STOCHASTIC DYNAMIC RESPONSE OF A SIMPLIFIED NONLINEAR FLUID MODEL FOR VISCOELASTIC MATERIALS

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ABSTRACT

In the present study, we propose a simplified nonlinear fluid model to characterize the complex nonlinear response of some viscoelastic materials. Recently, the viscoelastic modeling has been utilized by many researchers to simulate some parts of human body in bioengineering and to represent many material properties in mechanical engineering, electronic engineering and construction engineering. Occasionally it is almost impossible to evaluate the constant parameters in the model in the deterministic sense, therefore, the damping coefficient of the dashpot and the spring constants of the linear and nonlinear springs are considered as stochastic to model the stochastic properties of the viscoelastic materials. After some transformations, the closed-form solution can be obtained for the mean value of the displacement of the simplified nonlinear fluid model, subjected to constant rate of displacement. Based on the closed-form solution, the proposed method generates the stochastic dynamic response of the simplified nonlinear model, which plays an important role in performing the reliability analysis of the nonlinear system.

Keywords: Simplified nonlinear fluid model, Viscoelastic materials, Closed-form solution, Stochastic dynamic response.

1. INTRODUCTION

Because of its simplicity and closed-form solution, a linear spring-dashpot model is excellent to represent many materials. However, the linear model can not produce the nonlinear elastic responses which are exhibited in many viscoelastic materials (Drozdov [1]; Lakes [2]). Recently, the viscoelastic modeling has been used by many researchers to simulate some parts of human body in bioengineering. Among others, the following researchers have already contributed to the development of this field (Geil [3]; Weiss *et al*. [4]; Koop and Lewis [5]; Allen and Athanasiou [6]; Ciarletta *et al*. [7]; Guedes *et al*. [8]). Besides, the viscoelastic modeling has been adopted by many researchers to characterize material properties particularly in mechanical engineering, aerospace engineering, material engineering, electronic engineering, mechanics and construction engineering. Recently the following researchers have made contributions to the development of this field (Li and Weng [9]; Ju and Liu [10]; Wang and Xiao [11]; Kumar and Talreja [12]; Liu and Subhash [13]).

In the present study, we propose a simplified spring-dashpot model to characterize the complex nonlinear response of some viscoelastic materials. First of all, we use the standard Maxwell fluid model to get the softening behavior of the material, then we add a nonlinear spring in parallel to the linear spring as shown in Fig. 1. Occasionally it is almost impossible to evaluate the constant parameters in the model in deterministic sense, therefore, the damping coefficient of the dashpot and the spring constants of the linear and nonlinear springs are considered as stochastic to represent the stochastic properties of the viscoelastic materials. Of course, finding a closed-form solution to the proposed nonlinear model is quite challenging, the goal of this study is to get the stochastic dynamic responses of simplified nonlinear fluid model analytically or partially analytically.

2. MODEL DERIVATIONS

In Fig. 1, the following equation can be obtained by using equilibrium

$$c\dot{x} = Ld + Nd^2 \tag{1}$$

where x and d are the displacements of the dashpot and springs respectively. L denotes the linear spring constant and N defines the nonlinear spring constant, c is the damping coefficient of the linear dashpot.

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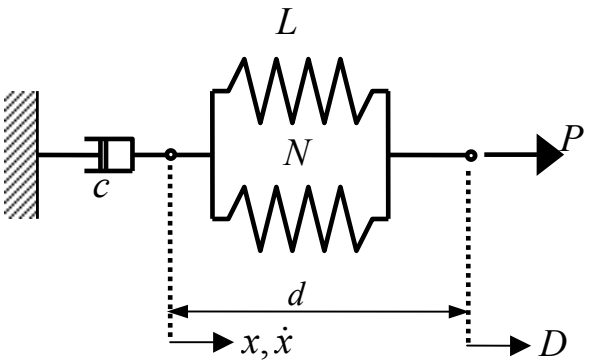


Fig. 1 A nonlinear model, consisting of a linear spring (L) and a nonlinear spring (N) in parallel, in series with a linear dashpot (c). This model is the extension of the model adopted in Ref. [15]

