{P}^n)$, from which $c_n(\Omega) = (-2)^n$. However, from formula (3·8), one has that $c_n(\Omega) = -(n+1)$, giving the desired contradiction.

Now we consider $\mathrm{Ind}\,(\mathscr{T}(k))$. If $\mathrm{Ind}\,(\mathscr{T}(k)) = \oplus \{D(b_j) : 1 \leqslant j \leqslant n\}$ then, in $K_0(\mathbb{P}^n)$, we have

$$[\mathscr{T}(k)] = \sum_1^n [\mathscr{O}(b_j)].$$

Tensoring up by $\mathscr{O}(-k)$ and then taking duals gives rise to the following formula:

$$[\Omega] = \sum_1^n [\mathscr{O}(k - b_j)].$$

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But, as has already been shown, this cannot hold in $K_0(\mathbb{P}^n)$, and the proof is complete.

This may not seem so surprising to the reader who knows that the bundles $\mathscr{T}(k)$ and $\Omega(k)$ are indecomposable. However, as the next proposition shows, there is more to the splitting of locally free D-modules than one may be led to assume from similar results about bundles.

PROPOSITION 3·10. *If $n > 1$ is odd then* Ind $(\mathscr{T})$ *is not indecomposable in the category of locally free $\mathscr{D}(\mathbb{P}^n)$-modules.*

Proof. By [12], p. 77, there exists an exact sequence of locally free $\mathscr{O}_{\mathbb{P}^n}$-modules of the form

$$0 \to \mathscr{N}(1) \to \mathscr{T} \to \mathscr{O}(2) \to 0$$

whenever n is odd. Since $\mathscr{T}$ is indecomposable, this sequence of bundles does not split. However, applying the functor Ind to the above sequence, one obtains

$$0 \to \text{Ind}\,(\mathscr{N}(1)) \to \text{Ind}\,(\mathscr{T}) \to D(2) \to 0.$$

But $D(2)$ is a projective left $\mathscr{D}(\mathbb{P}^n)$-module for $n \geqslant 2$. Hence this latter sequence splits. Thus Ind $(\mathscr{T})$ is not indecomposable in the category of locally free $\mathscr{D}(\mathbb{P}^n)$-modules.

4. *Odds and ends*

In this section a number of miscellaneous results and conjectures about locally free $\mathscr{D}(\mathbb{P}^n)$-modules are collected. These are of two kinds. First we consider the existence of projective $\mathscr{D}(\mathbb{P}^n)$-modules which are not locally free and show how these modules can be constructed. Next we prove that certain locally free modules admit the additional structure of a quadratic form. The hope is that such a structure may be useful in establishing new invariants which could be used for settling questions about the splitting of locally free D-modules. A number of unanswered questions raised by our research on $\mathscr{D}(\mathbb{P}^n)$-modules are collected at the end of the section.

Let P be a left ideal of $\mathscr{D}(\mathbb{P}^n)$. The singularity set of P is the set of all points of $\mathbb{P}^n$ at which the corresponding stalk of $\mathscr{D}_{\mathbb{P}^n} \otimes P$ is not cyclic. The singularity set is denoted by $\mathscr{S}(P)$.

PROPOSITION 4·1. *If P is a left ideal of $\mathscr{D}(\mathbb{P}^1)$ then $\mathscr{S}(P)$ is finite.*

Proof. Let $\mathscr{D}(\mathbb{P}^1) \subseteq \mathbb{C}[t, \partial]$ be the standard embedding of $\mathscr{D}(\mathbb{P}^1)$ in the first Weyl algebra. As is well known, $\mathbb{C}(t)[\partial]$ is a principal ideal domain. Hence there exist $a \in P$ and $f \in \mathbb{C}[t] \backslash 0$ such that $f^{-1}a$ generates $\mathbb{C}[t, f^{-1}][\partial]P$. Thus a point of $\mathscr{S}(P)$ is either a root of f or the point at infinity. Hence $\mathscr{S}(P)$ is finite.

