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Absolute summability functions for Hausdorff methods

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1. Following Lorentz, we suppose throughout that $\Omega(n)$ is a non-negative non-decreasing function of the non-negative integer n such that $\Omega(n)\to\infty$ as $n\to\infty$. Consider the summability method given by the sequence-to-sequence transformation corresponding to the matrix $A = (a_{nk})$. We say that $\Omega(n)$ is a summability function for A (or absolute summability function for A) if the following holds: Any bounded sequence $\{s_n\}$ such that the number of values of ν with $\nu \leqslant n$, $s_\nu \neq 0$ does not exceed $\Omega(n)$ is summable A (or is absolutely summable A, respectively). These definitions are due to Lorentz (4), (6). We shall be concerned with the case in which A is a regular Hausdorff method, say $A = H = (H, \mu_n)$. Then H is given by the matrix (h_{nk}) with

$$h_{nk} = \begin{cases} \binom{n}{k} \Delta^{n-k}\mu_k & (0 \leqslant k \leqslant n) \\ 0 & (k > n) \end{cases}$$

with $$\mu_n = \int_0^1 t^n \, d\chi(t) \quad \text{where} \quad \chi(t) \in BV[0,1],$$

$\chi(0) = \chi(0+) = 0$, $\chi(1) = 1$; (see e.g. (1), chapter XI). We shall suppose throughout that these conditions are satisfied. It is known that H is then necessarily also absolutely regular.

Summability functions for H were investigated in (3), (5); here we consider absolute summability functions. The results of (3), (5) show that the set of summability functions depends on the behaviour of $\chi(t)$; roughly speaking, the 'smoother' $\chi(t)$ is, the bigger is this set. No such result can be true for absolute summability functions. For it follows from the results of (6) (theorems 8, 9) that for some values of α the set of absolute summability functions for the Cesàro method C_α is strictly contained in that for the Euler method E_t, while for other values of α the set of absolute summability functions for C_α strictly contains that for E_t. However, one general theorem can be asserted. It was shown in ((6), theorem 8) that when $\alpha \geqslant 1$, the set of absolute summability functions for C_α is precisely the set of those $\Omega(n)$ for which

$$\sum_{n=1}^{\infty} \frac{\Omega(n)}{n^2} < \infty. \tag{1}$$

Assuming only that $\chi(t)$ satisfies a certain weak restriction in the neighbourhood of the origin, we show that this is the largest possible set of absolute summability functions. More precisely, we shall prove the following result.

Theorem. *Suppose that, as* $t\to 0+$,

$$\int_0^t |d\chi(u)| = O[(\log(1/t))^{-r}], \tag{2}$$

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where r is a constant, $r > 1$. In order that $\Omega(n)$ should be an absolute summability function for H, it is necessary that (1) *be satisfied.*

We shall suppose throughout the rest of this paper that $\chi(t)$ is a real valued function of t. The result for the case in which $\chi(t)$ is allowed to take complex values can easily be deduced from this case.

2. We write, for $0 \leqslant k \leqslant n$, $0 \leqslant t \leqslant 1$,

$$p(n,k;t) = \binom{n}{k} t^k (1-t)^{n-k};$$

thus the H-transform of $\{s_k\}$ is $\{u_n\}$, where

$$u_n = \int_0^1 \left\{ \sum_{k=0}^{n} p(n,k;t)\, s_k \right\} d\chi(t). \tag{3}$$

It is trivial (and familiar) that

$$p(n,k;t) \geqslant 0; \quad \sum_{k=0}^{n} p(n,k;t) = 1. \tag{4}$$

Some other properties of $p(n,k;t)$ which will be required may conveniently be stated as a lemma.

LEMMA 1. (i) *Let a, b be given, $0 < a < b < 1$. Then given any positive numbers ϵ, η (with η sufficiently small) there exists n_0 such that, for $n > n_0$,*

$$\sum_{an \leqslant k \leqslant bn} p(n,k;t) \text{ is } \begin{cases} \leqslant \epsilon & \text{if} \quad 0 \leqslant t \leqslant a-\eta \quad \text{or} \quad b+\eta \leqslant t \leqslant 1; \\ \geqslant 1-\epsilon & \text{if} \quad a+\eta \leqslant t \leqslant b-\eta. \end{cases}$$

