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Density of hyperbolicity for rational maps with Cantor Julia sets

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Abstract. In this paper, taking advantage of quasi-conformal surgery, we prove that each non-hyperbolic rational map with a Cantor Julia set can be approximated by hyperbolic rational maps with Cantor Julia sets of the same degree.

1. *Introduction and statement*

1.1. *Main theorem and scheme of the proof.*

Recall that a rational map of degree greater than one is said to be hyperbolic if there is a smooth conformal metric ρ defined in a neighborhood of the Julia set $J(f)$ such that $\|f'(z)\|_\rho > C > 1$ for all $z \in J(f)$. It is well known that a rational map is hyperbolic if and only if every critical point belongs to the Fatou set $F(f)$ and is attracted to an attracting cycle.

One central conjecture in the complex holomorphic dynamics is the following.

HYPERBOLICITY CONJECTURE. *Each rational map can be approximated by hyperbolic rational maps with the same degree.*

Equivalently, this means that hyperbolic rational maps are dense among all the rational maps of a given degree. In the family of real quadratic polynomials, Graczyk–Świątek and Lyubich proved that any polynomial can be approximated by hyperbolic maps (see [**GS, L**]). Kozlovski *et al* proved that any real polynomial of degree $d \geq 2$ with a connected Julia set and only real critical points can be approximated by hyperbolic real polynomials of degree d with real critical points and connected Julia sets (see [**KSS**]). By the enhanced nest constructed in [**KSS**] and the Kahn–Lyubich covering lemma [**KL**], Kozlovski and

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van Strien proved that any non-renormalizable polynomial which has only hyperbolic periodic points can be approximated by a hyperbolic polynomial of the same degree (see [**KS**]).

In this paper, we focus on rational maps with Cantor Julia sets and prove the following result relative to the hyperbolicity conjecture.

THEOREM 1. *Any non-hyperbolic rational map with a Cantor Julia set of degree $d \geq 2$ can be approximated arbitrarily by hyperbolic rational maps with Cantor Julia sets of degree d.*

If $J(f)$ is a Cantor set, then f has only one Fatou component, which is either a completely invariant attracting domain or a parabolic domain. So, we will prove Theorem 1 in two cases: the attracting case and the parabolic case.

For the attracting case, we construct a sequence of quasi-regular maps F_n by using the original rational map f, which is uniformly convergent to f in $\widehat{\mathbb{C}}$. According to the quasi-conformal surgery lemma, we prove that for each n, F_n is quasi-conformal conjugate with a hyperbolic rational map f_n whose Julia set is also a Cantor set. Moreover, we show that f_n is locally convergent to a rational map $\widetilde{f}$ in $\widehat{\mathbb{C}}$ of the degree of f. Finally, combining some rigidity results and normalized conditions, we can conclude that actually $\widetilde{f} = f$. For the parabolic case, we apply the plumbing surgery used first by Cui in [**Cui**] to convert the parabolic fixed point into an attracting point. To the attracting point, we apply quasi-conformal surgery and approximation as in the attracting case and then get the conclusion.

1.2. *Preliminary results.* First, we state a theorem which gives a sufficient and necessary condition for judging whether the Julia set of a rational map is a Cantor set. Let f be a rational map with a completely invariant attracting or parabolic Fatou domain F_0, and denote $\widehat{\mathbb{C}} - F_0$ by K_f. We call this set K_f the filled-in Julia set for the rational map f. In this instance, $J(f) = \partial F_0 = \partial K_f$. We have the following theorem.

THEOREM A. [**Zh1**] *The Julia set $J(f)$ is a Cantor set if and only if each critical component of K_f is not periodic.*

This is a generalized version of the Branner–Hubbard conjecture, which was raised by Branner and Hubbard (see [**BH**]) and settled by Qiu–Yin and Kozlovski–van Strien independently (see [**KS, QY**]). Obviously, the 'only if' part of Theorem A is trivial; it is sufficient to prove the sufficiency part of the theorem.

The attracting case of the proof of Theorem A is similar to the polynomial case; here the completely invariant Fatou domains can be regarded as the completely invariant Fatou domains of polynomials containing the infinity. So, we only need to deal with the parabolic case. Now, we assume that 0 is the parabolic fixed point in the Julia set. When we construct the Branner–Hubbard puzzle, we will meet with the problem that the boundaries of puzzle pieces intersect with the Julia set, or the boundary of some piece touches the boundary of other pieces at the preimages of 0, which makes degenerate annuli appear.

We first show that the components of the filled-in Julia set containing the preimages of 0 are single points; here we will use the local dynamics of the parabolic fixed points. Next, for any other point x belonging to K_f, we prove that there is a nested sequence of topological disks

$$P'_{n_1}(x) \supset P_{n_1}(x) \supset P'_{n_2}(x) \supset P_{n_2}(x) \supset \cdots \supset P'_{n_k}(x) \supset P_{n_k}(x) \supset \cdots$$

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containing x such that all the $\mathrm{mod}(P'_{n_k}(x) - \overline{P_{n_k}(x)})$ are uniformly bounded away from zero. Then it follows from Grötzsch's inequality that the component of the filled-in Julia set containing x is a single point. For details, one can refer to [**Zh1**].

In proving Theorem 1, we need two rigidity theorems for rational maps with Cantor Julia sets.

Definition 1. A Julia set $J(f)$ of a rational map f is said to be dynamically removable if every homeomorphism h which is a conjugation between f and another rational map g, and is quasi-conformal off $J(f)$, is globally quasi-conformal.

THEOREM B. [**Zh2**] *Each Cantor Julia set of a rational map is dynamically removable.*

Suppose that two rational maps f and g with Cantor Julia sets are topologically conjugate via a homeomorphism $h_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. We can modify the conjugation h_0 and obtain a new topological conjugation h such that h is quasi-conformal off $J(f)$ and coincides with h_0 in $J(f)$.

We decompose the points in $J(f)$ into three parts, X_0, X_1 and X_2, where X_0 is at most a countable set and X_1 is of zero Lebesgue measure.

Suppose that U is a simply connected domain and $\omega \in U$. The *Shape* of U about ω, denoted by $\mathrm{Shape}(U, \omega)$, is defined as

$$\mathrm{Shape}(U, \omega) = \frac{\max_{z\in\partial U} d(\omega, z)}{\min_{z\in\partial U} d(\omega, z)} = \frac{\max_{z\in\partial U} d(\omega, z)}{d(\omega, \partial U)}.$$

Let Ω be a topological disk. For a homeomorphism $\phi : \Omega \to \widetilde{\Omega}$ and $x \in \Omega$, let

$$\underline{H}(\phi, x) = \liminf_{r\to 0} \frac{\sup_{|y-x|=r} |\phi(y) - \phi(x)|}{\inf_{|y-x|=r} |\phi(y) - \phi(x)|} \in [1, \infty].$$

We prove that for any $x \in X_1$, $\underline{H}(h, x) < \infty$. In other words, there exists a sequence of geometrical disks centered at x such that their h-images have bounded shapes about $h(x)$. Moreover, for any $x \in X_2$, there is a constant $M \geq 1$ such that there exists a sequence of puzzle pieces U_n shrinking to x with $\mathrm{Shape}(U_n, x) \leq M$. Since the topological conjugation h preserves the combinatorial information of the puzzle pieces, we also have $\mathrm{Shape}(h(U_n), h(x)) \leq M_0$, where $M_0 \geq 1$ is a constant.

Once these facts are confirmed, we are able to use the area–length method introduced by Kozlovski *et al* when they resolve the rigidity of real polynomials (see [**KSS**, Appendix 1]), we show that h is quasi-conformal in $J(f)$. Namely, h is in fact globally quasi-conformal (see [**Zh2**]).