Two remarks are in order. First, the singularity set of a left ideal of $\mathscr{D}(\mathbb{P}^n)$ is not necessarily finite when $n \geqslant 2$. Second, the construction below gives uncountably many non-isomorphic, projective, left ideals of $\mathscr{D}(\mathbb{P}^1)$ which are not locally free.

THEOREM 4·2. *Let $\mathscr{S}$ be a finite subset of $\mathbb{P}^1$. Then there exists a projective left ideal of $\mathscr{D}(\mathbb{P}^1)$ whose singularity set is $\mathscr{S}$.*

Proof. Let $\mathscr{S} = \{a_0, \ldots, a_m\}$. As usual, the open set $\{[x_0 : x_1] \in \mathbb{P}^1 : x_0 \neq 0\}$ is denoted by U_0. The symbol ∞ denotes the point $[0 : 1]$. To simplify the notation, let

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$D = \mathscr{D}(\mathbb{P}^1)$. Choosing coordinates appropriately, it can be assumed that $\mathscr{S} \subseteq U_0$ and that $a_0 = 0$. Consider the homomorphism $\pi: D^2 \to D(m+1)$ defined by $(a, b) \mapsto a\partial + b\prod_0^m (t - a_i)$. Let $P = \mathrm{Ker}\,(\pi)$. Recall from Proposition 1·5 that $D(m+1)$ has projective dimension at most one. Thus to prove that P is a projective module it is enough to show that π is surjective. This can be checked locally. That $\pi|_{U_0}$ is surjective follows from the fact that

$$\mathscr{D}(U_0)\,\partial + \mathscr{D}(U_0)\prod_0^m (t - a_i) = \mathscr{D}(U_0).$$

To handle the stalk at ∞ note first that $(1 - a_i t^{-1})$ is a unit of the stalk at ∞ of the structure sheaf $\mathscr{O}_{\mathbb{P}^1}$, for every $i \neq 0$. Thus, denoting by $\mathscr{D}_\infty$, the stalk of $\mathscr{D}_{\mathbb{P}^1}$, we have

$$\mathscr{D}_\infty \prod_0^m (t - a_i) = \mathscr{D}_\infty \prod_1^m (1 - a_i t^{-1})\, t^{m+1} = \mathscr{D}_\infty t^{m+1},$$

completing the proof that π is surjective.

We proceed to calculate the singularity set of P. For every $b \in \mathbb{P}^1$, set

$$P_b = \mathscr{D}_b \otimes P = \mathscr{D}_b\,\partial \cap \mathscr{D}_b \sum_0^m (t - a_i).$$

There are several cases to be considered. First, $P_\infty = \mathscr{D}_\infty\,\partial \cap \mathscr{D}_\infty\, t^m$ is clearly cyclic. So is P_b if $b \notin \mathscr{S} \cup \{\infty\}$, because, for these b it follows that $P_b = \mathscr{D}_b\,\partial \cap \mathscr{D}_b$. However, for $0 \leqslant i \leqslant m$, the module $P_{a_i} = \mathscr{D}_{a_i}\,\partial \cap \mathscr{D}_{a_i}(t - a_i)$ is not cyclic by [**16**], theorem 1·2. Thus P is projective and has singularity set $\mathscr{S}$, as expected.

We now turn to the question of existence of quadratic structures for locally free D-modules. We review briefly what is meant by a quadratic structure, in this context. For details see [**11**]. Let A be a ring and M a finitely generated, projective, left A-module. The dual $M^* = \mathrm{Hom}_A(M, A)$ has a natural right module structure. However, if A has an anti-automorphism σ, then M^* can be turned into a left A-module with the following action:

$$a\theta = \theta\sigma(a) \quad \text{for } a \in A \text{ and } \theta \in M^*.$$

Let M^*_σ denote M^* with this left action. We say that M has a *σ-quadratic structure* if there exists an isomorphism $\alpha: M^*_\sigma \to M$. The isomorphism α is called a *quadratic form* (with respect to σ).