Suppose the model is under a stretched displacement at a constant rate a, the length equations can be written as

$$D = D_0 + a(t - t_0) \tag{2}$$

$$D = D_0 + x + d \tag{3}$$

where t is the time and the subscript denotes the start of elongation. Therefore, a test starting at time $t_0 = 0$ yields the following equation

$$at = x + d \tag{4}$$

Substituting Eq. (4) into Eq. (1) gives

$$c\dot{x} = L(at - x) + N(at - x)^2 \tag{5}$$

In the present investigation, the damping coefficient of the dashpot and the spring constants of the linear and nonlinear springs are considered as stochastic to simulate the random properties of the viscoelastic materials. All the stochastic parameters are assumed as Gaussian distributed. Applying the perturbation technique, the following equations can be written:

$$c = \bar{c} + \eta c_1 + \eta^2 c_2 + \ldots \tag{6}$$

$$L = \bar{L} + \eta L_1 + \eta^2 L_2 + \ldots \tag{7}$$

$$N = \bar{N} + \eta N_1 + \eta^2 N_2 + \ldots \tag{8}$$

where $\bar{c}$ is the mean value of the damping coefficient, η is a zero-mean small parameter and c_i are parameters of order i, the similar definitions can be denoted in Eqs. (7 ~ 8). By the same token, the displacement x of the springs can be written as follows:

$$x = \bar{x} + \eta x_1 + \eta^2 x_2 + \ldots \tag{9}$$

Substituting Eqs. (6 ~ 9) into Eq. (5) and applying the perturbation technique to Eq. (5), we can achieve the following equation based on the zero order of η:

$$\dot{\bar{x}} = A(t)\bar{x}^2 + B(t)\bar{x} + C(t) \tag{10}$$

where

$$A(t) = s \tag{11}$$

$$B(t) = -(r + 2sat) \tag{12}$$

$$C(t) = at(r + sat) \tag{13}$$

here $r = \bar{L}/\bar{c}$, $s = \bar{N}/\bar{c}$ are adopted in the above equations.

Based on the first order of η, we can obtain the following equation:

$$\dot{x}_1 + \frac{g(t)}{\bar{c}} x_1 = \frac{1}{\bar{c}} f(t) \tag{14}$$

where

$$g(t) = \bar{L} + 2at\bar{N} - 2\bar{N}\bar{x} \tag{15}$$

$$f(t) = L_1 at + N_1(at)^2 + N_1\bar{x}^2 - L_1\bar{x} - N_1(2at\bar{x}) - c_1\dot{\bar{x}} \tag{16}$$

It is noted that it is possible to obtain the closed-form solution for the displacement $\bar{x}$ in Eq. (10), then x_1 can be computed by using Eq. (14) readily. In general, the solution to Eq. (14) must be calculated numerically due to its complexity. Finally the statistical quantities of the displacement x can be computed by using Eq. (9).

3. CLOSED-FORM SOLUTION

Normally speaking, it is very difficult or almost impossible to find the closed-form solution to any nonlinear differential equation except some special problems. In our case, Eq. (10) is a nonlinear non-homogeneous first-order differential equation, which happens to be a typical Riccati equation. To solve it, first try a particular solution as follows:

$$\bar{x}_p = \alpha t + \beta \tag{17}$$

Substituting Eq. (17) into Eq. (10) yields

$$\alpha = s(\alpha t + \beta)^2 - (r + 2sat)(\alpha t + \beta) + at(r + sat) \tag{18}$$

Based on the zero order of t, we can obtain:

$$\alpha = s\beta^2 - r\beta \tag{19}$$

Based on the first order of t, we can get:

$$0 = 2s\alpha\beta - r\alpha - 2sa\beta + ra \tag{20}$$

Based on the second order of t, we can achieve:

$$0 = s\alpha^2 - 2sa\alpha + sa^2 \tag{21}$$

Solving the above equations simultaneously yields:

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$$\alpha = a \tag{22}$$

$$\beta = \frac{r \pm \sqrt{r^2 + 4sa}}{2s} \tag{23}$$

Obviously, $\bar{x}_p = at + \beta$ is a particular solution to Eq. (10).