Definition 2. We say that a rational map f admits an *invariant line field* on the Julia set $J(f)$ if there is a measurable Beltrami differential $\mu(z)(d\bar{z}/dz)$ in $\widehat{\mathbb{C}}$ such that $f^*\mu = \mu$ almost everywhere, $|\mu| = 1$ in a positive measure subset E of $J(f)$ and $\mu = 0$ in $\widehat{\mathbb{C}} - E$.

In [**YZ**], the second and third authors proved the following theorem.

THEOREM C. [**YZ**] *Let f be a rational map with a Cantor Julia set. Then f carries no invariant line fields on its Julia set.*

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Notation. $\mathbb{C}$ denotes the complex plane; $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere; $\mathbb{C}^* = \mathbb{C} - \{0\}$ denotes the punctured complex plane. A disk in general is denoted by $\mathbb{D}(a, r) = \{z : |z - a| < r\}$, specifically $\mathbb{D}(r) = \mathbb{D}(0, r)$, and $\mathbb{D} = \mathbb{D}(0, 1)$ denotes the unit disk. Denote by $\mathbb{A}_r = \{z : 1 < |z| < r\}$ the standard geometrical annulus.

For a rational map f, let $C(f)$ be the set of the critical points and $\mathrm{Crit}(f) = C(f) \cap J(f)$. Moreover, the post-critical set is denoted by

$$P(f) = \overline{\bigcup_{n \geq 1} \{f^n(c) : c \in C(f)\}}.$$

2. *Proof of Theorem 1 in the attracting case*

In this section, we always assume that the Julia set $J(f)$ of a rational map f is a Cantor set. The Fatou set $F(f)$ has only one component; it is an attracting basin.

2.1. *Branner–Hubbard puzzle.* We assume that ∞ is the fixed attracting point. Take a simply connected neighborhood $U_0 \subset F(f)$ of ∞ such that $U_0 \subset\subset f^{-1}(U_0)$. Of course, we can take a disk $\{z : |z| > R\}$ as U_0, where R is a sufficiently large number and $\{z : |z| = R\}$ is disjoint from the critical orbits of f. Let U_n be the component of $f^{-n}(U_0)$ containing ∞. Then $U_n \subset\subset U_{n+1}$ and $F(f) = \bigcup_{n=0}^{\infty} U_n$. For a large enough integer N_0, $f^{-n}(U_{N_0})$ has only one component for any $n \geq 0$. The set $f^{-n}(\widehat{\mathbb{C}} - \overline{U_{N_0}})$ is the disjoint union of a finite number of topological disks. For each $n \geq 0$, let $\mathbf{P_n}$ be the collection of all components of $f^{-n}(\widehat{\mathbb{C}} - \overline{U_{N_0}})$, which are called puzzle pieces of depth n. For two arbitrary different puzzle pieces P_1 and P_2, there are three possibilities: $\overline{P_1} \cap \overline{P_2} = \emptyset$, $\overline{P_1} \subset P_2$ and $P_1 \supset \overline{P_2}$.

For any point $x \in J(f)$ and any $n \geq 0$, there is only one $P_n(x) \in \mathbf{P_n}$ containing x. Thus each point $x \in J(f)$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$ and $\bigcap_{n \geq 0} P_n(x) = \{x\}$ since $J(f)$ is a Cantor set.

2.2. *Quasi-conformal surgery and approximation.* The terminology 'quasi-conformal surgery' was first introduced in complex dynamic systems by Shishikura. In [**Sh**], he proved that the number of periodic cycles of any rational map which are not repelling does not exceed $2d - 2$, where d is the degree of the rational map. A similar technique was applied in proving the straightening theorem of polynomial-like maps and in proving the no-wandering Fatou domain theorem of rational maps (see [**DH, Su**]).

We first state some preliminary lemmas in this subsection; all these lemmas are proved in [**CP**] using different terminology.

LEMMA 1. *Given $d \geq 1$, let $U \ni p$, $V \ni q$ be two simply connected domains whose boundaries are both Jordan curves. Then there exists a holomorphic proper map $g : U \to V$ of degree d satisfying $g(p) = q$ such that p is the only branched point in U. Moreover, if ∂U and ∂V are smooth Jordan curves, then $g : \partial U \to \partial V$ is smooth.*

Proof. Suppose $\phi_U : U \to \mathbb{D}$ and $\phi_V : V \to \mathbb{D}$ are conformal maps with $\phi_U(p) = 0$ and $\phi_V(q) = 0$. Then ϕ_U and ϕ_V respectively extend to be homeomorphisms from $\overline{U}$ and $\overline{V}$ onto $\mathbb{D}$. Let $P(z) = z^d$. Lifting $P(z)$ by ϕ_U and ϕ_V, we get the desired map g.

If ∂U and ∂V are smooth curves, then ϕ_U and ϕ_V extend to smooth homeomorphisms in $\overline{U}$ and $\overline{V}$, so $g : \partial U \to \partial V$ is smooth. □

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Definition 3. A quasi-regular map is the composite of a quasi-conformal map and a holomorphic map.

LEMMA 2. *Let A_i be annuli with smooth inner boundaries I_i and smooth outer boundaries O_i, $i = 1, 2$. Suppose $h_1 : I_1 \to I_2$ and $h_2 : O_1 \to O_2$ are both d-fold differentiable covering maps. Then there exists a d-fold quasi-regular map $R : \overline{A}_1 \to \overline{A}_2$ with no branched points such that $R|_{I_1} = h_1$ and $R|_{O_1} = h_2$.*

Proof. Suppose $\phi_i : A_i \to \mathbb{A}_{r_i}$ are conformal maps, $i = 1, 2$. ϕ_i can be extended to the boundaries of A_i such that $\phi_i : I_i \to S^1$ and $\phi_i : O_i \to \{z \in \widehat{\mathbb{C}} : |z| = r_i\}$ are differentiable. Then

$$H_1 := \phi_2 \circ h_1 \circ \phi_1^{-1} : S^1 \to S^1$$

and

$$H_2 := \phi_2 \circ h_2 \circ \phi_1^{-1} : \{z \in \widehat{\mathbb{C}} : |z| = r_1\} \to \{z \in \widehat{\mathbb{C}} : |z| = r_2\}$$

are both d-fold differentiable covering maps. Let $P(z) = z^d$. There exist diffeomorphisms

$$Q_1 : S^1 \to S^1$$

and

$$Q_2 : \{z \in \widehat{\mathbb{C}} : |z| = r_1\} \to \{z \in \widehat{\mathbb{C}} : |z| = r_2^{1/d}\}$$

such that $P \circ Q_1 = H_1$ on S^1 and $P \circ Q_2 = H_2$ on $\{z \in \widehat{\mathbb{C}} : |z| = r_1\}$.