The enveloping algebra $U(\mathfrak{sl}(n+1))$ has a natural anti-automorphism, known as the Chevalley anti-automorphism. This induces an anti-automorphism σ on $\mathscr{D}(\mathbb{P}^n)$. In terms of generators it is described as follows:

$$\begin{aligned} &\partial_j \mapsto -t_j \pi \quad \text{for } 1 \leqslant j \leqslant n \\ &t_i \partial_j \mapsto t_j \partial_i \quad \text{for } 1 \leqslant i \neq j \leqslant n \\ &t_i \pi \mapsto -\partial_i \quad \text{for } 1 \leqslant i \leqslant n. \end{aligned}$$

The next theorem shows that some locally free $\mathscr{D}(\mathbb{P}^n)$-modules have a unique quadratic structure for the anti-automorphism above. The uniqueness is proved separately, as it holds in greater generality. First let us consider what it means for a fractional ideal of $\mathscr{D}(\mathbb{P}^n)$ to have a σ-quadratic structure. Notice that the Chevalley anti-automorphism σ of $\mathscr{D}(\mathbb{P}^n)$ can be extended to its ring of fractions $Q(\mathbb{P}^n)$. Let M

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be a fractional, left ideal of $\mathscr{D}(\mathbb{P}^n)$. Then $\sigma(M) \subseteq Q(\mathbb{P}^n)$ has a canonical right $\mathscr{D}(\mathbb{P}^n)$-module structure obtained from the multiplication in $Q(\mathbb{P}^n)$. Similarly, $\sigma(M^*)$ has a left $\mathscr{D}(\mathbb{P}^n)$-module structure. It can now be shown that $M \cong M^*_\sigma$ if and only if $\sigma(M) \cong M^*$. This is also equivalent to $M \cong \sigma(M^*)$.

PROPOSITION 4·3. *Let M be a finitely generated, fractional, left ideal of $\mathscr{D}(\mathbb{P}^n)$. Any two σ-quadratic structures on M are homothetically equivalent.*

Proof. The dual M^* can be identified with the set $\{q \in Q(\mathbb{P}^n) : Mq \in \mathscr{D}(\mathbb{P}^n)\}$. Suppose that M has two different σ-quadratic structures. These are given by elements $a, b \in Q(\mathbb{P}^n)$ such that $\sigma(M) = aM^* = bM^*$. Thus $(M^*\mathscr{D}(U_0)) = a^{-1}b(M^*\mathscr{D}(U_0))$ in $Q(\mathbb{P}^n)$. Recall that $\mathscr{D}(U_0)$ is isomorphic to the nth Weyl algebra. But from [17], lemma 4·6 it follows that the automorphisms of a right ideal of the Weyl algebra are all constant. Hence $a^{-1}b \in \mathbb{C}$ is a constant and the quadratic forms differ only by a homothety, as claimed.

THEOREM 4·4. *Every rank one, locally free, projective $\mathscr{D}(\mathbb{P}^n)$-module has a unique (up to homothety) quadratic structure arising from the Chevalley anti-automorphism.*

Proof. From Proposition 1·5 it follows that the modules in the statement are exactly the $D(k)$, for $k \leqslant n$. Thus they are fractional ideals, and the uniqueness follows from Proposition 4·3. We must prove that a quadratic structure exists. This is obviously true if $k = 0$. Consider first the case $1 \leqslant k \leqslant n$. In the notation of [5], §4, one has that $D(k)^* = M(-k) \dots M(-1)$. Thus, as a right $\mathscr{D}(\mathbb{P}^n)$-module, $D(k)^*$ is generated by the elements

$$(\pi + k - 1) \dots (\pi - r)\, \partial_1^{i_1} \dots \partial_n^{i_n},$$

where $-k \leqslant r \leqslant 0$, and $i_1, \dots, i_n$ are non-negative integers satisfying $i_1 + \dots + i_n = -r$. Applying the Chevalley anti-automorphism to the above elements, one concludes that $\sigma(D(k)^*)$ is generated by

$$(t_n \pi)^{i_n} \dots (t_1 \pi)^{i_1} (\pi + k - 1) \dots (\pi - r).$$

Using the formulae

$$t_i^j(\pi - m) = (\pi - m - j)\, t_i^j, \quad \partial_i^j(\pi - m) = (\pi - m + j)\, \partial_i^j,$$

repeatedly, the latter becomes

$$t_n^{i_n} \dots t_1^{i_1}[\pi \dots (\pi + k - 1)].$$

Thus we have proved that

$$\sigma(D(k)^*) = D(k)\pi \dots (\pi + k - 1) \cong D(k),$$

as claimed.

