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# Maximizing weighted sums of binomial coefficients using generalized continued fractions

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Let $m$, $r \in \mathbb{Z}$ and $\omega \in \mathbb{R}$ satisfy $0 \leqslant r \leqslant m$ and $\omega \geqslant 1$. Our main result is a generalized continued fraction for an expression involving the partial binomial sum $s_m(r) = \sum_{i=0}^{r} \binom{m}{i}$. We apply this to create new upper and lower bounds for $s_m(r)$ and thus for $g_{\omega,m}(r) = \omega^{-r} s_m(r)$. We also bound an integer $r_0 \in \{0, 1, \ldots, m\}$ such that $g_{\omega,m}(0) < \cdots < g_{\omega,m}(r_0 - 1) \leqslant g_{\omega,m}(r_0)$ and $g_{\omega,m}(r_0) > \cdots > g_{\omega,m}(m)$. For real $\omega \geqslant \sqrt{3}$ we prove that $r_0 \in \{\lfloor \frac{m+2}{\omega+1} \rfloor, \lfloor \frac{m+2}{\omega+1} \rfloor + 1\}$, and also $r_0 = \lfloor \frac{m+2}{\omega+1} \rfloor$ for $\omega \in \{3, 4, \ldots\}$ or $\omega = 2$ and $3 \nmid m$.

*Keywords:* partial sum; binomial coefficients; continued fraction; bounds

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## 1. Introduction

Given a real number $\omega \geqslant 1$ and integers $m$, $r$ satisfying $0 \leqslant r \leqslant m$, set

$$s_m(r) := \sum_{i=0}^{r} \binom{m}{i} \quad \text{and} \quad g(r) = g_{\omega,m}(r) := \omega^{-r} s_m(r), \tag{1.1}$$

where the binomial coefficient $\binom{m}{i}$ equals $\prod_{k=1}^{i} \frac{m-k+1}{k}$ for $i > 0$ and $\binom{m}{0} = 1$. The weighted binomial sum $g_{\omega,m}(r)$ and the partial binomial sum $s_m(r) = g_{1,m}(r)$ appear in many formulas and inequalities, e.g. the cumulative distribution function $2^{-m} s_m(r)$ of a binomial random variable with $p = q = \frac{1}{2}$ as in remark 5.3, and the Gilbert–Varshamov bound [**6**, Theorem 5.2.6] for a code $C \subseteq \{0, 1\}^n$. Partial sums of binomial coefficients are found in probability theory, coding theory, group theory, and elsewhere. As $s_m(r)$ cannot be computed exactly for most values of $r$, it is desirable for certain applications to find simple sharp upper and lower bounds for

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$s_m(r)$. Our interest in bounding $2^{-r}s_m(r)$ was piqued in [4] by an application to Reed–Muller codes RM$(m, r)$, which are linear codes of dimension $s_m(r)$.

Our main result is a generalized continued fraction $a_0 + \mathop{\mathcal{K}}_{i=1}^{r} \frac{b_i}{a_i}$ (using Gauss' Kettenbruch notation) for $Q := \frac{(r+1)}{s_m(r)}\binom{m}{r+1}$. From this we derive useful approximations to $Q$, $2 + \frac{Q}{r+1}$, and $s_m(r)$, and with these find a maximizing input $r_0$ for $g_{\omega,m}(r)$.

The *jth tail* of the generalized continued fraction $\mathop{\mathcal{K}}_{i=1}^{r} \frac{b_i}{a_i}$ is denoted by $\mathcal{T}_j$ where

$$\mathcal{T}_j := \mathop{\mathcal{K}}_{i=j}^{r} \frac{b_i}{a_i} = \cfrac{b_j}{a_j + \cfrac{b_{j+1}}{a_{j+1} + \cfrac{b_{j+2}}{\ddots \\ a_{r-1} + \cfrac{b_r}{a_r}}}} = \frac{b_j}{a_j + \mathcal{T}_{j+1}} \quad \text{and } 1 \leqslant j \leqslant r. \tag{1.2}$$

If $\mathcal{T}_j = \frac{B_j}{A_j}$, then $\mathcal{T}_j = \frac{b_j}{a_j + \mathcal{T}_{j+1}}$ shows $b_jA_j - a_jB_j = \mathcal{T}_{j+1}B_j$. By convention we set $\mathcal{T}_{r+1} = 0$.

It follows from $\binom{m}{r-i} = \binom{m}{r}\prod_{k=1}^{i}\frac{r-k+1}{m-r+k}$ that $x^i\binom{m}{r} \leqslant \binom{m}{r-i} \leqslant y^i\binom{m}{r}$ for $0 \leqslant i \leqslant r$ where $x := \frac{1}{m}$ and $y := \frac{r}{m-r+1}$. Hence $\frac{1-x^{r+1}}{1-x}\binom{m}{r} \leqslant s_m(r) \leqslant \frac{1-y^{r+1}}{1-y}\binom{m}{r}$. These bounds are close if $\frac{r}{m}$ is near 0. If $\frac{r}{m}$ is near $\frac{1}{2}$ then better approximations involve the Berry–Esseen inequality [7] to estimate the binomial cumulative distribution function $2^{-m}s_m(r)$. The cumulative distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt$ is used in remark 5.3 to show that $|2^{-m}s_m(r) - \Phi(\frac{2r-m}{\sqrt{m}})| \leqslant \frac{0.4215}{\sqrt{m}}$ for $0 \leqslant r \leqslant m$ and $m \neq 0$. Each binomial $\binom{m}{i}$ can be estimated using Stirling's approximation as in [10, p. 2]: $\binom{m}{i} = \frac{C_i^m}{\sqrt{2\pi p(1-p)m}}\left(1 + O\left(\frac{1}{m}\right)\right)$ where $C_i = \frac{1}{p^p(1-p)^{1-p}}$ and $p = p_i = i/m$. However, the sum $\sum_{i=0}^{r}\binom{m}{i}$ of binomials is harder to approximate. The preprint [11] discusses different approximations to $s_m(r)$.

Sums of binomial coefficients modulo prime powers, where $i$ lies in a congruence class, can be studied using number theory, see [5, p. 257]. Theorem 1.1 below shows how to find excellent rational approximations to $s_m(r)$ via generalized continued fractions.

THEOREM 1.1. *Fix $r$, $m \in \mathbb{Z}$ where $0 \leqslant r \leqslant m$ and recall that $s_m(r) = \sum_{i=0}^{r}\binom{m}{i}$.*

(a) *If $b_i = 2i(r+1-i)$, $a_i = m - 2r + 3i$ for $0 \leqslant i \leqslant r$, then*

$$Q := \frac{(r+1)\binom{m}{r+1}}{s_m(r)} = a_0 + \mathop{\mathcal{K}}_{i=1}^{r} \frac{b_i}{a_i}.$$

(b) *If $1 \leqslant j \leqslant r$, then $\mathcal{T}_j = R_j/R_{j-1} > 0$ where the sum $R_j := 2^j j! \sum_{k=0}^{r-j}\binom{r-k}{j}\binom{m}{k}$ satisfies $b_jR_{j-1} - a_jR_j = R_{j+1}$. Also, $(m-r)\binom{m}{r} - a_0R_0 = R_1$.*

Since $s_m(m) = 2^m$, it follows that $s_m(m-r) = 2^m - s_m(r-1)$ so we focus on values of $r$ satisfying $0 \leqslant r \leqslant \lfloor \frac{m}{2} \rfloor$. Theorem 1.1 allows us to find a sequence of successively sharper upper and lower bounds for $Q$ (which can be made arbitrarily

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tight), the coarsest being $m-2r \leqslant Q \leqslant m-2r+\frac{2r}{m-2r+3}$ for $1 \leqslant r < \frac{m+3}{2}$, see proposition 2.3 and corollary 2.4.