We can construct a quasi-conformal map $H : \overline{\mathbb{A}}_{r_1} \to \overline{\mathbb{A}}_{r_2^{1/d}}$ such that $H = H_1$ on $\{z \in \widehat{\mathbb{C}} : |z| = r_1\}$ and $H = H_2$ on $\{z \in \widehat{\mathbb{C}} : |z| = r_2^{1/d}\}$. Then

$$R = \phi_2^{-1} \circ P \circ H \circ \phi_1$$

is the desired map. □

LEMMA 3. *Assume $\{f_n\}$ is a sequence of rational maps with constant degree $d \geq 1$ and D is a non-empty open subset of $\widehat{\mathbb{C}}$. If f_n converges to a map f in D uniformly, then f is a restriction of a rational map of degree $d_0 \leq d$. Moreover, f_n converges to f in $\widehat{\mathbb{C}}$ uniformly provided $d_0 = d$.*

Proof. We may assume $\infty \in D$ and $f_n(\infty) \to 1$ as $n \to \infty$ through composing a Möbius transformation. For sufficiently large n, f_n has the form

$$f_n(z) = k_n \frac{(z - a_1^n) \cdots (z - a_d^n)}{(z - b_1^n) \cdots (z - b_d^n)},$$

where $k_n \to 1$ as $n \to \infty$. Both $\{|a_i^n|\}$ and $\{|b_i^n|\}$ are bounded from above. Choose a subsequence, f_{n_k}, $n_k \to \infty$, such that

$$(a_1^{n_k}, \ldots, a_d^{n_k}; b_1^{n_k}, \ldots, b_d^{n_k}) \to (a_1, \ldots, a_d; b_1, \ldots, b_d).$$

If $a_i \neq b_j$ for all $1 \leq i, j \leq d$, let

$$\widetilde{f}(z) = \frac{(z - a_1) \cdots (z - a_d)}{(z - b_1) \cdots (z - b_d)},$$

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and then f_{n_k} converges uniformly to $\widetilde{f}$ in $\widehat{\mathbb{C}}$. Since $f_n|_D$ converges uniformly to f, we have $f = \widetilde{f}$ in D and f_n converges uniformly to f in $\widehat{\mathbb{C}}$.

Otherwise, suppose $a_i \neq b_j$ for $1 \leq i, j \leq d_0$ and $a_k = b_k$ for $d < k \leq d_0$; let

$$\widetilde{f}(z) = \frac{(z-a_1)\cdots(z-a_{d_0})}{(z-b_1)\cdots(z-b_{d_0})}.$$

Then f_{n_k} converges to $\widetilde{f}$ in any compact set in $\widehat{\mathbb{C}} - X$, where $X \subset \widehat{\mathbb{C}}$ consists of the points $a_{d_0+1}, \ldots, a_d$. So, $f = \widetilde{f}$ in D is the restriction of a rational map of degree $d_0 < d$. □

For each $n \geq 0$, let U_n be the complement of the union of all puzzle pieces of depth n, that is

$$U_n = \widehat{\mathbb{C}} - \bigcup_{P_n \in \mathbf{P}_n} P_n.$$

Now, we will take advantage of the technique of quasi-conformal surgery to construct a sequence of d-fold quasi-regular maps $F_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying the following properties.

(1) Define $F_n = f$ in $\widehat{\mathbb{C}} - \bigcup_{c \in \mathrm{Crit}(f)} P_n(c)$ for each $n \geq 0$.

(2) For each $c \in \mathrm{Crit}(f)$, $n \geq 0$, take two simply connected domains $V_n, W_n \subset P_{n-1}(f(c))$ whose boundaries are smooth Jordan curves satisfying the following conditions.

   (i) $P_n(f(c))$ is compactly contained in V_n and V_n is disjoint from any other puzzle pieces of depth n contained in $P_{n-1}(f(c))$. In other words, $V_n - P_n(f(c)) \subset F(f)$.

   (ii) $W_n \supset V_n$ contains each puzzle piece $P_n \subset P_{n-1}(f(c))$ of depth n, that is, $P_{n-1}(f(c)) - W_n \subset F(f)$.

   By Lemma 1, there exists a $\deg_c f$-fold holomorphic proper map $g_n : \overline{P_{n+1}(c)} \to \overline{W_n}$ with $g_n(c) \in V_n - \overline{P_n(f(c))}$. Define $F_n = g_n$ in $P_{n+1}(c)$.

(3) Denote the annuli $P_n(c) - \overline{P_{n+1}(c)}$ and $P_{n-1}(f(c)) - \overline{W_n}$ by $A_n(c)$ and $A_{n-1}(f(c))$, respectively.

From the construction above, we know that f is defined on one boundary component $\partial P_n(c)$ of $A_n(c)$, and g_n is defined on another boundary component $\partial P_{n+1}(c)$ of $A_n(c)$. Therefore, by Lemma 2, there exists a quasi-regular map $R_n : A_n(c) \to A_{n-1}(f(c))$ with no branched points in $A_n(c)$ such that its boundary values are f and g_n. Define $F_n = R_n$ in $A_n(c)$ (see Figure 1).

It is easy to verify that the quasi-regular maps F_n we constructed above have an excellent property: for any $z \in \widehat{\mathbb{C}}$ and any $n \geq 0$, there is at most one point in $\mathrm{orb}(z) = \{z, F_n(z), F_n^2(z), \ldots\}$ located in the non-holomorphic domain of F_n. In other words, every point enters the non-holomorphic domain at most one time under the iteration.

LEMMA 4. (Surgery principle (see [**Ah**] and also [**Sh**])) *Suppose that $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map and σ is a bounded measurable conformal structure such that $g^*\sigma = \sigma$, almost everywhere outside a measurable set X. If each orbit of g passes through X at most a bounded number of times, then there exists a quasi-conformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $h = \varphi \circ g \circ \varphi^{-1}$ is a rational map.*

For the proof of this lemma, readers can consult [**Ah**, Appendix 2]. In our special situation, we may choose $\sigma = \sigma_0$, where σ_0 is the standard conformal structure.

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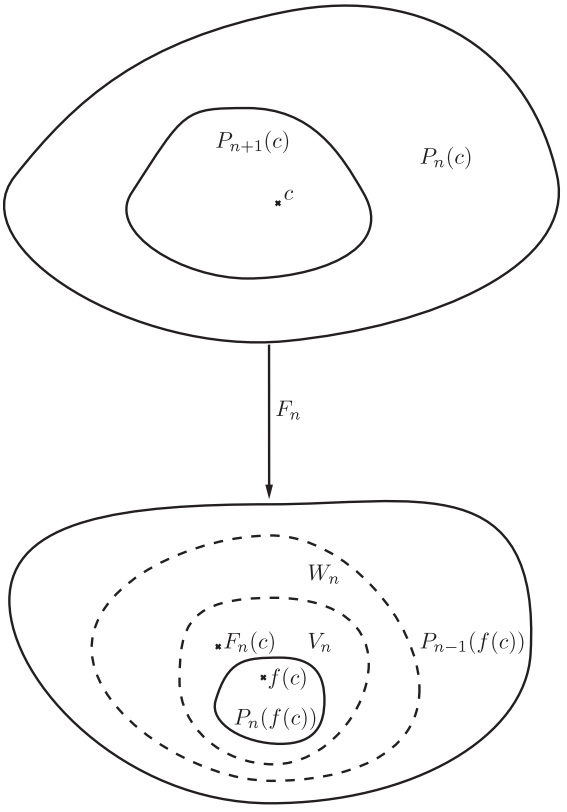


FIGURE 1. Quasi-conformal surgery.

Therefore, as a result of Lemma 4, we can find a sequence of quasi-conformal maps $\varphi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $f_n = \varphi_n \circ F_n \circ \varphi_n^{-1}$ are rational maps, and all the maps φ_n satisfy the normalization condition that $\varphi_n(\infty) = \infty$ and $\varphi_n'(\infty) = 1$; such a φ_n is exclusive for each n. Moreover, according to the proof of Lemma 4, we can make the conjugation φ_n holomorphic in U_n for each $n \geq 0$, that is, $\partial\varphi_n/\partial\bar{z} = 0$ in U_n.

Point y is called a branched point of a continuous map f if there is no neighborhood V of y such that $f|_V$ is injective. For a rational map f, the branched points are the critical points of f.

For each n, when constructing the quasi-regular map F_n, we can make all the critical points of f be branched points of F_n, and F_n has no other branched points.