(ii) *Let β be constant, $0 < \beta < 1$. Then, uniformly in*

$$1 \leqslant K \leqslant \tfrac{1}{2}n, \quad (K/n)^\beta \leqslant t \leqslant 1,$$

we have

$$\sum_{k=0}^{K} p(n,k;t) = O\left(\frac{1}{n^{1-\beta}}\right). \tag{5}$$

The result (i), which is an easy consequence of ((1), Theorem 138) is given in ((2), Lemma 2). For (ii), we note that since, for fixed n and k, the function $p(n,k;t)$ is decreasing in $(k/n, 1)$ and hence in $((K/n)^\beta, 1)$ for all $k \leqslant K$ it is enough to consider the case

$$t = (K/n)^\beta. \tag{6}$$

Now

$$\sum_{k=0}^{n} p(n,k;t)\,(nt-k)^2 = nt(1-t),$$

and hence

$$\sum_{k=0}^{K} p(n,k;t)\,(nt-k)^2 < nt(1-t) \leqslant nt. \tag{7}$$

But if (6) holds, then for every term in the sum (7),

$$\frac{k}{nt} \leqslant \frac{K}{nt} = \left(\frac{K}{n}\right)^{1-\beta} \leqslant 2^{\beta-1}$$

so that

$$(nt-k)^2 \geqslant n^2t^2(1-2^{\beta-1})^2 = cn^2t^2, \quad \text{say.}$$

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It therefore follows from (7) that the sum on the left of (5) is less than or equal to

$$\frac{1}{cnt} = \frac{1}{c} n^{\beta-1} K^{-\beta} \leqslant \frac{n^{\beta-1}}{c} \quad \text{since} \quad K \geqslant 1.$$

This is, of course, a very crude argument, but the result suffices for our applications. We also require another lemma.

LEMMA 2. *Suppose that $\chi(t)$ is continuous at $t = 1$. Then there are constants λ, t_1 and t_2 with $0 < \lambda < 1$, $0 < t_2 < t_1 < 1$ such that t_1, t_2, λt_1 and λt_2 are points of continuity of $\chi(t)$, and such that*

$$\chi(t_1) - \chi(\lambda t_1) > \chi(t_2) - \chi(\lambda t_2).$$

Proof. Suppose that the conclusion is false. Then, for all λ, t_1, t_2 satisfying the above inequalities for which $t_1, t_2, \lambda t_1, \lambda t_2$ are points of continuity of $\chi(t)$, we have

$$\chi(t_1) - \chi(\lambda t_1) \leqslant \chi(t_2) - \chi(\lambda t_2). \tag{8}$$

Suppose, as we may, that $\chi(t)$ is normalized; that is to say that, for $0 < t < 1$, we have $\chi(t) = \frac{1}{2}\{\chi(t+) + \chi(t-)\}$. It will then follow that (8) continues to hold when the assumption that $t_1, t_2, \lambda t_1, \lambda t_2$ are points of continuity is omitted. Now since $\chi(t) \to 1$ as $t \to 1-$, we may choose $t_1 (0 < t_1 < 1)$ such that $\chi(t_1) > 0$. Since $\chi(t) \to 0$ as $t \to 0+$, we can choose λ such that $\chi(\lambda t_1) < \chi(t_1)$. Write $\chi(t_1) - \chi(\lambda t_1) = d$; thus $d > 0$. Applying (8) successively with $t_2 = \lambda t_1, \lambda^2 t_1, \ldots$, we deduce that $\chi(\lambda^n t_1) \leqslant \chi(t_1) - nd$ for all n. But this contradicts the boundedness of $\chi(t)$.

3. *Proof of the theorem.* If $\chi(t)$ has a discontinuity at $t = 1$, then it was shown in (4) that H has no summability functions. Thus, a fortiori, it has no absolute summability functions, so that there is nothing to prove. Again, it was shown in (4) that every summability function for H, and thus, a fortiori, every absolute summability function, satisfies the condition

$$\Omega(n) = o(n). \tag{9}$$

Thus we may suppose that $\chi(t)$ is continuous at $t = 1$, and that (9) holds. These assumptions will be made throughout the rest of the proof. We also suppose, as we clearly may, that $\Omega(n)$ is an integer valued function. We suppose that (1) is false, and show that then $\Omega(n)$ is not an absolute summability function.

Let λ, t_1 and t_2 be as in Lemma 2. Let β be any fixed number with $0 < \beta < 1$. Let a be an integer, to be chosen later, with $a \geqslant 2$. Following Lorentz, we write $p_\nu = a^\nu$ and $q_\nu = \Omega(p_\nu) - \Omega(p_{\nu-1})$. It follows from ((6), Lemma 3) that

$$\sum_{\nu=1}^{\infty} \frac{q_\nu}{p_\nu} = \infty. \tag{10}$$

We also note that it follows from (9) that

$$q_\nu = o(p_\nu). \tag{11}$$

We may therefore suppose that ν_0 has been chosen such that, for each $\nu \geqslant \nu_0$,