The case $k \leqslant -1$ requires a similar calculation on generators and will be omitted.

The reader is warned that it is not true that only locally free projective ideals have a σ-quadratic structure. In fact we make the following conjecture:

Conjecture 4·5. All finitely generated, projective, left $\mathscr{D}(\mathbb{P}^n)$-modules have a σ-quadratic structure.

Here are a few other questions raised in the course of our research and that we have not been able to answer.

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Question 4·6. Are there left ideals I of $\mathscr{D}(\mathbb{P}^n)$ with $\mathbf{m}I = I$ that cannot be generated by two elements?

If the answer is affirmative this would allow one to simplify and generalize Theorem 3·6. An old unanswered question is whether all left ideals of a simple, noetherian ring can be generated by 2 elements. Even though $\mathscr{D}(\mathbb{P}^n)$ is not simple, the twisted ring $\mathscr{D}_{-1}(\mathbb{P}^n)$ is simple. For a definition of the latter, see [5].

Question 4·7. Can all left ideals of $\mathscr{D}_{-1}(\mathbb{P}^n)$ be generated by two elements?

The first author would like to thank the University of Leeds for its hospitality and CNPq (Brazil), SERC (UK) for financial support. This work was carried out as part of the SERC grant GR/E25511 under which the second author was supported by a SERC Research Assistantship.

REFERENCES

[1] E. Andronikof. Systèmes déterminés et systèmes normaux d'équations aux dérivées partielles. *C. R. Acad. Sci. Paris Sér. I Math.* **293** (1981), 257–260.
[2] H. Bass. Finitistic dimension and a homological generalisation of semiprimary rings. *Trans. Amer. Math. Soc.* **95** (1960), 466–488.
[3] A. Beilinson and J. Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 15–18.
[4] S. C. Coutinho. Basic element theory of noncommutative Noetherian rings. *J. Algebra* **144** (1991), 24–42.
[5] S. C. Coutinho and M. P. Holland. Differential operators on smooth varieties. In *Séminaire d'Algèbra Malliavin*, Lecture notes in Math. vol. 1404 (Springer-Verlag, 1989), pp. 201–219.
[6] S. C. Coutinho and M. P. Holland. K-theory of twisted differential operators. *J. London Math. Soc.* (2), to appear.
[7] A. Grothendieck. Sur la classification des fibres holomorphes sur la sphère de Riemann. *Amer. J. Math.* **79** (1957), 121–138.
[8] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Math. vol. 52 (Springer-Verlag, 1977).
[9] T. J. Hodges. K-theory of D-modules and primitive factors of enveloping algebras of semisimple Lie algebras. *Bull. Sci. Math.* (2) **113** (1989), 85–88.
[10] M. P. Holland and J. T. Stafford. Differential operators on rational projective curves. *J. Algebra* **147** (1992), 176–244.
[11] B. A. Magurn, W. Van der Kallen and L. N. Vaserstein. Absolute stable rank and Witt cancellation for non-commutative rings. *Invent. Math.* **911** (1988), 525–542.
[12] C. Okonek, M. Schneider and H. Spindler. *Vector Bundles on Complex Projective Spaces*, Progress in Math. no. 3 (Birkhäuser, 1980).
[13] J. T. Stafford. Module structure of Weyl algebras. *J. London Math. Soc.* (2) **18** (1977), 429–442.
[14] J. T. Stafford. Generating modules efficiently: algebraic K-theory for noncommutative Noetherian rings. *J. Algebra* **69** (1981), 312–346.
[15] J. T. Stafford. Homological properties of the enveloping algebra $U(SL_2)$. *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 29–37.
[16] J. T. Stafford. Stably free, projective right ideals. *Compositio Math.* **54** (1985), 63–78.
[17] J. T. Stafford. Endomorphisms of right ideals of the Weyl algebra. *Trans. Amer. Math. Soc.* **299** (1987), 623–639.

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