The fact that the tails $\mathcal{T}_1, \ldots, \mathcal{T}_r$ are all positive is unexpected as $b_i/a_i$ is negative if $\frac{m+3i}{2} < r$. This fact is crucial for approximating $\mathcal{T}_1 = \mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$, see theorem 1.3. Theorem 1.1 implies that $\mathcal{T}_1\mathcal{T}_2\cdots\mathcal{T}_r = R_r/R_0$. Since $R_0 = s_m(r)$, $R_r = 2^r r!$, $\mathcal{T}_j = \frac{b_j}{a_j+\mathcal{T}_{j+1}}$ and $\prod_{j=1}^{r} b_j = 2^r(r!)^2$, the surprising factorizations below follow *c.f.* remark 2.1.

COROLLARY 1.2. *We have* $s_m(r)\prod_{j=1}^{r}\mathcal{T}_j = 2^r r!$ *and* $r!s_m(r) = \prod_{j=1}^{r}(a_j+\mathcal{T}_{j+1})$.

Suppose that $\omega > 1$ and write $g(r) = g_{\omega,m}(r)$. We extend the domain of $g(r)$ by setting $g(-1) = 0$ and $g(m+1) = \frac{g(m)}{\omega}$ in keeping with (1.1). It is easy to prove that $g(r)$ is a *unimodal* function *c.f.* [**2**, § 2]. Hence there exists some $r_0 \in \{0, 1, \ldots, m\}$ that satisfies

$$\begin{aligned} &g_{\omega,m}(-1) < \cdots < g_{\omega,m}(r_0-1) \leqslant g_{\omega,m}(r_0) \quad \text{and} \\ &g_{\omega,m}(r_0) > \cdots > g_{\omega,m}(m+1). \end{aligned} \tag{1.3}$$

As $g(-1) < g(0) = 1$ and $\left(\frac{2}{\omega}\right)^m = g(m) > g(m+1) = \frac{2^m}{\omega^{m+1}}$, both chains of inequalities are non-empty. The chains of inequalities (1.3) serve to define $r_0$.

We use theorem 1.1 to show that $r_0$ is commonly close to $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$. We always have $r' \leqslant r_0$ (by lemma 3.3) and though $r_0 - r'$ approaches $\frac{m}{2}$ as $\omega$ approaches 1 (see remark 4.4), if $\omega \geqslant \sqrt{3}$ then $0 \leqslant r_0 - r' \leqslant 1$ by the next theorem.

THEOREM 1.3. *If* $\omega \geqslant \sqrt{3}$, $m \in \{0, 1, \ldots\}$ *and* $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$, *then* $r_0 \in \{r', r'+1\}$, *that is*

$$g(0) < \cdots < g(r'-1) \leqslant g(r'), \quad \textit{and} \quad g(r'+1) > g(r'+2) > \cdots > g(m).$$

Sharp bounds for $Q$ seem powerful: they enable short and elementary proofs of results that previously required substantial effort. For example, our proof in [**4**, Theorem 1.1] for $\omega = 2$ of the formula $r_0 = \lfloor \frac{m}{3} \rfloor + 1$ involved a lengthy argument, and our first proof of theorem 1.4 below involved a delicate induction. By this theorem there is a unique maximum, namely $r_0 = r' = \lfloor \frac{m+2}{\omega+1} \rfloor$ when $\omega \in \{3, 4, 5, \ldots\}$ and $\omega \neq m+1$, *c.f.* remark 4.2. In particular, strict inequality $g_{\omega,m}(r'-1) < g_{\omega,m}(r')$ holds.

THEOREM 1.4. *Suppose that* $\omega \in \{3, 4, 5, \ldots\}$ *and* $r' = \lfloor \frac{m+2}{\omega+1} \rfloor$. *Then*

$$g_{\omega,m}(0) < \cdots < g_{\omega,m}(r'-1) \leqslant g_{\omega,m}(r') > g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m),$$

*with equality if and only if* $\omega = m+1$.

Our motivation was to analyse $g_{\omega,m}(r)$ by using estimates for $Q$ given by the generalized continued fraction in theorem 1.1. This gives tighter estimates than the method involving partial sums used in [**4**]. The plots of $y = g_{\omega,m}(r)$ for $0 \leqslant r \leqslant m$ are highly asymmetrical if $\omega - 1$ and $m$ are small. However, if $m$ is large the plots exhibit an 'approximate symmetry' about the vertical line $r = r_0$ (see figure 1).

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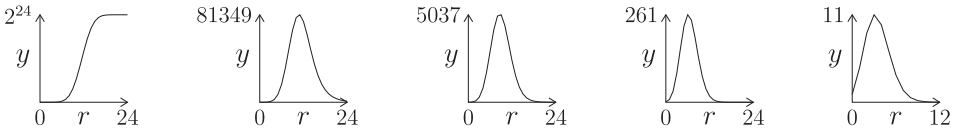



Figure 1. Plots of $y = g_{\omega,24}(r)$ for $0 \leqslant r \leqslant 24$ with $\omega \in \{1, \frac{3}{2}, 2, 3\}$, and $y = g_{3,12}(r)$

Our observation that $r_0$ is close to $r'$ for many choices of $\omega$ was the starting point of our research.

Byun and Poznanović [**2**, Theorem 1.1] compute the maximizing input, call it $r^*$, for the function $f_{m,a}(r) := (1+a)^{-r} \sum_{i=0}^{r} \binom{m}{i} a^i$ where $a \in \{1, 2, \ldots\}$. Their function equals $g_{\omega,m}(r)$ only when $\omega = 1 + a = 2$. Some of their results and methods are similar to those in [**4**] which studied the case $\omega = 2$. They prove that $r^* = \lfloor \frac{a(m+1)+2}{2a+1} \rfloor$ provided $m \notin \{3, 2a+4, 4a+5\}$ or $(a, m) \neq (1, 12)$ when $r^* = \lfloor \frac{a(m+1)+2}{2a+1} \rfloor - 1$.

In Section 2 we prove theorem 1.1 and record approximations to our generalized continued fraction expansion. When $m$ is large, the plots of $y = g_{\omega,m}(r)$ are reminiscent of a normal distribution with mean $\mu \approx \frac{m}{\omega+1}$. Section 3 proves key lemmas for estimating $r_0$, and applies theorem 1.1 to prove theorem 1.4. Non-integral values of $\omega$ are considered in Section 4 where theorem 1.3 is proved. In Section 5 we estimate the maximum height $g(r_0)$ using elementary methods and estimations, see lemma 5.1. A 'statistical' approximation to $s_m(r)$ is given in remark 5.3, and it is compared in remark 5.4 to the 'generalized continued fraction approximations' of $s_m(r)$ in proposition 2.3.

## 2. Generalized continued fraction formulas

In this section we prove theorem 1.1, namely that $Q := \frac{r+1}{s_m(r)} \binom{m}{r+1} = a_0 + \mathcal{T}_1$ where $\mathcal{T}_1 = \mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$. The equality $s_m(r) = \frac{r+1}{a_0+\mathcal{T}_1} \binom{m}{r+1}$ is noted in corollary 2.2.

A version of theorem 1.1(a) was announced in the SCS2022 Poster room, created to run concurrently with vICM 2022, see [**9**].