$$q_\nu \leqslant p_\nu \left(\frac{1}{\lambda} - 1\right) - 1. \tag{12}$$

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For each $\nu \geqslant \nu_0$, let S_ν be a set (to be chosen later) of just q_ν of those integers k for which

$$p_\nu \leqslant k \leqslant (p_\nu/\lambda) - 1. \tag{13}$$

We note that, by (12), there are at least q_ν integers satisfying (13), so that it is possible to choose such a set S_ν. We suppose that $a > 1/(\lambda t_2) > 1/\lambda$; then the sets (13) are non-overlapping. Now choose

$$s_k = \begin{cases} 1 & (k \in S_\nu,\ \nu = \nu_0, \nu_0+1, \nu_0+2, \ldots) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the number of values of $k \leqslant n$ for which $s_k = 1$ does not exceed $\Omega(n)$. Thus it is enough to show that the sets S_ν can be chosen so that $s = \{s_k\}$ is not absolutely summable H.

Let $u = \{u_n\} = \{u(n)\}$ be given by (3). Write

$$n_\nu = [p_\nu/(\lambda t_1)], \quad m_\nu = [p_\nu/(\lambda t_2)].$$

Then it is enough to show that, provided that the sets S_ν are suitably chosen,

$$\sum_{\nu=0}^{\infty} [u(n_\nu) - u(m_\nu)] \tag{14}$$

diverges to ∞.

We write Σ_ν to denote a sum taken over the range (13), and consider first

$$\int_0^1 \{\Sigma_\nu p(n_\nu, k; t)\}\, d\chi(t). \tag{15}$$

Since the expression in curly brackets does not exceed 1, and since λt_1 and t_1 are points of continuity of $\chi(t)$, it follows that the contribution to (15) of the intervals $(\lambda t_1 - \eta, \lambda t_1 + \eta)$ and $(t_1 - \eta, t_1 + \eta)$, (where $\eta > 0$), can be made arbitrarily small by taking η sufficiently small. Having fixed η, it follows from Lemma 1 (i) that we can make the expression in curly brackets arbitrarily close to 0 in $[0, \lambda t_1 - \eta]$ and $[t_1 + \eta, 1]$ and arbitrarily close to 1 in $[\lambda t_1 + \eta, t_1 - \eta]$ by choosing ν sufficiently large. Thus the expression (15) tends to $\chi(t_1) - \chi(\lambda t_1)$ as $\nu \to \infty$. Similarly,

$$\int_0^1 \{\Sigma_\nu p(m_\nu, k; t)\}\, d\chi(t) \to \chi(t_2) - \chi(\lambda t_2)$$

as $\nu \to \infty$. Now let α be a positive number chosen so that

$$\alpha((1/\lambda) - 1) < (\chi(t_1) - \chi(\lambda t_1)) - ((\chi(t_2) - \chi(\lambda t_2)).$$

We can choose such an α, since the expression on the right is positive by Lemma 2; we note that α can be chosen independently of a. It then follows that we can suppose ν_0 chosen so that, for $\nu \geqslant \nu_0$, we have

$$\int_0^1 \{\Sigma_\nu (p(n_\nu, k; t) - p(m_\nu, k; t))\}\, d\chi(t) \geqslant \alpha\left(\frac{1}{\lambda} - 1\right). \tag{16}$$

Now let $w(\nu, \mu)$ denote the contribution to the sum defining $u(n_\nu) - u(m_\nu)$ of the values of k in the set S_μ. Since we have taken $a > 1/(\lambda t_2)$, we see that $p_{\nu+1} > m_\nu > n_\nu$, and thus $w(\nu, \mu) = 0$ for $\mu > \nu$. Hence

$$u(n_\nu) - u(m_\nu) = \sum_{\mu=0}^{\nu} w(\nu, \mu). \tag{17}$$

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We now show that the divergence of (14) to $+\infty$, and hence the theorem, will follow if we show that

$$\sum_{\mu=\nu_0}^{\infty}\left\{w(\mu,\mu)-\sum_{\nu=\mu+1}^{\infty}|w(\nu,\mu)|\right\}=\infty. \tag{18}$$

For, since H is absolutely regular, the columns of H are absolutely convergent, whence it follows that, for fixed μ,

$$\sum_{\nu=\mu}^{\infty}|w(\nu,\mu)|<\infty.$$

Thus (18) is equivalent to the assertion that

$$\sum_{\mu=0}^{\infty}\left\{w(\mu,\mu)-\sum_{\nu=\mu+1}^{\infty}|w(\nu,\mu)|\right\}=\infty. \tag{19}$$

Now, given any N,

$$\begin{aligned}\sum_{\nu=0}^{N}[u(n_\nu)-u(m_\nu)] &\geqslant \sum_{\nu=0}^{N}\left\{w(\nu,\nu)-\sum_{\mu=0}^{\nu-1}|w(\nu,\mu)|\right\}\\ &=\sum_{\nu=0}^{N}w(\nu,\nu)-\sum_{\mu=0}^{N-1}\sum_{\nu=\mu+1}^{N}|w(\nu,\mu)|\\ &\geqslant \sum_{\nu=0}^{N}w(\nu,\nu)-\sum_{\mu=0}^{N}\sum_{\nu=\mu+1}^{\infty}|w(\nu,\mu)|.\end{aligned}$$

But, assuming (19), this is arbitrarily large for sufficiently large N.