For any $c \in \mathrm{Crit}(f)$, because of the construction of F_n, it is easy to see that $F_n^k(c) \to \infty$ as $n \to \infty$. Since f_n and F_n are quasi-conformally conjugate, the branched points of f_n tend to ∞ under the iteration. Thus all f_n are hyperbolic rational maps with Cantor Julia sets.

PROPOSITION 1. *The family* $\{\varphi_n\}$ *is normal in* U_0.

Proof. Take $R > 0$ large enough such that $\mathbb{D}(R)$ contains all the puzzle pieces of depth 0. Then $\{\varphi_n\}$ are conformal in $\widehat{\mathbb{C}} - \overline{\mathbb{D}(R)}$. For each n, we can assume that $\varphi_n(\infty) = \infty$

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and $\varphi_n'(\infty) = 1$, so we can normalize $\{\varphi_n\}$ such that

$$\varphi_n(z) = z + \sum_{k=1}^{\infty} \frac{b_k(n)}{z^k}$$

in a neighborhood of the infinity. Let

$$h_n(z) = \frac{1}{R}\varphi_n(Rz), \quad |z| > 1.$$

Then $h_n \in \sum_0$, a special class of univalent functions. It is well known that $|h_n(z)| \le 2|z|$ for $|z| > 1$; consequently, $|\varphi_n(z)| \le 2R|z|$ for $|z| > R$. Therefore, $\varphi_n(z)$ are locally uniformly bounded in $\widehat{\mathbb{C}} - \overline{\mathbb{D}(R)}$, and then $\{\varphi_n\}$ is a normal family in $\widehat{\mathbb{C}} - \overline{\mathbb{D}(R)}$.

On the other hand, the image of $\widehat{\mathbb{C}} - \overline{\mathbb{D}(R)}$ under $\omega = \varphi_n(z)$ covers the domain $|\omega| > 2R$, so $|\varphi_n(z)| \le 2R$ when $|z| \le R$ and $z \in U_0$. $\varphi_n(z)$ is uniformly bounded in $\overline{\mathbb{D}(R)} \cap U_0$.

Briefly speaking, $\{\varphi_n\}$ is a normal family in U_0. □

Using the same argument, we can prove that $\{\varphi_n\}$ is a normal family in U_n for each $n \ge 1$ by choosing subsequences again and again. Finally, we can find a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ such that $\{\varphi_{n_k}\}$ is a normal family in $F(f)$. So, we can choose a subsequence, say itself, such that φ_{n_k} is locally convergent to a conformal map φ in $F(f)$ since φ cannot be a constant map by $\varphi'(\infty) = 1$.

PROPOSITION 2. *$\{\varphi_{n_k}^{-1}\}$ is locally uniformly convergent to φ^{-1} in $\varphi(F(f))$.*

Proof. For arbitrary $\infty \ne \omega_0 \in \varphi(F(f))$, let $z_0 = \varphi^{-1}(\omega_0)$. For an appropriately small $\rho > 0$, φ_{n_k} are uniformly convergent to φ in $\overline{\mathbb{D}(z_0, \rho)} \subset F(f)$. Let $\gamma = \{z : |z - z_0| = \rho\}$ and $\delta = \min_{z\in\gamma} |\varphi(z) - \omega_0|$. There exists an integer $N \ge 1$ such that for all $k \ge N$, $|\varphi_{n_k}(z) - \varphi(z)| \le \delta/2$. Moreover, we notice that for any $\omega \in \mathbb{D}(\omega_0, \delta/2)$, $|\varphi(z) - \omega| > \delta/2$, $z \in \gamma$. By Rouché's theorem, for all $k \ge N$, we have

$$\left\{\omega : |\omega - \omega_0| < \frac{\delta}{2}\right\} \subset \varphi_{n_k}(\mathbb{D}(z_0, \rho)).$$

In other words, for any given $\omega_0 \in \varphi(F(f))$ and appropriately small ρ, there exists a positive number $\delta/2$ such that $\varphi_{n_k}(z)$ take each value ω in $\{\omega : |\omega - \omega_0| < \delta/2\}$ when $\{z : |z - z_0| < \rho\}$. This implies that $\varphi_{n_k}^{-1}(\omega)$ converges to $\varphi^{-1}(\omega_0)$ at $\omega = \omega_0$. Because ω_0 was selected arbitrarily, $\varphi_{n_k}^{-1}(\omega)$ is locally uniformly convergent to $\varphi^{-1}(\omega)$ in $\varphi(F(f))$. □

PROPOSITION 3. *$\{F_n\}$ is uniformly convergent to f in $\widehat{\mathbb{C}}$.*

The proof of Proposition 3 is trivial since $J(f)$ is a Cantor set.

PROPOSITION 4. *$\{f_n\}$ has a subsequence uniformly convergent to a rational map $\tilde{f}$ in $\widehat{\mathbb{C}}$.*

Proof. By the previous propositions, $f_{n_k} = \varphi_{n_k} \circ F_{n_k} \circ \varphi_{n_k}^{-1}$ converges uniformly to a map $\varphi \circ f \circ \varphi^{-1}$ in $\varphi(U_0)$. By Lemma 3, $\tilde{f} = \varphi \circ f \circ \varphi^{-1}$ is a restriction of a rational map. Note that $\deg(f : U_0 \to f(U_0)) = d$ and φ is univalent in U_0. Applying Lemma 3 again, we obtain that $\tilde{f}$ is a rational map of degree d and f_{n_k} converges uniformly to $\tilde{f}$ in $\widehat{\mathbb{C}}$. □

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Summing up the previous few propositions, we can easily see the following three.

(1) $\widetilde{f} = \varphi \circ f \circ \varphi^{-1}$ in $\varphi(F(f))$, where φ is conformal in $F(f)$.

(2) $\infty \in \varphi(F(f))$ is an attracting fixed point of $\widetilde{f}$ and $\varphi(F(f))$ is the completely invariant Fatou component of $\widetilde{f}$ containing ∞. We denote $\varphi(F(f))$ by $F_{\widetilde{f}}(\infty)$.

(3) The Julia set $J(\widetilde{f})$ of $\widetilde{f}$ is disconnected.

2.3. *Completion of the proof.* We define $K_{\widetilde{f}} = \widehat{\mathbb{C}} - F_{\widetilde{f}}(\infty)$.

CLAIM 1. *The Julia set $J(\widetilde{f})$ is a Cantor set.*

Proof. We can construct the Branner–Hubbard puzzle for $\widetilde{f}$ by the method introduced in §2.1. For each $\widetilde{x} \in K_{\widetilde{f}}$ and any $n \geq 0$, there is only one puzzle piece $\widetilde{P}_n(\widetilde{x}) \in \widetilde{\mathbf{P}}_\mathbf{n}$ containing $\widetilde{x}$. Thus each point $\widetilde{x} \in K_{\widetilde{f}}$ determines a nested sequence $\widetilde{P}_0(\widetilde{x}) \supset \widetilde{P}_1(\widetilde{x}) \supset \cdots$ and $\bigcap_{n\geq 0} \widetilde{P}_n(\widetilde{x}) = K_{\widetilde{f}}(\widetilde{x})$. Denote by $\mathcal{K}_f$ and $\mathcal{K}_{\widetilde{f}}$ all the connected components of $K_f = J(f)$ and $K_{\widetilde{f}}$, respectively. Now, we will construct a one-to-one mapping between $\mathcal{K}_f$ and $\mathcal{K}_{\widetilde{f}}$ using the conjugation $\psi = \varphi^{-1} : F_{\widetilde{f}}(\infty) \to F(f)$. Let $\widetilde{P}_0 \supset \widetilde{P}_1 \supset \cdots$ be any sequence of nested puzzle pieces and $\widetilde{K} = \bigcap_{n\geq 0} \widetilde{P}_n$. Denote by Γ_n the bounded component of $\widehat{\mathbb{C}} - \psi(\partial \widetilde{P}_n)$. Then, by the conjugacy of ψ, $\bigcap_{n\geq 0} \Gamma_n$ is a connected component of K_f, which will be denoted by K. For an arbitrary component $\widetilde{K} \in \mathcal{K}_{\widetilde{f}}$, define a map $\Psi : \mathcal{K}_{\widetilde{f}} \to \mathcal{K}_f$ such that $\Psi(\widetilde{K}) = K$. We define the degree of $\widetilde{f}$ in $\widetilde{K}$ as $\deg \widetilde{f}|_{\widetilde{K}} = \lim_{n\to\infty} \deg \widetilde{f}|_{\partial \widetilde{P}_n}$. Correspondingly, $\deg f|_K = \lim_{n\to\infty} \deg f|_{\partial \Gamma_n}$. It is easy to verify that Ψ is a one-to-one map and $\deg \widetilde{f}|_{\widetilde{K}} = \deg f|_K$ by the conformality of ψ. In order to prove that $J(\widetilde{f})$ is a Cantor set, we have to show that each component of $K_{\widetilde{f}}$ is a single point.