*Proof of theorem* 1.1. Set $R_{-1} = Q\, s_m(r) = (r+1)\binom{m}{r+1} = (m-r)\binom{m}{r}$ and

$$R_j = 2^j j! \sum_{k=0}^{r-j} \binom{r-k}{j} \binom{m}{k} \quad \text{for } 0 \leqslant j \leqslant r+1.$$

Clearly $R_0 = s_m(r)$, $R_{r+1} = 0$ and $R_j > 0$ for $0 \leqslant j \leqslant r$. We will prove in the following paragraph that the quantities $R_j$, $a_j = m - 2r + 3j$, and $b_j = 2j(r+1-j)$ satisfy the following $r+1$ equations, where the first equation (2.1) is atypical:

$$R_{-1} - a_0 R_0 = R_1, \tag{2.1}$$

$$b_j R_{j-1} - a_j R_j = R_{j+1} \quad \text{where } 1 \leqslant j \leqslant r. \tag{2.2}$$

Assuming (2.2) is true, we prove by induction that $\mathcal{T}_j = R_j / R_{j-1}$ holds for $r+1 \geqslant j \geqslant 1$. This is clear for $j = r+1$ since $\mathcal{T}_{r+1} = R_{r+1} = 0$. Suppose

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that $1 \leqslant j \leqslant r$ and $\mathcal{T}_{j+1} = R_{j+1}/R_j$ holds. We show that $\mathcal{T}_j = R_j/R_{j-1}$ holds. Using (2.2) and $R_j > 0$ we have $b_j R_{j-1}/R_j - a_j = R_{j+1}/R_j = \mathcal{T}_{j+1}$. Hence $R_j/R_{j-1} = b_j/(a_j + \mathcal{T}_{j+1}) = \mathcal{T}_j$, completing the induction. Equation (2.1) gives $Q = R_{-1}/R_0 = a_0 + R_1/R_0 = a_0 + \mathcal{T}_1$ as claimed. Since $R_j > 0$ for $0 \leqslant j \leqslant r$, we have $\mathcal{T}_j = R_j/R_{j-1} > 0$ for $1 \leqslant j \leqslant r$. This proves the first half of theorem 1.1(b), and the recurrence $\mathcal{T}_j = b_j/(a_j + \mathcal{T}_{j+1})$ for $1 \leqslant j \leqslant r$, proves part (a).

We now show that (2.1) holds. The identity $R_0 = 2^0 0! \sum_{k=0}^{r} \binom{m}{k} = s_m(r)$ gives

$$\begin{aligned} R_{-1} - a_0 R_0 &= (r+1)\binom{m}{r+1} - (m-2r)\sum_{i=0}^{r}\binom{m}{i} \\ &= (r+1)\binom{m}{r+1} - \sum_{i=0}^{r}(-i+m-i-2r+2i)\binom{m}{i} \\ &= \sum_{i=0}^{r}\left[(i+1)\binom{m}{i+1} - (m-i)\binom{m}{i}\right] + 2\sum_{i=0}^{r-1}(r-i)\binom{m}{i}. \end{aligned}$$

As $(i+1)\binom{m}{i+1} = (m-i)\binom{m}{i}$, we get $R_{-1} - a_0 R_0 = 2\sum_{k=0}^{r-1}\binom{r-k}{1}\binom{m}{k} = R_1$.

We next show that (2.2) holds. In order to simplify our calculations, we divide by $C_j := 2^j j!$. Using $(j+1)\binom{r-k}{j+1} = (r-k-j)\binom{r-k}{j}$ gives

$$\begin{aligned} \frac{R_{j+1}}{C_j} &= \sum_{k=0}^{r-j-1} 2(j+1)\binom{r-k}{j+1}\binom{m}{k} \\ &= \sum_{k=0}^{r-j} 2(r-k-j)\binom{r-k}{j}\binom{m}{k} \\ &= \sum_{k=0}^{r-j+1}(j-k)\binom{r-k}{j}\binom{m}{k} - \sum_{k=0}^{r-j}(k-2r+3j)\binom{r-k}{j}\binom{m}{k} \end{aligned}$$

noting that the term with $k = r-j+1$ in the first sum is zero as $\binom{j-1}{j} = 0$. Using the abbreviation $L = \sum_{k=0}^{r-j}(k-2r+3j)\binom{r-k}{j}\binom{m}{k}$ and using the identity $j\binom{r-k}{j} = (r+1-j-k)\binom{r-k}{j-1}$ gives

$$\begin{aligned} \frac{R_{j+1}}{C_j} &= \sum_{k=0}^{r-j+1}\left[(r+1-j-k)\binom{r-k}{j-1} - k\binom{r-k}{j}\right]\binom{m}{k} - L \\ &= \sum_{k=0}^{r-j+1}\left[(r+1-j)\binom{r-k}{j-1} - k\binom{r-k}{j-1} - k\binom{r-k}{j}\right]\binom{m}{k} - L \\ &= \sum_{k=0}^{r-j+1}\left[(r+1-j)\binom{r-k}{j-1} - k\binom{r-k+1}{j}\right]\binom{m}{k} - L. \end{aligned}$$

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However, $k\binom{m}{k} = (m-k+1)\binom{m}{k-1}$, and therefore,

$$\sum_{k=0}^{r-j+1} k\binom{r-k+1}{j}\binom{m}{k} = \sum_{k=1}^{r-j+1}(m-k+1)\binom{r-k+1}{j}\binom{m}{k-1}$$
$$= \sum_{\ell=0}^{r-j}(m-\ell)\binom{r-\ell}{j}\binom{m}{\ell}.$$

Thus

$$\frac{R_{j+1}}{C_j} = \sum_{k=0}^{r-j+1}(r-j+1)\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}(m-k)\binom{r-k}{j}\binom{m}{k} - L$$
$$= \sum_{k=0}^{r-j+1}(r-j+1)\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}(m-k+k-2r+3j)\binom{r-k}{j}\binom{m}{k}$$
$$= \sum_{k=0}^{r-j+1}(r-j+1)\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}\overbrace{(m-2r+3j)}^{a_j}\binom{r-k}{j}\binom{m}{k}$$
$$= \frac{\overbrace{2j(r-j+1)}^{b_j}\,2^{j-1}(j-1)!}{C_j}\sum_{k=0}^{r-j+1}\binom{r-k}{j-1}\binom{m}{k} - \sum_{k=0}^{r-j}a_j\binom{r-k}{j}\binom{m}{k}$$

Hence $\frac{R_{j+1}}{C_j} = \frac{b_j R_{j-1}}{C_j} - \frac{a_j R_j}{C_j}$ for $1 \leqslant j \leqslant r$. When $j = r$, our convention gives $R_{r+1} = 0$. This proves part (b) and completes the proof of part (a). □

Remark 2.1. View $m$ as an indeterminant, so that $r!s_m(r)$ is a polynomial in $m$ over $\mathbb{Z}$ of degree $r$. The factorization $r!s_m(r) = \prod_{j=1}^{r}(a_j + \mathcal{T}_{j+1})$ in corollary 1.2 involves the rational functions $a_j + \mathcal{T}_{j+1}$. However, theorem 1.1(b) gives $\mathcal{T}_{j+1} = \frac{R_{j+1}}{R_j}$, so that $a_j + \mathcal{T}_{j+1} = \frac{a_j R_j + R_{j+1}}{R_j} = \frac{b_j R_{j-1}}{R_j}$. This determines the numerator and denominator of the rational function $a_j + \mathcal{T}_{j+1}$, and explains why we have $\prod_{j=1}^{r}(a_j + \mathcal{T}_{j+1}) = \frac{R_0}{R_r}\prod_{j=1}^{r} b_j = r!s_m(r)$. This is different from, but reminiscent of, the ratio $p_{j+1}/p_j$ described on p. 26 of [8]. ◇

Corollary 2.2. *If $r$, $m \in \mathbb{Z}$ and $0 < r < m$, then*

$$s_m(r) := \sum_{i=0}^{r}\binom{m}{i} = \frac{(r+1)\binom{m}{r+1}}{m-2r+\mathcal{T}_1} \quad \text{where } \mathcal{T}_1 = \mathcal{K}_{i=1}^{r}\frac{2i(r+1-i)}{m-2r+3i} > 0.$$

If $r = 0$, then $s_m(r) = \frac{(r+1)\binom{m}{r+1}}{m-2r+\mathcal{T}_1}$ is true, but $\mathcal{T}_1 = \mathcal{K}_{i=1}^{r}\frac{2i(r+1-i)}{m-2r+3i} = 0$.