The general solution contains the complementary solution and particular solution in the following:

$$\bar{x}(t) = \bar{x}_c(t) + \bar{x}_p(t) = \bar{x}_c(t) + at + \beta \tag{24}$$

Substituting Eq. (24) into Eq. (10) yields

$$a + \dot{\bar{x}}_c = s\beta^2 + 2s\beta\bar{x}_c + s\bar{x}_c^2 - r\beta - r\bar{x}_c \tag{25}$$

Using Eqs. (22) ~ (23), Eq. (25) becomes

$$\dot{\bar{x}}_c = (2s\beta - r)\,\bar{x}_c + s\bar{x}_c^2 \tag{26}$$

By letting $\bar{x}_c(t) = 1/\bar{z}(t)$, Eq. (26) can be transformed into a linear first order ordinary differential equation as follows:

$$\dot{\bar{z}}(t) + h\bar{z}(t) = -s \tag{27}$$

where $h = 2s\beta - r$

Utilizing the initial condition, $\bar{x}(0) = 0$, to Eq. (24), $\bar{z}(0) = -1/\beta$, and solving Eq. (27) yields

$$\bar{z}(t) = \left(\frac{s}{h} - \frac{1}{\beta}\right)e^{-ht} - \frac{s}{h} \tag{28}$$

The complementary solution gives

$$\bar{x}_c(t) = \frac{h\beta e^{ht}}{\beta s(1 - e^{ht}) - h} \tag{29}$$

Finally, the general closed-form solution is

$$\bar{x}(t) = at + \beta + \frac{h\beta e^{ht}}{\beta s(1 - e^{ht}) - h} \tag{30}$$

where

$\beta = \dfrac{r \pm \sqrt{r^2 + 4sa}}{2s}$

$r = \bar{L}/\bar{c} \quad s = \bar{N}/\bar{c}, \quad h = 2s\beta - r$

a = constant rate of displacement.

It is understood that once the closed-form solution for the mean value of the displacement has been obtained, it is fairly straightforward to compute the mean value of the velocity and force in closed-form.

4. STOCHASTIC DYNAMIC RESPONSE

After $\bar{x}$ and x_1 are estimated, the stochastic dynamic response of the proposed model can be calculated without any difficulties. For example, the variance of the displacement and the variance of the force can be expressed as follows:

$$Var\{x\} = \sigma_x^2 = E[\eta^2 x_1^2] = E[\eta^2](x_1^2) \tag{31}$$

$$Var\{P\} = \sigma_x^2 = E[\eta^2][c_1\dot{\bar{x}} + \bar{c}\dot{x}_1]^2 \tag{32}$$

where $E[\eta^2]$ is the mean square value of η, σ_x and σ_p denote the standard deviation of the displacement and standard deviation of the force respectively.

Besides, the coefficient of variation of the displacement is defined as:

$$\text{C.O.V. of } x = \frac{\sigma_x(t)}{\bar{x}(t)} \tag{33}$$

Needless to say, the coefficient of variation of the other quantities can be defined in a similar way.

5. VERIFICATION OF THE PRESENT MODEL

In order to verify the validity of the present model, all the parameters in the model were considered as deterministic, and the results of the present model were compared with some experimental results (Herzog and Leonard [14]) and the results from another model (Forcinito *et al*. [15]). In Forcinito's model, the system was composed of a linear spring in series with a linear dashpot, however, there was no nonlinear spring in his model. On the other hand, a second-order nonlinear spring was added in our model. In these experiments, the cat soleus forces were enhanced following stretching and were depressed following shortening compared to the corresponding isometric forces. The cat muscle was passively stretched to the initial length, activated until the isometric force was completely developed, then the muscle was allowed to shorten 4mm at a constant speed, and eventually was maintained isometrically at its final length while the activation was kept. In order to match the experimental results, it was quite necessary to choose the corresponding parameters of the present model $\bar{L}, \bar{N}$ and $\bar{c}$. It turned out that if $\bar{L} = 7.0$N/mm, $\bar{N} =$ 1.0N/mm/mm and $\bar{c} = 0.757$Ns/mm were adopted, then the results of the present model matched the experimental results and the results from Forcinito's model as shown in Figs. 2 ~ 5. Figure 2 shows the resulting fit for shortening speed of $a = -8$mm/s, it can be found that the present model is slightly better than Forcinito's model due to the additional nonlinear spring attached on the present model. In Fig. 3, the shortening speed is changed to $a = -20$mm/s and the results from the present study show good agreements with those from experiments and Forcinito's model. The same procedure as applied to the shortening contractions was adopted to match the lengthening contractions, and the results are presented in Figs. 4 ~ 5. Similar conclusions can be drawn as in Figs. 2 ~ 3.