As a result of Theorem A, it is sufficient to show that $K_{\widetilde{f}}$ has no periodic critical component in our case. Assume that there is a component $\widetilde{K} \in \mathcal{K}_{\widetilde{f}}$ such that $f^n(\widetilde{K})$ is periodic and contains critical points for some $n \geq 0$. Without loss of generality, we can suppose that $\widetilde{K}$ is periodic and critical. By the construction of the one-to-one map Ψ, we know that $K = \Psi(\widetilde{K})$ is also a periodic critical component of K_f, but this contradicts Theorem A since $J(f)$ is a Cantor set. Therefore, there is no such periodic critical component of $K_{\widetilde{f}}$, which implies that $J(\widetilde{f})$ is also a Cantor set. □

First, we will extend φ to the whole Riemann sphere such that it is a topological conjugation between f and $\widetilde{f}$. For each $\widetilde{x} \in J(\widetilde{f})$, define $\psi(\widetilde{x}) = \bigcap_{n\geq 0} \Gamma_n(\widetilde{x})$, where $\Gamma_n(\widetilde{x})$ is the bounded component of $\widehat{\mathbb{C}} - \partial \widetilde{P}_n(\widetilde{x})$. $\bigcap_{n\geq 0} \Gamma_n(\widetilde{x})$ is a single point; denote it by x. It is easy to show that $\psi : J(\widetilde{f}) \to J(f)$ is a one-to-one map, ans so is the map $\varphi : J(f) \to J(\widetilde{f})$. For any positive number $\epsilon > 0$ and each $x_0 \in J(f)$, there exists an integer N such that $\operatorname{diam}(\widetilde{P}_n(\widetilde{x}_0)) < \epsilon$ when $n \geq N$. Let $\delta = d(x, \partial P_N(x_0))$. Then, for any x with $|x - x_0| < \delta$, we have $|\varphi(x) - \varphi(x_0)| < \epsilon$. So, φ is continuous in $\widehat{\mathbb{C}}$. Moreover, $\widetilde{f} \circ \varphi(x) = \varphi \circ f(x)$ for all $x \in J(f)$ by the continuity of φ. Hence, $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a topological conjugation between f and $\widetilde{f}$ which is conformal in $F(f)$. By Theorem B, we conclude that φ is a quasi-conformal conjugation; in fact, in particular, it is conformal in $F(f)$.

Let $\mu_\varphi(z)$ be the complex dilatation of the quasi-conformal map φ, $\mu_\varphi(z) = (\partial\varphi/\partial\bar{z})/(\partial\varphi/\partial z)$. The maps f and $\widetilde{f}$ are quasi-conformally conjugate via φ, $\varphi \circ f = \widetilde{f} \circ \varphi$. Then we can conclude that $f^*(\mu_\varphi) = \mu_\varphi$ and $|\mu_\varphi(f(z))| = |\mu_\varphi(z)|$ by a simple computation.

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Suppose that $|\mu_\varphi(z)| > 0$ in a subset E of $J(f)$ with positive measure; then we have $|\mu_\varphi(z)/|\mu_\varphi(z)|| = 1$ and

$$f^*\left(\frac{\mu_\varphi(z)}{|\mu_\varphi(z)|}\right) = \frac{\mu_\varphi(f(z))}{|\mu_\varphi(f(z))|} \cdot \frac{\overline{f'(z)}}{f'(z)} = \frac{\mu_\varphi(z)}{|\mu_\varphi(z)|}$$

in E. Thus $(\mu_\varphi(z)/|\mu_\varphi(z)|)(d\bar{z}/dz)$ is an invariant line field on $J(f)$. This contradicts Theorem C. It follows that $\mu_\varphi = 0$ almost everywhere in $\widehat{\mathbb{C}}$, therefore φ is conformal in $\widehat{\mathbb{C}}$ up to a set of zero measure.

Finally, we obtain a globally conformal conjugation φ between f and $\widetilde{f}$ satisfying the normalization condition in $\widehat{\mathbb{C}}$, which hints that the conjugation φ is the identity map and therefore $f = \widetilde{f}$. The proof of Theorem 1 in the attracting case is completed.

3. *Proof of Theorem 1 in the parabolic case*

In this section, we assume that the Julia set $J(f)$ of a rational map f is a Cantor set and the Fatou set $F(f)$ is a parabolic basin. We suppose that the origin is the parabolic fixed point and the point ∞ lies in the Fatou set. Let $C(f)$ denote the set of critical points of f and $P(f)$ be the post-critical set of f.

3.1. *Local behavior of the parabolic fixed point.*

In this subsection, we will review the definitions of petals and sepals associated with the parabolic point 0. For the proof of the statements and more details, one may refer to [**Cui, Mil**].

By the Leau–Fatou flower theorem, there exists a Jordan domain $P \subset F(f)$ such that the following properties hold.

(1) $0 \in \partial P$, $f(\overline{P}) \subset P \cup \{0\}$.
(2) $f : P \to f(P)$ is conformal.
(3) f^n locally uniformly in P converges to the origin as $n \to \infty$.
(4) For any $z \in F(f)$, there exists an integer N such that $f^N(z) \in P$.

Such a domain P is called an *attracting petal*.

There are two Jordan domains S_r and S_l in $F(f)$ such that:

(1) following the direction of the attracting axis, S_l and S_r locate on the left and right, respectively, and $0 \in \overline{S}_l \cap \overline{S}_r$; and
(2) $f(\overline{S}_l) = \overline{S}_l$ and $f(\overline{S}_r) = \overline{S}_r$.

We call S_l and S_r *left* and *right sepals*, respectively. The union of S_l and S_r is called a *sepal*.

In this paper, we deal with rational maps whose Julia sets are Cantor sets. Since there is only one Fatou component of such maps, clearly, there is a unique attracting petal attached to the parabolic fixed point. In other words, the parabolic fixed point here is not degenerate.

3.2. *Plumbing surgery.*

The surgery will be done in three steps.