We will need some additional tools such as proposition 2.3 and corollary 2.4 below in order to prove theorem 1.3.

Since $s_m(m-r) = 2^m - s_m(r-1)$ approximating $s_m(r)$ for $0 \leqslant r \leqslant m$ reduces to approximating $s_m(r)$ for $0 \leqslant r \leqslant \lfloor \frac{m}{2} \rfloor$. Hence the hypothesis $r < \frac{m+3}{2}$ in proposition 2.3 and corollary 2.4 is not too restrictive. Proposition 2.3 generalizes [8, Theorem 3.3].

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Let $\mathcal{H}_j := \mathcal{K}_{i=1}^{j} \frac{b_i}{a_i}$ denote the *jth head* of the fraction $\mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$, where $\mathcal{H}_0 = 0$.

PROPOSITION 2.3. *Let $b_i = 2i(r+1-i)$ and $a_i = m - 2r + 3i$ for $0 \leqslant i \leqslant r$. If $r < \frac{m+3}{2}$, then $a_0 + \mathcal{H}_r = \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ can be approximated using the following chain of inequalities*

$$a_0 + \mathcal{H}_0 < a_0 + \mathcal{H}_2 < \cdots < a_0 + \mathcal{H}_{2\lfloor r/2 \rfloor} < a_0 + \mathcal{H}_{2\lfloor (r-1)/2 \rfloor + 1} < \cdots < a_0 + \mathcal{H}_3 < a_0 + \mathcal{H}_1.$$

*Proof.* Note that $r$ equals either $2\lfloor r/2 \rfloor$ or $2\lfloor (r-1)/2 \rfloor + 1$, depending on its parity.

We showed in the proof of theorem 1.1 that $\frac{(r+1)\binom{m}{r+1}}{s_m(r)} = a_0 + \mathcal{H}_r = a_0 + \mathcal{K}_{i=1}^{r} \frac{b_i}{a_i}$. Since $r < \frac{m+3}{2}$, we have $a_i > 0$ and $b_i > 0$ for $1 \leqslant i \leqslant r$ and hence $\frac{b_i}{a_i} > 0$. A straightforward induction (which we omit) depending on the parity of $r$ proves that $\mathcal{H}_0 < \mathcal{H}_2 < \cdots < \mathcal{H}_{2\lfloor r/2 \rfloor} < \mathcal{H}_{2\lfloor (r-1)/2 \rfloor + 1} < \cdots < \mathcal{H}_3 < \mathcal{H}_1$. For example, if $r = 3$, then

$$\mathcal{H}_0 = 0 < \frac{b_1}{a_1 + \dfrac{b_2}{a_2}} < \frac{b_1}{a_1 + \dfrac{b_2}{a_2 + \dfrac{b_3}{a_3}}} < \frac{b_1}{a_1} = \mathcal{H}_1.$$

proves $\mathcal{H}_0 < \mathcal{H}_2 < \mathcal{H}_3 < \mathcal{H}_1$ as the tails are positive. Adding $a_0$ proves the claim. □

In asking whether $g_{\omega,m}(r)$ is a unimodal function, it is natural to consider the ratio $g_{\omega,m}(r+1)/g_{\omega,m}(r)$ of successive terms. This suggests defining

$$t(r) = t_m(r) := \frac{s_m(r+1)}{s_m(r)} = 1 + \frac{\binom{m}{r+1}}{s_m(r)} = 1 + \frac{Q}{r+1}. \tag{2.3}$$

We will prove in lemma 3.1 that $t(r)$ is a strictly decreasing function that determines when $g_{\omega,m}(r)$ is increasing or decreasing, and $t_m(r_0 - 1) \geqslant \omega > t_m(r_0)$ determines $r_0$.

COROLLARY 2.4. *We have $m - 2r \leqslant \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ for $r \geqslant 0$, and*

$$\frac{(r+1)\binom{m}{r+1}}{s_m(r)} \leqslant m - 2r + \frac{2r}{m-2r+3} \quad \text{for} \quad 0 \leqslant r < \frac{m+3}{2}.$$

*Hence $\frac{m+2}{r+1} \leqslant t_m(r) + 1$ for $r \geqslant 0$, and*

$$\frac{m+2}{r+1} \leqslant t_m(r) + 1 \leqslant \frac{m+2}{r+1} + \frac{2r}{(r+1)(m-2r+3)} \quad \text{for } 0 \leqslant r < \frac{m+3}{2}.$$

*Also $\frac{m+2}{r+1} < t_m(r) + 1$ for $r > 0$, and the above upper bound is strict for $1 < r < \frac{m+3}{2}$.*

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Table 1. *Values of* $g_{w,m}(r)$

| $r$ | 0 | 1 | 2 | 3 | $\cdots$ | $m-2$ | $m-1$ | $m$ |
|---|---|---|---|---|---|---|---|---|
| $g_{\omega,m}(r)$ | 1 | $\frac{m+1}{\omega}$ | $\frac{m^2+m+2}{2\omega^2}$ | $\frac{m^3+5m+6}{6\omega^3}$ | $\cdots$ | $\frac{2^m-m-1}{\omega^{m-2}}$ | $\frac{2^m-1}{\omega^{m-1}}$ | $(\frac{2}{\omega})^m$ |

*Proof.* We proved $Q = \frac{(r+1)\binom{m}{r}}{s_m(r)} = (m-2r) + \mathcal{K}_{i=1}^{r} \frac{2i(r+1-i)}{m-2r+3i}$ in theorem 1.1. Hence $m-2r = \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ if $r=0$ and $m-2r < \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ if $1 \leqslant r < \frac{m+3}{2}$ by proposition 2.3. Clearly $m-2r < 0 \leqslant \frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ if $\frac{m+3}{2} \leqslant r \leqslant m$. Similarly $\frac{(r+1)\binom{m}{r+1}}{s_m(r)} = m-2r+\frac{2r}{m-2r+3}$ if $r=0$, 1, and again proposition 2.3 shows that $\frac{(r+1)\binom{m}{r+1}}{s_m(r)} < m-2r+\frac{2r}{m-2r+3}$ if $1 < r < \frac{m+3}{2}$. The remaining inequalities (and equalities) follow similarly since $t_m(r)+1 = 2+\frac{\binom{m}{r+1}}{s_m(r)}$ and $2+\frac{m-2r}{r+1} = \frac{m+2}{r+1}$. □

## 3. Estimating the maximizing input $r_0$

Fix $\omega > 1$. In this section we consider the function $g(r) = g_{\omega,m}(r)$ given by (1.1). As seen in table 1, it is easy to compute $g(r)$ if $r$ is near 0 or $m$. For $m$ large and $r$ near 0, we have 'sub-exponential' growth $g(r) \approx \frac{m^r}{r!\omega^r}$. Similarly for $r$ near $m$, we have exponential decay $g(r) \approx \frac{2^m}{\omega^r}$. The middle values require more thought.

On the other hand, the plots $y = g(r)$, $0 \leqslant r \leqslant m$, exhibit a remarkable visual symmetry when $m$ is large. The relation $s_m(m-r) = 2^m - s_m(r-1)$ and the distorting scale factor of $\omega^{-r}$ shape the plots. The examples in figure 1 show an approximate left–right symmetry about a maximizing input $r \approx \frac{m}{\omega+1}$. It surprised the authors that in many cases there exists a simple exact formula for the maximizing input (it is usually unique as corollary 3.2 suggests). In figure 1 we have used different scale factors for the $y$-axes. The maximum value of $g_{\omega,m}(r)$ varies considerably as $\omega$ varies (*c.f.* lemma 5.1), so we scaled the maxima (rounded to the nearest integer) to the same height.