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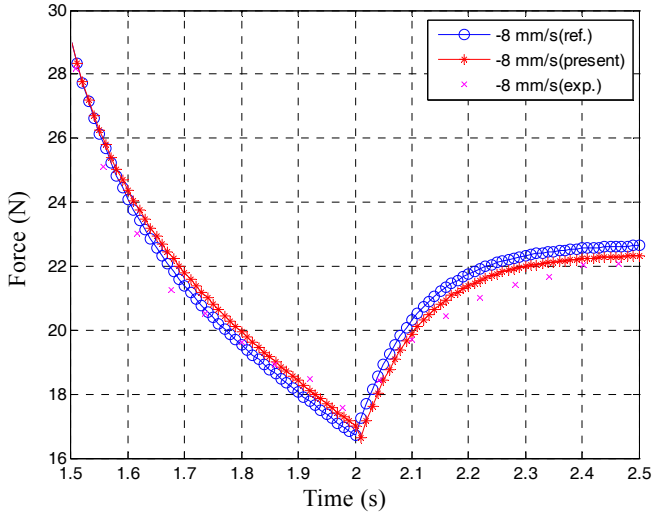


Fig. 2 Shorting at constant speed: Comparison between experimental results, the present model and Forcinito's model. (a = –8mm/s)

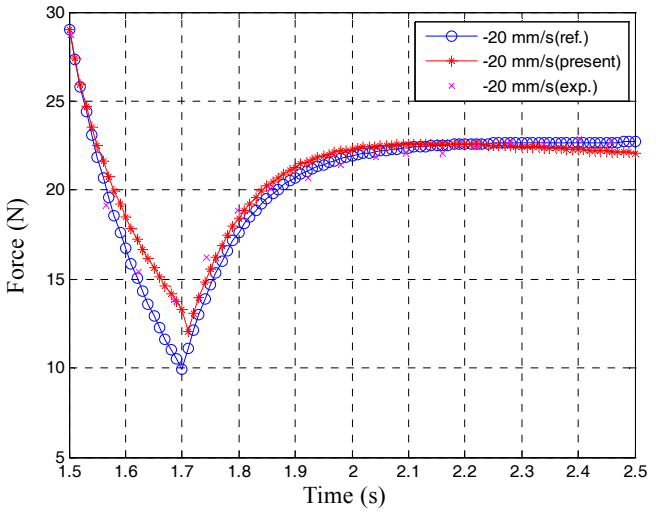


Fig. 3 Shorting at constant speed: Comparison between experimental results, the present model and Forcinito's model. (a = –20mm/s)

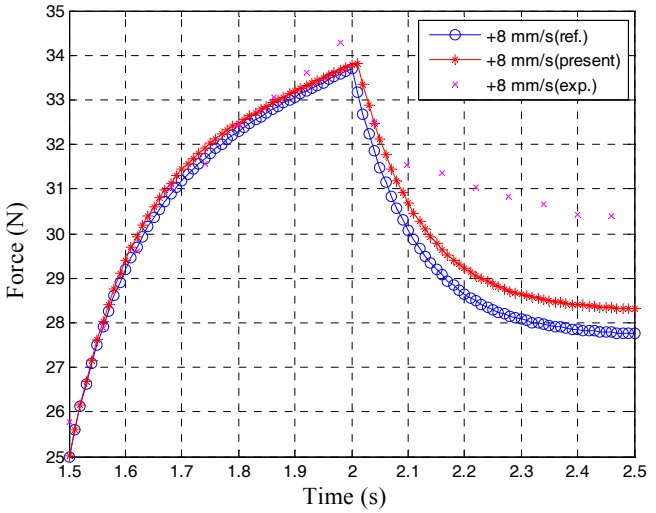


Fig. 4 Lengthening at constant speed: Comparison between experimental results, the present model and Forcinito's model. (a = +8mm/s)