Step 1: Choice of sepals. By the parabolic linearization theorem, there is a conformal embedding α from P into the complex plane $\mathbb{C}$ such that

$$\alpha(f(z)) = 1 + \alpha(z)$$

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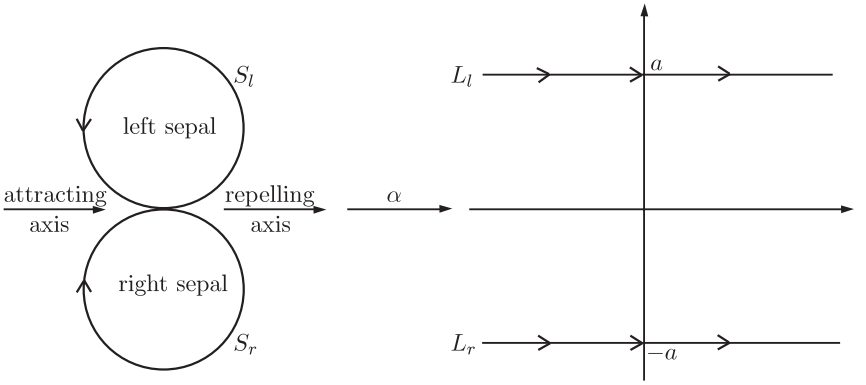


FIGURE 2. Left and right sepals.

for $z \in P$. The linearizing coordinate $\alpha(z)$ is usually referred to as an *attracting Fatou coordinate* in P. Define an equivalent relation in $F(f)$ by $z_1 \sim z_2$ if and only if $f^n(z_1) = f^m(z_2)$ for non-negative integers n, m. $P/\sim$ is conformally isomorphic to the infinite cylinder $\mathbb{C}/\mathbb{Z}$, which is called an *attracting cylinder*. We know that $\mathbb{C}/\mathbb{Z}$ is conformally isomorphic to $\mathbb{C}^*$. For simplicity, we identify $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$.

The map α can be extended to a holomorphic branched covering from $F(f)$ onto $\mathbb{C}$ such that

$$\alpha(f(z)) = 1 + \alpha(z)$$

for $z \in F(f)$. The set of the critical points of α is $\bigcup_{n\geq 0}\{f^{-n}(c) : c \in C(f)\}$. Take two horizontal lines $L_+ = \{z \in \mathbb{C} : \operatorname{Im} z = a > 0\}$ and $L_- = \{z \in \mathbb{C} : \operatorname{Im} z = -a\}$ disjoint from the critical values of α. Then there is a connected component of $\alpha^{-1}(L_+)$ (respectively $\alpha^{-1}(L_-)$) which is a Jordan arc landing at 0 bounding a *left sepal* (respectively a *right sepal*) S_l (respectively S_r) (see Figure 2). Let $S = S_l \cup S_r$. We can choose S small enough such that it is disjoint from $P(f)$.

$(F(f) - \bigcup_{n\geq 0}\{f^{-n}(z) : z \in P(f)\})/\langle f\rangle$ is a Riemann surface isomorphic to $\mathbb{C}^*$ with finitely many but at least one punctures, which are called marked points. Denote the projection by π_f. This is a holomorphic covering map. Note that each connected component of $\pi_f(S)$ is a 1-punctured disk.

Suppose $\pi_f(S_l) = \pi_f(S_r) = \mathbb{D}^* = \mathbb{D} - \{0\}$. Let $S(t) = \pi_f^{-1}(\mathbb{D}^*(t)) \cap S$, where $\mathbb{D}^*(t) = \mathbb{D}(t) - \{0\}$, $t < 1$. Then $S(t)$ is also a sepal. Let $S_l(t)$ (respectively $S_r(t)$) be the left (respectively right) sepal.

Step 2: Plumbing surgery in a petal. Define a map $h(z) = t/z$ in $\{z \in \mathbb{C} : t < |z| < 1\}$. Lift h by π_f and we get a conformal map, still denoted by h, from $S_l - S_l(t)$ onto $S_r - S_r(t)$ with $h \circ f = f \circ h$ in $S_l - S_l(t)$ (see Figure 3).

Take a vertical line β in the attracting cylinder which joins the two punctures 0 and ∞, and is disjoint from the marked points. There is a lift of β under π_f bounding an attracting petal U_0. Note that ∂U_0 intersects with each of ∂S_r, ∂S_l, $\partial S_r(t)$, $\partial S_l(t)$ at just one point (see Figure 3). We choose h such that it maps $\partial U_0 \cap (\overline{S_l} - \overline{S_l(t)})$ onto $\partial U_0 \cap (\overline{S_r} - \overline{S_r(t)})$. Define an equivalent relation on $U_0 - \overline{S(t)}$ by $z_1 \sim z_2$ if $z_1 = z_2$

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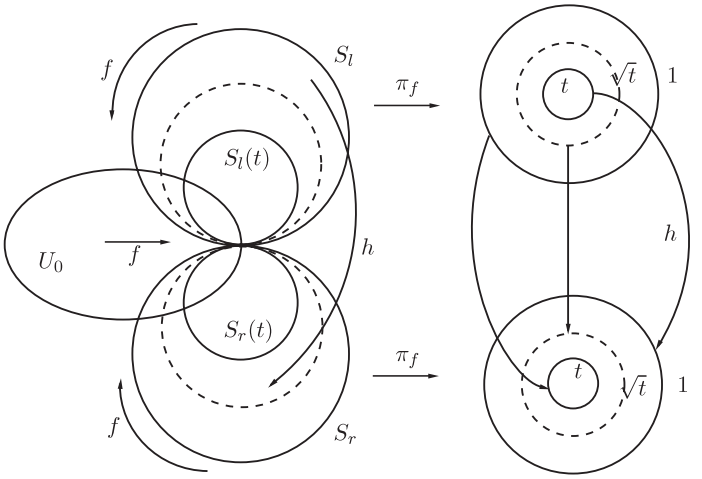


FIGURE 3. Definition of the map h.

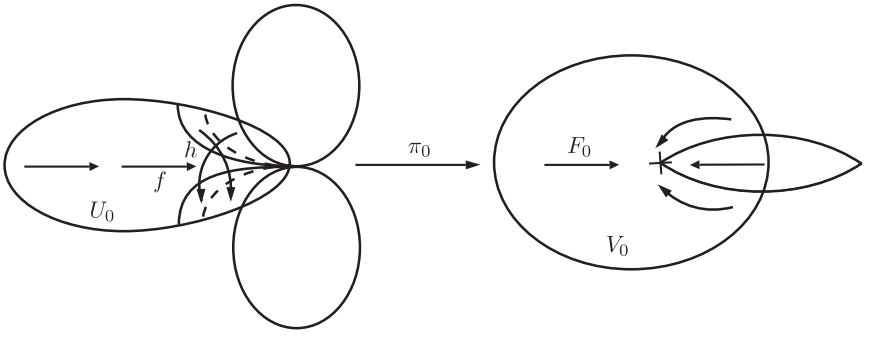


FIGURE 4. Plumbing surgery in U_0.

or $h(z_1) = z_2$. Then $U_0 - \overline{S(t)}/\sim$ is a Riemann surface with genus zero which is homeomorphic to a 1-punctured simply connected domain in $\mathbb{C}$. It is not difficult to prove that $U_0 - \overline{S(t)}/\sim$ is holomorphically isomorphic to a finitely connected domain V_0 in $\mathbb{C}$. Let $\pi_0 : U_0 - \overline{S(t)} \to V_0$ be the projection. Since $f \circ h = h \circ f$ in $S_l - S_l(t)$, there is a holomorphic map F_0 from V_0 onto $\pi_0(f(U_0) - \overline{S(t)})$ such that $F_0 \circ \pi_0 = \pi_0 \circ f$ in $U_0 - \overline{S(t)}$. Then $F_0(V_0) \subset\subset V_0$ and F_0 is conformal in V_0. So,

$$\sum_{k=0}^{\infty} \operatorname{mod}(F_0^k(V_0) - \overline{F_0^{k+1}(V_0)}) = \infty \tag{3.1}$$

and V_0 is a 1-punctured simply connected domain (see Figure 4).