LEMMA 3.1. *Recall that* $g(r) = \omega^{-r}s_m(r)$ *by* (1.1) *and* $t(r) = \frac{s_m(r+1)}{s_m(r)}$ *by* (2.3).

(a) $t(r-1) > t(r) > \frac{m-r}{r+1}$ *for* $0 \leqslant r \leqslant m$ *where* $t(-1) := \infty$;

(b) $g(r) < g(r+1)$ *if and only if* $t(r) > \omega$;

(c) $g(r) \leqslant g(r+1)$ *if and only if* $t(r) \geqslant \omega$;

(d) $g(r) > g(r+1)$ *if and only if* $\omega > t(r)$;

(e) $g(r) \geqslant g(r+1)$ *if and only if* $\omega \geqslant t(r)$;

(f) *if* $\omega > 1$ *then some* $r_0 \in \{0, \ldots, m\}$ *satisfies* $t(r_0-1) \geqslant \omega > t(r_0)$, *and this condition is equivalent to*

$$g(0) < \cdots < g(r_0-1) \leqslant g(r_0) \quad \text{and} \quad g(r_0) > \cdots > g(m).$$

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*Proof.* (a) We prove, using induction on $r$, that $t(r-1) > t(r) > \binom{m}{r+1}/\binom{m}{r}$ holds for $0 \leqslant r \leqslant m$. These inequalities are clear for $r = 0$ as $\infty > m+1 > m$. For real numbers $\alpha$, $\beta$, $\gamma$, $\delta > 0$, we have $\alpha\delta - \beta\gamma > 0$ if and only if $\frac{\alpha}{\beta} > \frac{\alpha+\gamma}{\beta+\delta} > \frac{\gamma}{\delta}$; that is, the mediant $\frac{\alpha+\gamma}{\beta+\delta}$ of $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$ lies strictly between $\frac{\alpha}{\beta}$ and $\frac{\gamma}{\delta}$. If $0 < r \leqslant m$, then by induction

$$t(r-1) > \frac{\binom{m}{r}}{\binom{m}{r-1}} = \frac{m-r+1}{r} > \frac{m-r}{r+1} = \frac{\binom{m}{r+1}}{\binom{m}{r}}.$$

Applying the 'mediant sum' to $t(r-1) = \frac{s_m(r)}{s_m(r-1)} > \frac{\binom{m}{r+1}}{\binom{m}{r}}$ gives

$$\frac{s_m(r)}{s_m(r-1)} > \frac{s_m(r) + \binom{m}{r+1}}{s_m(r-1) + \binom{m}{r}} = \frac{s_m(r+1)}{s_m(r)} = t(r) > \frac{\binom{m}{r+1}}{\binom{m}{r}}.$$

Therefore $t(r-1) > t(r) > \binom{m}{r+1}/\binom{m}{r} = \frac{m-r}{r+1}$ completing the induction, and proving (a).

(b,c,d,e) The following are equivalent: $g(r) < g(r+1)$; $\omega s_m(r) < s_m(r+1)$; and $\omega < t(r)$. The other claims are proved similarly by replacing $<$ with $\leqslant$, $>$, $\geqslant$.

(f) Observe that $t(m) = \frac{s_m(m+1)}{s_m(m)} = \frac{2^m}{2^m} = 1$. By part (a), the function $y = t(r)$ is decreasing for $-1 \leqslant r \leqslant m$. Since $\omega > 1$, there exists an integer $r_0 \in \{0, \ldots, m\}$ such that $\infty = t(-1) > \cdots > t(r_0 - 1) \geqslant \omega > t(r_0) > \cdots > t(m) = 1$. By parts (b,c,d,e) an equivalent condition is $g(0) < \cdots < g(r_0 - 1) \leqslant g(r_0)$ and $g(r_0) > \cdots > g(m)$. □

The following is an immediate corollary of lemma 3.1(f).

COROLLARY 3.2. *If $t(r_0 - 1) > \omega$, then the function $g(r)$ in (1.1) has a unique maximum at $r_0$. If $t(r_0 - 1) = \omega$, then $g(r)$ has two equal maxima, one at $r_0 - 1$ and one at $r_0$.*

As an application of theorem 1.1 we show that the largest maximizing input $r_0$ for $g_{\omega,m}(r)$ satisfies $\lfloor \frac{m+2}{\omega+1} \rfloor \leqslant r_0$. There are at most two maximizing inputs by corollary 3.2.

LEMMA 3.3. *Suppose that $\omega > 1$ and $m \in \mathbb{Z}$, $m \geqslant 0$. If $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$, then*

$$g(-1) < g(0) < \cdots < g(r'-1) \leqslant g(r'), \quad \text{and} \quad g(-1) < \cdots < g(r'-1) < g(r')$$

*if $r' > 1$ or $\omega \neq m+1$.*

*Proof.* The result is clear when $r' = 0$. If $r' = 1$, then $r' \leqslant \frac{m+2}{\omega+1}$ gives $\omega \leqslant m+1$ or $g(0) \leqslant g(1)$. Hence $g(0) < g(1)$ if $\omega \neq m+1$. Suppose that $r' > 1$. By lemma 3.1(c,f) the chain $g(0) < \cdots < g(r')$ is equivalent to $g(r'-1) < g(r')$, that is $t(r'-1) > \omega$. However, $t(r'-1) + 1 > \frac{m+2}{r'}$ by corollary 2.4 and $r' \leqslant \frac{m+2}{\omega+1}$ implies $\frac{m+2}{r'} \geqslant \omega + 1$. Hence $t(r'-1) + 1 > \omega + 1$, so that $t(r'-1) > \omega$ as desired. □

*Proof of theorem 1.4.* Suppose that $\omega \in \{3, 4, \ldots\}$. Then $g(0) < \ldots < g(r'-1) \leqslant g(r')$ by lemma 3.3 with strictness when $\omega \neq m+1$. If $\omega = m+1$,

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then $r' = \lfloor \frac{m+2}{\omega+1} \rfloor = 1$ and $g(0) = g(1)$ as claimed. It remains to show that $g(r') > g(r'+1) > \cdots > g(m)$. However, we need only prove that $g(r') > g(r'+1)$ by lemma 3.1(f), or equivalently $\omega > t(r')$ by lemma 3.1(d).

Clearly $\omega \geqslant 3$ implies $r' \leqslant \frac{m+2}{\omega+1} \leqslant \frac{m+2}{4}$. As $0 \leqslant r' < \frac{m+3}{2}$, corollary 2.4 gives

$$\frac{m+2}{r'+1} + \frac{2r'}{(r'+1)(m-2r'+3)} \geqslant t(r') + 1.$$

Hence $\omega + 1 > t(r') + 1$ holds if $\omega + 1 > \frac{m+2}{r'+1} + \frac{2r'}{(r'+1)(m-2r'+3)}$. Since $\omega + 1$ is an integer, we have $m + 2 = r'(\omega + 1) + c$ where $0 \leqslant c \leqslant \omega$. It follows from $0 \leqslant r' \leqslant \frac{m+2}{4}$ that $\frac{2r'}{m-2r'+3} < 1$. This inequality and $m + 2 \leqslant r'(\omega + 1) + \omega$ gives

$$m + 2 + \frac{2r'}{m - 2r' + 3} < r'(\omega + 1) + \omega + 1 = (r' + 1)(\omega + 1).$$

Thus $\omega + 1 > \frac{m+2}{r'+1} + \frac{2r'}{(r'+1)(m-2r'+3)} \geqslant t(r') + 1$, so $\omega > t(r')$ as required. □

Remark 3.4. The proof of theorem 1.4 can be adapted to the case $\omega = 2$. If $m + 2 = 3r' + c$ where $c \leqslant \omega - 1 = 1$, then $\frac{2r'}{m-2r'+3} = \frac{2r'}{r'+c+1} < 2$, and if $c = \omega = 2$, then a sharper $\mathcal{H}_2$-bound must be used. This leads to a much shorter proof than [4, Theorem 1.1]. ◇

## 4. Non-integral values of $\omega$

In this section, we prove that the maximum value of $g(r)$ is $g(r')$ or $g(r'+1)$ if $\omega \geqslant \sqrt{3}$. Before proving this result (theorem 1.3), we shall prove two preliminary lemmas.