In order to obtain (18), we first consider

$$\begin{aligned}&\Sigma_\mu\left\{\int_0^1[p(n_\mu,k;t)-p(m_\mu,k;t)]\,d\chi(t)-\sum_{\nu=\mu+1}^{\infty}\int_0^1|p(n_\nu,k;t)-p(m_\nu,k;t)|\,|\,d\chi(t)|\right\}\\ &\quad=\int_0^1\{\Sigma_\mu[p(n_\mu,k;t)-p(m_\mu,k;t)]\}\,d\chi(t)\\ &\qquad-\sum_{\nu=\mu+1}^{\infty}\int_0^1\{\Sigma_\mu|p(n_\nu,k;t)-p(m_\nu,k;t)|\}\,|d\chi(t)|.\end{aligned} \tag{20}$$

An estimate for the first term on the right of (20) is given by (16). Consider a typical term in the second sum on the right of (20). We split the range of integration into $[0,t_1^\beta a^{(\mu-\nu)\beta}]$ and $[t_1^\beta a^{(\mu-\nu)\beta},1]$. By (4) and (2), the contribution of the first range does not exceed

$$2\int_0^{t_1^\beta a^{(\mu-\nu)\beta}}|d\chi(t)|=O\left\{\left(\log\frac{1}{t_1^\beta a^{(\mu-\nu)\beta}}\right)^{-r}\right\}=O\left\{\frac{1}{(\nu-\mu)^r(\log a)^r}\right\}.$$

In the range of summation of Σ_μ we have

$$k\leqslant\frac{p_\mu}{\lambda}<\tfrac{1}{2}n_\nu<\tfrac{1}{2}m_\nu,$$

so that, by Lemma 1 (ii) we have, uniformly in the second range of integration,

$$\Sigma_\mu|p(n_\nu,k;t)-p(m_\nu,k;t)|=O\left\{\frac{1}{a^{(\nu-\mu)(1-\beta)}}\right\}.$$

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Thus, for sufficiently large μ the expression (20) is greater than or equal to

$$\alpha\left(\frac{1}{\lambda}-1\right)-\sum_{\nu=\mu+1}^{\infty} O\left\{\frac{1}{(\nu-\mu)^r(\log a)^r}\right\}-\sum_{\nu=\mu+1}^{\infty} O\left\{\frac{1}{a^{(\nu-\mu)(1-\beta)}}\right\}.$$

Since α and the constants implied in the 'O's are independent of a, we may arrange, by choosing a sufficiently large, that this expression is greater than or equal to

$$\tfrac{1}{2}\alpha\{(1/\lambda)-1\}.$$

Now the number of integers k occurring in the sum Σ_μ does not exceed $p_\mu\{(1/\lambda)-1\}$. Since the set S_μ is to include exactly q_μ of these integers we see that, for sufficiently large μ, we may suppose S_μ chosen so that

$$\begin{aligned} w(\mu,\mu)-\sum_{\nu=\mu+1}^{\infty}|w(\nu,\mu)| &\geqslant \Sigma_\mu s_k\left\{\int_0^1\{p(n_\nu,k;t)-p(m_\nu,k;t)\}\,d\chi(t)\right. \\ &\quad\left.-\sum_{\nu=\mu+1}^{\infty}\int_0^1|p(n_\nu,k;t)-p(m_\nu,k;t)|\,|d\chi(t)|\right\} \\ &\geqslant \tfrac{1}{2}\alpha\frac{q_\mu}{p_\mu}. \end{aligned}$$

The result (18) then follows from (10), and the proof of the theorem is completed.

One further remark may be made. Suppose that H limits every C_1-limitable sequence. Then H limits absolutely every absolutely C_1-limitable sequence; hence every absolute summability function for C_1 must be an absolute summability function for H. Thus, by the result of ((6), Theorem 8) already mentioned, the set of summability functions of H must include the set of those $\Omega(n)$ which satisfy (1). If we assume further that (2) holds, then it follows from our theorem that the set of absolute summability functions is precisely the set of those $\Omega(n)$ which satisfy (1).

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