Since h maps the unique closed geodesic in the annulus $\{z \in \mathbb{C} : t < |z| < 1\}$ to itself, then $\pi_0 : U_0 - \overline{S(\sqrt{t})} \to V_0$ is conformal, where $S(\sqrt{t}) = \pi_f^{-1}(\mathbb{D}^*(\sqrt{t})) \cap S$.

Step 3: Plumbing surgery in U_n. Take a sequence of numbers $\{t_n\}$ such that $1 = t_0 > t_1 > \cdots > 0$ and $\lim_{n\to\infty} t_n = 0$. Let U_n be the connected component of $f^{-n}(U_0)$

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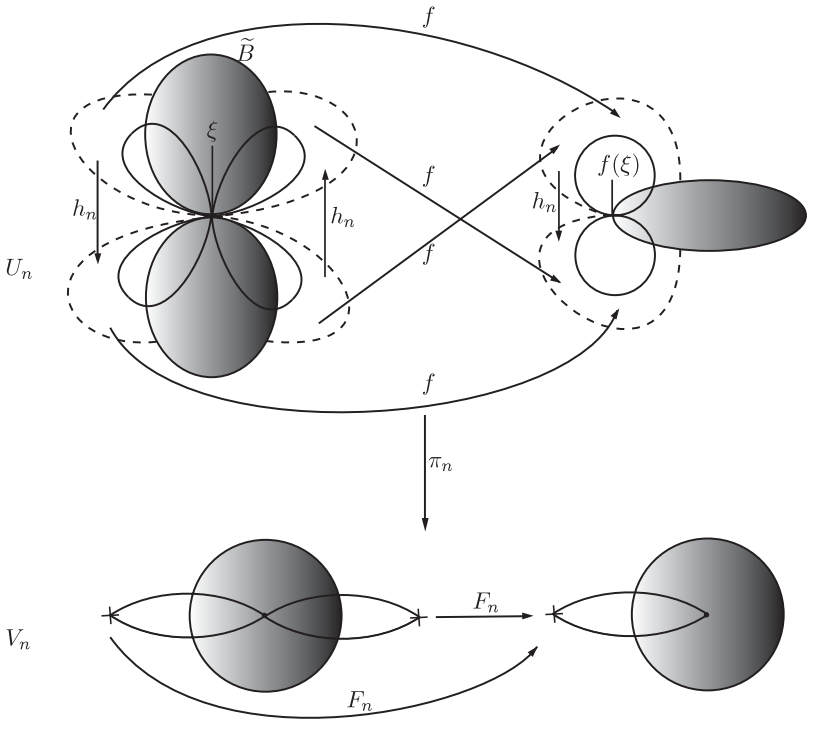


FIGURE 5. Surgery at the point $\xi \in f^{-n}(0)$.

containing U_0 for each $n > 0$. Then $U_{n-1} \subset U_n$, $F(f) = \bigcup_{n>0} U_n$ and $f : U_n \to U_{n-1}$ is a holomorphic proper map. There exists an integer N such that for all $n \geq N$:

(1) $\deg(f : U_n \to U_{n-1}) = \deg(f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}) = d$; and

(2) $F(f) \cap C(f) \subset U_n$.

Fix $n \geq N + 2$. Let $S(t_n) = \pi_f^{-1}(\mathbb{D}^*(t_n)) \cap S$, $S_l(t_n) \subset S(t_n)$ be the left sepal and $S_r(t_n) \subset S(t_n)$ be the right sepal with $\pi_f(S_l(t_n)) = \pi_f(S_r(t_n)) = \mathbb{D}^*(t_n)$. We can construct a conformal map $h_n : S_l - S_l(t_n) \to S_r - S_r(t_n)$ just as the construction of h. There is a conformal map, still denoted by h_n, mapping $f^{-n}(S_l) - f^{-n}(S_l(t_n))$ onto $f^{-n}(S_r) - f^{-n}(S_r(t_n))$ such that $f^n \circ h_n = h_n \circ f^n$ in $(f^{-n}(S_l) - f^{-n}(S_l(t_n))) \cap U_n$. Fix a point $\xi \in f^{-n}(0)$. Then $\xi \in \partial U_n$. Suppose ξ lies in the connected component B of ∂U_n. If $B - \{\xi\}$ is connected, then the map h_n is unique. Otherwise, h_n is not unique and we choose h_n as follows. Let A denote a connected component of $f^{-n}(S_l) - f^{-n}(S_l(t_n))$ with $\xi \in \partial A$. Then A intersects a connected component $\widetilde{B}$ of $B - \{\xi\}$. We choose h_n such that it maps A onto the connected component of $f^{-n}(S_r) - f^{-n}(S_r(t_n))$ which is adjacent to A and intersects $\overline{\widetilde{B}}$ at a unique point ξ; see Figure 5, where U_n and V_n denote the complements of the domains marked in gray.

Define an equivalence relation in $U_n - \overline{f^{-n}(S(t_n))}$ by $z_1 \sim z_2$ if $z_1 = z_2$ or $h_n(z_1) = z_2$. Then $U_n - \overline{f^{-n}(S(t_n))}/\sim$ is holomorphically isomorphic to a finitely connected domain V_n in $\mathbb{C}$. Let $\pi_n : U_n - \overline{f^{-n}(S(t_n))} \to V_n$ be the projection. There exists a holomorphic

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map F_n mapping V_n onto $\pi_n(U_{n-1} - \overline{f^{-(n-1)}(S(t_n))})$ such that $F_n \circ \pi_n = \pi_n \circ f$ in $U_n - \overline{f^{-n}(S(t_n))}$. Using the map F_n and equation (3.1), we can easily see that the map F_n can be extended to a holomorphic map in the union of V_n and those punctured points, and that it has an attracting fixed point. Note that π_n is conformal from $U_n - \overline{f^{-n}(S(\sqrt{t_n}))}$ onto V_n. Thus F_n is a holomorphic proper map of degree d (see Figure 5).

3.3. *Quasi-conformal surgery and approximation.* Now fix $n \geq N + 2$.

Let

$$\widetilde{V}_{n-1} = \pi_n(U_{n-1} - \overline{f^{-(n-1)}(S(t_n))}).$$

Using a similar method to that used in the attracting case, we can extend

$$F_n : \pi_n(U_{n-1} - \overline{f^{-(n-1)}(S(t_n))}) \to \pi_n(U_{n-2} - \overline{f^{-(n-2)}(S(t_n))})$$

to a quasi-regular map of degree d from $\widehat{\mathbb{C}}$ onto itself satisfying the following.

(1) For each branch point c of F_n, $F_n^k(c) \to \eta$ as $k \to \infty$, where η is the attracting fixed point.

(2) For each $z \in \widehat{\mathbb{C}}$, there is at most one point in the orbit $\{z, F_n(z), \ldots\}$ at which F_n is not holomorphic.

Then there exists a quasi-conformal map ϕ_n from $\widehat{\mathbb{C}}$ onto itself such that $f_n = \phi_n \circ F_n \circ \phi_n^{-1}$ is a rational map and $\partial\phi_n/\partial\bar{z} = 0$ in $\widetilde{V}_{n-1}$. It is not difficult to prove that f_n is a hyperbolic rational map with degree d and $J(f_n)$ is a Cantor set. Let $\Phi_n = \phi_n \circ \pi_n$. Then Φ_n is univalent in $U_{n-1} - \overline{f^{-(n-1)}(S(\sqrt{t_n}))}$ and $\Phi_n \circ f = f_n \circ \Phi_n$ in $U_{n-1} - \overline{f^{-(n-1)}(S(\sqrt{t_n}))}$. Moreover, we may choose appropriate π_n and ϕ_n such that Φ_n satisfies the normalization condition.