Lemma 4.1. *Suppose that* $\omega > 1$ *and* $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$. *If* $\frac{m+2}{r'+1} \geqslant \sqrt{3} + 1$, *then*

$$g(-1) < g(0) < \cdots < g(r'-1) \leqslant g(r'), \quad \text{and} \quad g(r'+1) > g(r'+2) > \cdots > g(m).$$

*Proof.* It suffices, by lemma 3.1(f) and lemma 3.3 to prove that $g(r'+1) > g(r'+2)$. The strategy is to show $\omega > t(r'+1)$, that is $\omega + 1 > t(r'+1) + 1$. However, $\omega + 1 > \frac{m+2}{r'+1}$, so it suffices to prove that $\frac{m+2}{r'+1} \geqslant t(r'+1) + 1$. Since $r' + 1 \leqslant \frac{m+2}{\sqrt{3}+1} < \frac{m+2}{2}$, we can use corollary 2.4 and just prove that $\frac{m+2}{r'+1} \geqslant \frac{m+2}{r'+2} + \frac{2r'+2}{(r'+2)(m-2r'+1)}$. This inequality is equivalent to $\frac{m+2}{r'+1} \geqslant \frac{2r'+2}{m-2r'+1}$. However, $\frac{m+2}{r'+1} \geqslant \sqrt{3} + 1$, so we need only show that $\sqrt{3} + 1 \geqslant \frac{2(r'+1)}{m-2r'+1}$, or equivalently $m - 2r' + 1 \geqslant (\sqrt{3} - 1)(r' + 1)$. This is true since $\frac{m+2}{r'+1} \geqslant \sqrt{3} + 1$ implies $m - 2r' + 1 \geqslant (\sqrt{3} - 1)r' + \sqrt{3} > (\sqrt{3} - 1)(r' + 1)$. □

Remark 4.2. The strict inequality $g(r'-1) < g(r')$ holds by lemma 3.3 if $r' > 1$ or $\omega \neq m + 1$. It holds vacuously for $r' = 0$. Hence adding the additional hypothesis that $\omega \neq m + 1$ if $r' = 1$ to lemma 4.1 (and theorem 1.3), we may conclude that the inequality $g(r'-1) \leqslant g(r')$ is strict.

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REMARK 4.3. In lemma 4.1, the maximum can occur at $r'+1$. If $\omega = 2.5$ and $m = 8$, then $r' = \lfloor \frac{10}{3.5} \rfloor = 2$ and $\frac{m+2}{r'+1} = \frac{10}{3} \geqslant \sqrt{3}+1$ however $g_{2.5,8}(2) = \frac{740}{125} < \frac{744}{125} = g_{2.5,8}(3)$. ◇

REMARK 4.4. The gap between $r'$ and the largest maximizing input $r_0$ can be arbitrarily large if $\omega$ is close to 1. For $\omega > 1$, we have $r' = \lfloor \frac{m+2}{\omega+1} \rfloor < \frac{m+2}{2}$. If $1 < \omega \leqslant \frac{1}{1-2^{-m}}$, then $g(m-1) \leqslant g(m)$, so $r_0 = m$. Hence $r_0 - r' > \frac{m-2}{2}$.

REMARK 4.5. Since $r' \leqslant \lfloor \frac{m+2}{\omega+1} \rfloor < r'+1$, we see that $r'+1 \approx \frac{m+2}{\omega+1}$, so that $\frac{m+2}{r'+1} \approx \omega + 1$. Thus lemma 4.1 suggests that if $\omega \gtrsim \sqrt{3}$, then $g_{\omega,m}(r)$ may have a maximum at $r'$ or $r'+1$. This heuristic reasoning is made rigorous in theorem 1.3. ◇

REMARK 4.6. Theorem 1.1 can be rephrased as $t_m(r) = \frac{s_m(r+1)}{s_m(r)} = \frac{m-r+1}{r+1} + \frac{\mathcal{K}_m(r)}{r+1}$ where

$$\mathcal{K}_m(r) = \mathop{\mathcal{K}}_{i=1}^{r} \frac{2i(r+1-i)}{m-2r+3i} = \cfrac{2r}{m-2r+3+\cfrac{4r-4}{m-2r+6+\cfrac{6r-12}{\ddots\, m+r-3+\cfrac{2r}{m+r}}}}. \tag{4.1}$$

The following lemma repeatedly uses the expression $\omega > t_m(r+1)$. This is equivalent to $\omega > \frac{m-r}{r+2} + \frac{\mathcal{K}_m(r+1)}{r+2}$, that is $(\omega+1)(r+2) > m+2+\mathcal{K}_m(r+1)$. ◇

LEMMA 4.7. *Let $m \in \{0, 1, \ldots\}$ and $r' = \lfloor \frac{m+2}{\omega+1} \rfloor$. If any of the following three conditions are met, then $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ holds:* (a) *$\omega \geqslant 2$, or* (b) *$\omega \geqslant \frac{1+\sqrt{97}}{6}$ and $r' \neq 2$, or* (c) *$\omega \geqslant \sqrt{3}$ and $r' \notin \{2, 3\}$.*

*Proof.* The conclusion $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ holds trivially if $r'+1 \geqslant m$. Suppose henceforth that $r'+1 < m$. Except for the excluded values of $r'$, $\omega$, we will prove that $g_{\omega,m}(r'+1) > g_{\omega,m}(r'+2)$ holds, as this implies $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ by lemma 3.1(f). Hence we must prove that $\omega > t_m(r'+1)$ by lemma 3.1(d).

Recall that $r' \leqslant \frac{m+2}{\omega+1} < r'+1$. If $r' = 0$, then $m+2 < \omega+1$, that is $\omega > m+1 > t(1)$ as desired. Suppose now that $r' = 1$. There is nothing to prove if $m = r'+1 = 2$. Assume that $m > 2$. Since $m+2 < 2(\omega+1)$, we have $2 < m < 2\omega$. The last line of remark 4.6 and (4.1) give the desired inequality:

$$\omega > \frac{m}{2} \geqslant \frac{m-1}{3} + \frac{4}{3\left(m-1+\dfrac{4}{m+2}\right)} = t_m(2).$$

In summary, $g_{\omega,m}(r'+1) > \cdots > g_{\omega,m}(m)$ holds for all $\omega > 1$ if $r' \in \{0, 1\}$.

We next prove $g_{\omega,m}(r'+1) > g_{\omega,m}(r'+2)$, or equivalently $\omega > t_m(r'+1)$ for $r'$ large enough, depending on $\omega$. We must prove that $(\omega+1)(r'+2) > m+2+$

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$\mathcal{K}_m(r'+1)$ by remark 4.6. Writing $m+2=(\omega+1)(r'+\varepsilon)$ where $0\leqslant\varepsilon<1$, our goal, therefore, is to show $(\omega+1)(2-\varepsilon)>\mathcal{K}_m(r'+1)$. Using (4.1) gives

$$\mathcal{K}_m(r'+1)=\frac{2(r'+1)}{m-2(r'+1)+3+\mathcal{T}}=\frac{2(r'+1)}{(\omega+1)(r'+\varepsilon)-2(r'+1)+1+\mathcal{T}}$$

where $\mathcal{T}>0$ by theorem 1.1 as $r'>0$. Rewriting the denominator using

$$(\omega+1)(r'+\varepsilon)-2(r'+1)=(\omega-1)(r'+1)-(\omega+1)(1-\varepsilon),$$

our goal $(\omega+1)(2-\varepsilon)>\mathcal{K}_m(r'+1)$ becomes

$$(\omega+1)(2-\varepsilon)\,[(\omega-1)(r'+1)-(\omega+1)(1-\varepsilon)+1+\mathcal{T}]>2(r'+1).$$

Dividing by $(2-\varepsilon)(r'+1)$ and rearranging gives

$$(\omega^2-1)+\frac{(\omega+1)(1+\mathcal{T})}{r'+1}>\frac{2}{2-\varepsilon}+\frac{(\omega+1)^2(1-\varepsilon)}{r'+1}.$$

This inequality may be written $(\omega^2-1)+\lambda>\frac{2}{2-\varepsilon}+\mu(1-\varepsilon)$ where $\lambda=\frac{(\omega+1)(1+\mathcal{T})}{r'+1}>0$ and $\mu=\frac{(\omega+1)^2}{r'+1}>0$. We view $f(\varepsilon):=\frac{2}{2-\varepsilon}+\mu(1-\varepsilon)$ as a function of a real variable $\varepsilon$ where $0\leqslant\varepsilon<1$. However, $f(\varepsilon)$ is concave as the second derivative $f''(\varepsilon)=\frac{4}{(2-\varepsilon)^3}$ is positive for $0\leqslant\varepsilon<1$. Hence the maximum value occurs at an end point: either $f(0)=1+\mu$ or $f(1)=2$. Therefore, it suffices to prove that $(\omega^2-1)+\lambda>\max\{2,\,1+\mu\}$.

If $2\geqslant 1+\mu$, then the desired bound $(\omega^2-3)+\lambda>0$ holds as $\omega\geqslant\sqrt{3}$. Suppose now that $2<1+\mu$. We must show $(\omega^2-1)+\lambda>1+\mu$, that is $\omega^2-2>\mu-\lambda=\frac{(\omega+1)(\omega-\mathcal{T})}{r'+1}$. Since $\mathcal{T}>0$, a stronger inequality (that implies this) is $\omega^2-2\geqslant\frac{(\omega+1)\omega}{r'+1}$. The (equivalent) quadratic inequality $r'\omega^2-\omega-2(r'+1)\geqslant 0$ in $\omega$ is true provided $\omega\geqslant\frac{1+\sqrt{1+8r'(r'+1)}}{2r'}$. This says $\omega\geqslant 2$ if $r'=2$, and $\omega\geqslant\frac{1+\sqrt{97}}{6}$ if $r'=3$. If $r'\geqslant 4$, we have

$$\frac{1+\sqrt{1+8r'(r'+1)}}{2r'}=\frac{1}{2r'}+\sqrt{\frac{1}{4(r')^2}+2\left(1+\frac{1}{r'}\right)}\leqslant\frac{1}{8}+\sqrt{\frac{1}{64}+\frac{5}{2}}<\sqrt{3}.$$

The conclusion now follows from the fact that $2>\frac{1+\sqrt{97}}{6}>\sqrt{3}$. □

*Proof of theorem 1.3.* By lemma 4.1 it suffices to show that $g(r'+1)>g(r'+2)$ holds when $r'+1<m$ and $\omega\geqslant\sqrt{3}$. By lemma 4.7(a), we can assume that $\sqrt{3}\leqslant\omega<2$ and $r'\in\{2,3\}$. For these choices of $\omega$ and $r'$, we must show that $\omega>t_m(r'+1)$ by lemma 3.1 for all permissible choices of $m$. Since $(\omega+1)r'\leqslant m+2<(\omega+1)(r'+1)$, when $r'=2$ we have $5<2(\sqrt{3}+1)\leqslant m+2<9$ so that $4\leqslant m\leqslant 6$. However, $t_m(3)$ equals $\frac{16}{15}$, $\frac{31}{26}$, $\frac{19}{14}$ for these values of $m$. Thus $\sqrt{3}>t_m(3)$ holds as desired. Similarly, if $r'=3$, then $8<3(\sqrt{3}+1)\leqslant m+2<12$ so that $7\leqslant m\leqslant 9$. In this case $t_m(4)$ equals $\frac{40}{33}$, $\frac{219}{163}$, $\frac{191}{128}$ for these values of $m$. In each case $\sqrt{3}>t_m(4)$, so the proof is complete. □

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REMARK 4.8. We place remark 4.4 in context. The conclusion of theorem 1.3 remains true for values of $\omega$ smaller than $\sqrt{3}$ and not 'too close to 1' and $m$ is 'sufficiently large'. Indeed, by adapting the proof of lemma 4.7 we can show there exists a sufficiently large integer $d$ such that $m > d^4$ and $\omega > 1 + \frac{1}{d}$ implies $g(r' + d) > g(r' + d + 1)$. This shows that $r' \leqslant r_0 \leqslant r' + d$, so $r_0 - r' \leqslant d$. We omit the technical proof of this fact. ◇

REMARK 4.9. The sequence, $a_0 + \mathcal{H}_1, \ldots, a_0 + \mathcal{H}_r$ terminates at $\frac{r+1}{s_m(r)}\binom{m}{r+1}$ by theorem 1.1. We will not comment here on *how quickly* the alternating sequence in proposition 2.3 converges when $r < \frac{m+3}{2}$. If $r = m$, then $a_0 = -m$ and $\frac{m+1}{s_m(m+1)}\binom{m}{m+1} = 0$, so theorem 1.1 gives the curious identity $\mathcal{H}_m = \mathcal{K}_{i=1}^{m} \frac{2i(m+1-i)}{3i-m} = m$. If $\omega$ is less than $\sqrt{3}$ and 'not too close to 1', then we believe that $r_0$ is approximately $\lfloor \frac{m+2}{\omega+1} + \frac{2}{\omega^2-1} \rfloor$, *c.f.* remark 4.8.

## 5. Estimating the maximum value of $g_{\omega,m}(r)$

In this section we relate the size of the maximum value $g_{\omega,m}(r_0)$ to the size of the binomial coefficient $\binom{m}{r_0}$. In the case that we know a formula for a maximizing input $r_0$, we can readily estimate $g_{\omega,m}(r_0)$ using approximations, such as [**10**], for binomial coefficients.

LEMMA 5.1. *The maximum value $g_{\omega,m}(r_0)$ of $g_{\omega,m}(r)$, $0 \leqslant r \leqslant m$, satisfies*

$$\frac{1}{(\omega-1)\omega^{r_0}}\binom{m}{r_0+1} < g_{\omega,m}(r_0) \leqslant \frac{1}{(\omega-1)\omega^{r_0-1}}\binom{m}{r_0}.$$

*Proof.* Since $g(r_0)$ is a maximum value, we have $g(r_0 - 1) \leqslant g(r_0)$. This is equivalent to $(\omega-1)s_m(r_0-1) \leqslant \binom{m}{r_0}$ as $s_m(r_0) = s_m(r_0-1) + \binom{m}{r_0}$. Adding $(\omega-1)\binom{m}{r_0}$ to both sides gives the equivalent inequality $(\omega-1)s_m(r_0) \leqslant \omega\binom{m}{r_0}$. This proves the upper bound.