Since $\bigcup_{n>0} U_n = F(f)$, $\lim_{n\to\infty} t_n = 0$, and Φ_n is univalent in

$$U_{n-1} - \overline{f^{-(n-1)}(S(\sqrt{t_n}))} \quad \text{for all } n \geq N + 2,$$

we can prove that there exists a subsequence of $\{\Phi_n\}_{n\geq N+2}$, still denoted by $\{\Phi_n\}$, such that Φ_n locally uniformly converges to the univalent map Φ in $F(f)$ as $n \to \infty$. By Lemma 3, we easily conclude that f_n locally uniformly in $\widehat{\mathbb{C}}$ converges to a rational map $\widetilde{f}$ of degree d, and $\widetilde{f} = \Phi \circ f \circ \Phi^{-1}$ in $F(f)$. We claim that $\Phi(F(f))$ is a parabolic domain of $\widetilde{f}$. In fact, since $\widetilde{f}^{-1}(\Phi(F(f))) = \Phi(F(f))$, $\Phi(F(f))$ is contained in a Fatou domain V of $\widetilde{f}$. It is obvious that V is not a Siegel disk or a Herman ring. If V is an attracting domain, then $V/\langle\widetilde{f}\rangle$ is a torus. Since $\Phi(F(f))/\langle\widetilde{f}\rangle$ is an infinite cylinder, and a cylinder cannot be isomorphic to a torus or embedded in a torus, then V cannot be attracting. Hence, V is a parabolic domain and $V = \Phi(F(f))$.

3.4. *Completion of the proof in the parabolic case.*

CLAIM 2. *$J(\widetilde{f})$ is a Cantor Julia set.*

Proof. Since $\widetilde{f}^{-1}(V) = V$, by Theorem A, we just need to show that each periodic connected component of $\widehat{\mathbb{C}} - V$ does not contain any critical point of $\widetilde{f}$.

First, we construct the Branner–Hubbard puzzle for f in the parabolic case. Recall that 0 is the parabolic fixed point and ∞ is in the Fatou set. According to the Leau–Fatou flower theorem, there is a petal $U_0 \subset F(f)$ with $0 \in \partial U_0$ such that $\overline{U_0} \subset f^{-1}(U_0) \cup \{0\}$.

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Beginning from U_0, we will construct the puzzle as in the attracting case. There is an integer N_0 such that $f^{-n}(U_{N_0})$ has only one component for any $n \geq 0$. The set $f^{-n}(\widehat{\mathbb{C}} - \overline{U_{N_0}})$ is the disjoint union of a finite number of topological disks. For each $n \geq 0$, let $\mathbf{P_n}$ be the collection of all components of $f^{-n}(\widehat{\mathbb{C}} - \overline{U_{N_0}})$ which are called puzzle pieces of depth n. It is obvious that $(\bigcup_{k\geq 0} f^{-k}(0)) \cap \partial U_n = f^{-n}(0)$ because $\partial U_0 \cap J(f) = \{0\}$.

Recall that for the puzzle constructed in the attracting case, the boundaries of any two puzzle pieces do not touch each other, but here, for the parabolic case, the boundaries of two pieces may intersect. More precisely, given two pieces P_1 and P_2 with depths n_1 and n_2, respectively, assume $G := \partial P_1 \cap \partial P_2 \neq \emptyset$; if $n_1 = n_2$, then G consists of exactly one point z_0 contained in the set $(\bigcup_{m\geq 0} f^{-m}(C(f))) \cap f^{-(n_1+N_0)}(0)$; otherwise, assume $n_1 > n_2$, then $G \subset (f^{-(n_1+N_0)}(0) - f^{-(n_2+N_0)}(0)) \cup \{0\}$.

Each point $x \in J(f) - \bigcup_{n\geq 0} f^{-n}(0)$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$ and $\bigcap_{n\geq 0} P_n(x) = \{x\}$.

Take N_0 large enough such that U_{N_0} contains all critical points in the Fatou set and each puzzle piece contains at most one critical point.

Note that $\Phi(U_0)$ is a petal for $\widetilde{f}$. Beginning from $\Phi(U_0)$, we apply a similar method as above to construct the puzzle for $\widetilde{f}$. The property that $\widetilde{f} \circ \Phi = \Phi \circ f$ in $F(f)$ allows us to define a one-to-one correspondence induced by Φ between the puzzle pieces of the same depth for f and $\widetilde{f}$. Denote the correspondence by ξ. The definition of ξ is as follows. Let P be a puzzle piece of f. There is a unique puzzle piece satisfying that it has the same depth as P and its boundary contains $\Phi(\partial P \cap F(f))$. Set $\xi(P)$ to be this piece. Moreover, the univalence of the map Φ guarantees that $\deg(f : P \to f(P)) = \deg(\widetilde{f} : \xi(P) \to \xi(f(P)))$.

Let E be a periodic connected component of $\widehat{\mathbb{C}} - V$. Then either the parabolic point w_0 of $\widetilde{f}$ lies in ∂E or it does not. First suppose it does. By the condition that $J(f)$ is a Cantor set, we can conclude that:

(i) there exists a puzzle piece P with $0 \in \partial P$ such that $P \cap C(f) = \emptyset$; and

(ii) $\partial E \cap (\bigcup_{n\geq 0} \widetilde{f}^{-n}(w_0)) = \{w_0\}$.

Then it follows from (i) that $\xi(P)$ contains no critical points of $\widetilde{f}$, and from (ii) that $E - \{w_0\} \subset \xi(P)$. Hence, E contains no critical points of $\widetilde{f}$.

Suppose $w_0 \notin \partial E$. It is easy to see that E contains no points in $\widetilde{f}^{-n}(w_0)$ for all $n \geq 0$. So, we can find a sequence of puzzle pieces $\widetilde{P}_1 \supset \widetilde{P}_2 \supset \cdots \supset E$ for $\widetilde{f}$ such that $E = \bigcap_{n\geq 1} \widetilde{P}_n$ and $\widetilde{P}_1 - E$ contains no critical points of $\widetilde{f}$. For simplicity, assume $\widetilde{f}(E) = E$. There exists an integer $k_n \geq 1$ such that $\widetilde{f}^{k_n} : \widetilde{P}_{n+1} \to \widetilde{P}_n$. Then there is a sequence of puzzle pieces $\xi^{-1}(\widetilde{P}_1) \supset \xi^{-1}(\widetilde{P}_2) \supset \cdots$ for f such that $f^{k_n} : \xi^{-1}(\widetilde{P_{n+1}}) \to \xi^{-1}(\widetilde{P}_n)$ for all $n \geq 1$. We can easily prove that $\bigcap_{n\geq 1} \xi^{-1}(\widetilde{P}_n)$ is a connected component of $J(f)$ and hence a single point, denoted by ζ. Since $f^{k_n} : \xi^{-1}(\widetilde{P_{n+1}}) \to \xi^{-1}(\widetilde{P}_n)$, we have $f(\zeta) = \zeta$. Obviously, ζ is not a critical point and then $\xi^{-1}(\widetilde{P}_1)$ contains no critical points of f. So, $\widetilde{P}_1$ contains no critical points of $\widetilde{f}$ and hence E contains no critical points. □

Using the puzzles for f and $\widetilde{f}$, we can extend Φ to be a homeomorphism from $\widehat{\mathbb{C}}$ onto itself with $\widetilde{f} \circ \Phi = \Phi \circ f$ in $\widehat{\mathbb{C}}$. Then, by Theorem B, Φ is a quasi-conformal map. Combining with Theorem C, we can conclude that Φ is a Möbius map. Since Φ satisfies the normalization condition in $\widehat{\mathbb{C}}$, then Φ is equal to the identity map, that is $f = \widetilde{f}$.

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