Similar reasoning shows that the following are equivalent: (a) $g(r_0) > g(r_0+1)$; (b) $(\omega-1)s_m(r_0) > \binom{m}{r_0+1}$; and (c) $g_{\omega,m}(r_0) > \frac{1}{(\omega-1)\omega^{r_0}}\binom{m}{r_0+1}$. □

In theorem 1.3 the maximizing input $r_0$ satisfies $r_0 = r' + d$ where $d \in \{0, 1\}$. In such cases when $r_0$ and $d$ are known, we can bound the maximum $g_{\omega,m}(r_0)$ as follows.

COROLLARY 5.2. *Set $r' := \lfloor \frac{m+2}{\omega+1} \rfloor$ and $k := m + 2 - (\omega+1)r'$. Suppose that $r_0 = r' + d$ and $G = \frac{1}{(\omega-1)\omega^{r_0-1}}\binom{m}{r_0}$. Then*

$$0 \leqslant k < \omega + 1, d \geqslant 0 \quad \text{and} \quad 1 - \frac{1 + d - \frac{d+2-k}{\omega}}{r_0 + 1} < \frac{g_{\omega,m}(r_0)}{G} \leqslant 1.$$

*Proof.* By lemma 3.3, $r_0 = r' + d$ where $d \in \{0, 1, \ldots\}$. Since $r' = \lfloor \frac{m+2}{\omega+1} \rfloor$, we have $m + 2 = (\omega+1)r' + k$ where $0 \leqslant k < \omega + 1$. The result follows from lemma 5.1 and

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$m = (\omega+1)(r_0-d)+k-2$ as $\binom{m}{r_0+1} = \frac{m-r_0}{r_0+1}\binom{m}{r_0}$ and $\frac{m-r_0}{r_0+1}$ equals

$$\frac{\omega(r_0-d)-d+k-2}{r_0+1} = \omega - \frac{\omega+\omega d+d+2-k}{r_0+1} = \omega\left(1-\frac{1+d+\frac{d+2-k}{\omega}}{r_0+1}\right).$$

□

The following remark is an application of the Chernoff bound, *c.f.* [**11**, Section 4]. Unlike theorem 1.1, it requires the cumulative distribution function $\Phi(x)$, which is a non-elementary integral, to approximate $s_m(r)$. It seems to give better approximations only for values of $r$ near $\frac{m}{2}$, see remark 5.4.

REMARK 5.3. We show how the Berry–Esseen inequality for a sum of binomial random variables can be used to approximate $s_m(r)$. Let $B_1, \ldots, B_m$ be independent identically distributed binomial variables with a parameter $p$ where $0 < p < 1$, so that $P(B_i = 1) = p$ and $P(B_i = 0) = q := 1-p$. Let $X_i := B_i - p$ and $X := \frac{1}{\sqrt{mpq}}(\sum_{i=1}^m X_i)$. Then

$$E(X_i) = E(B_i) - p = 0, \quad E(X_i^2) = pq, \quad \text{and} \quad E(|X_i|^3) = pq(p^2+q^2).$$

Hence $E(X) = \frac{1}{\sqrt{mpq}}(\sum_{i=1}^m E(X_i)) = 0$ and $E(X^2) = \frac{1}{mpq}(\sum_{i=1}^m E(X_i^2)) = 1$. By [**7**, Theorem 2] the Berry–Esseen inequality applied to $X$ states that

$$\begin{aligned}|P(X \leqslant x) - \Phi(x)| &\leqslant \frac{Cpq(p^2+q^2)}{(pq)^{3/2}\sqrt{m}} \\ &= \frac{C(p^2+q^2)}{\sqrt{mpq}} \quad \text{for all } m \in \{1,2,\ldots\} \text{ and } x \in \mathbb{R},\end{aligned}$$

where the constant $C := 0.4215$ is close to the lower bound $C_0 = \frac{10+\sqrt{3}}{6\sqrt{2\pi}} = 0.4097\cdots$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,dt = \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$ is the cumulative distribution function for standard normal distribution.

Writing $B = \sum_{i=1}^m B_i$ we have $P(B \leqslant b) = \sum_{i=0}^{\lfloor b \rfloor}\binom{m}{i}p^i q^{m-i}$ for $b \in \mathbb{R}$. Thus $X = \frac{B-mp}{\sqrt{mpq}}$ and $x = \frac{b-mp}{\sqrt{mpq}}$ satisfy

$$\left|P(B \leqslant b) - \Phi\left(\frac{b-mp}{\sqrt{mpq}}\right)\right| \leqslant \frac{C(p^2+q^2)}{\sqrt{mpq}} \quad \text{for all } m \in \{1,2,\ldots\} \text{ and } b \in \mathbb{R}.$$

Setting $p = q = \frac{1}{2}$, and taking $b = r \in \{0, 1, \ldots, m\}$ shows

$$\left|2^{-m}s_m(r) - \Phi\left(\frac{2r-m}{\sqrt{m}}\right)\right| \leqslant \frac{0.4215}{\sqrt{m}} \quad \text{for } m \in \{1,2,\ldots\}.$$

REMARK 5.4. Let $a_0 + \mathcal{H}_k$ be the generalized continued fraction approximation to $\frac{(r+1)\binom{m}{r+1}}{s_m(r)}$ suggested by theorem 1.1, where $\mathcal{H}_k := \mathcal{K}_{i=1}^k \frac{b_i}{a_i}$, and $k$ is the depth of

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Table 2. *Upper bounds for* $|e_{m,r,k}|$ *and* $E_{m,r}$ *for* $m = 10^4$

| $r$ | $\lvert e_{m,r,3}\rvert$ | $\lvert e_{m,r,5}\rvert$ | $\lvert e_{m,r,21}\rvert$ | $E_{m,r}$ |
|---|---|---|---|---|
| 1000 | $2.3 \times 10^{-17}$ | $6.6 \times 10^{-25}$ | $5.7 \times 10^{-79}$ | 1 |
| 4500 | $1.3 \times 10^{-7}$ | $2.5 \times 10^{-10}$ | $7.1 \times 10^{-27}$ | 0.018 |
| 5000 | 0.93 | 0.86 | 0.24 | 0.008 |

the generalized continued fraction. We compare the following two quantities:

$$e_{m,r,k} := 1 - \frac{(r+1)\binom{m}{r+1}}{(a_0 + \mathcal{H}_k)s_m(r)} \quad \text{and} \quad E_{m,r} := \left|1 - \frac{2^m \Phi(\frac{2r-m}{\sqrt{m}})}{s_m(r)}\right| \leqslant \frac{0.4215 \cdot 2^m}{\sqrt{m}\, s_m(r)}.$$

The sign of $e_{m,r,k}$ is governed by the parity of $k$ by proposition 2.3. We shall assume that $r \leqslant \frac{m}{2}$. As $\frac{2^m}{s_m(r)}$ ranges from $2^m$ to about 2 as $r$ ranges from 0 to $\lfloor \frac{m}{2} \rfloor$, it is clear that the upper bound for $E_{m,r}$ will be huge unless $r$ satisfies $\frac{m-\varepsilon}{2} \leqslant r \leqslant \frac{m}{2}$ where $\varepsilon$ is 'small' compared to $m$. By contrast, the computer code [**3**] verifies that the same is true for $E_{m,r}$, and shows that $|e_{m,r,k}|$ is small, even when $k$ is tiny, when $0 \leqslant r < \frac{m-\varepsilon}{2}$, see table 2. Hence the 'generalized continued fraction' approximation to $s_m(r)$ is complementary to the 'statistical' approximation, as shown in table 2. The reader can extend table 2 by running the code [**3**] written in the MAGMA [**1**] language, using the online calculator http://magma.maths.usyd.edu.au/calc/, for example.

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