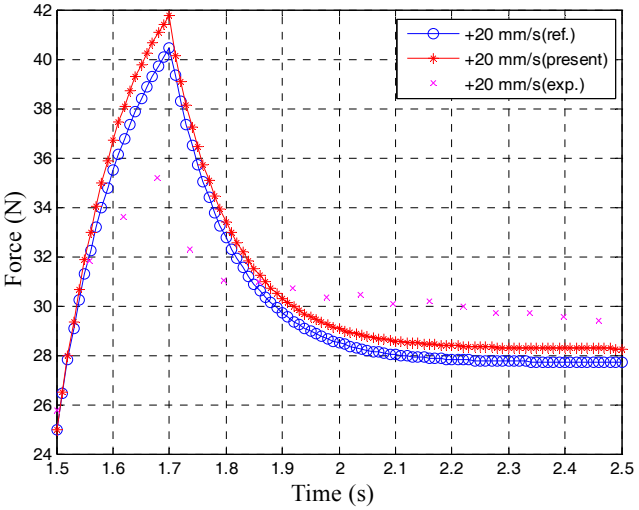


Fig. 5 Lengthening at constant speed: Comparison between experimental results, the present model and Forcinito's model. (a = +20mm/s)

6. SOME REMARKS

Generally speaking, it is quite easy to extend the present model into higher-order model without any difficulties. By adding higher order spring (such as third-order spring) to the linear and second-order springs in parallel, we can produce a system which is capable of expressing the responses with cubic components of the displacement. However, the closed-form solution for the mean value of the displacement will not be possible. Besides, we can also add higher order dashpot (such as second-order dashpot) to the linear dashpot in parallel to describe the responses with quadratic components of the velocity. Once again, there is no way to find the closed-form solution to the mean value of the displacement. It goes without saying that we can always estimate the solution to the higher-order model numerically, however, our goal is to present a simplified model to describe the nonlinear properties of the nonlinear system so that the closed-form solution to the mean value of the displacement is quite significant, since it is a basic solution for estimating the statistical response of some other quantities. Besides, if the stretched rate of the displacement is time dependent, then the numerical solution to the mean value of the displacement must be performed because getting the closed-form solution is out of the question. In order to get higher order statistical quantities, we must match the higher order of η, and then solve the equations successively; in that case numerical approximation will be unavoidable.

7. LIMITATIONS OF THE PROPOSED METHOD

As described in Section 3, the mean value of the displacement of the nonlinear model, subjected to constant

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rate displacement, can be solved analytically. However, there are some extreme cases in which the solutions become unstable. In order to find the stability limits, we increase a specific parameter by fixing all the other parameters until instabilities begin to occur. The stability limits of these extreme cases can be summarized in the following: (a) $\bar{L}/\bar{c} = 35.0$ (b) $\bar{N}/\bar{c} = 300.0$ (c) $a/\bar{c} = 300.0$. Therefore, it should be noted that whenever we want to utilize the present nonlinear model, we should always choose the parameter ratios which are less than the stability limits of the above three cases.

Another limitation is regarding the perturbation method, which is used to obtain the stochastic dynamic response of the nonlinear model. The approximate solutions solved by the perturbation techniques are valid only for the small values of the small parameters in most cases. For instance, the approximations computed by the method of multiple scales are uniformly valid provided that a specific system parameter is small as described in Ref. (Chang and Liu [16]). Therefore, it is fairly essential to check the validity of the approximations numerically or experimentally because there is no fixed way on how small the parameter should be. Generally speaking, in order to apply the perturbation technique, not only the small parameter must exist in the equation, but also its value must be appropriate and small enough to achieve a good approximation. Based on the proposed method, Eqs. 6 ~ 8 involve the small parameter η, therefore, the small parameter certainly exists in the equations. In the present study, η is assumed as a zero-mean small parameter and the standard deviation of the small parameter η is assumed as 0.1. First of all, the closed-form solutions for the mean values of the displacement, velocity and force are obtained, then the stochastic dynamic responses of the presented model are computed based on the closed-form solutions. Therefore, it is affirmed that the solutions based on the proposed method are accurate enough as long as the small parameter η is small enough. Actually, the stochastic dynamic responses based on the perturbation method are checked by Monte Carlo simulation, they show very good agreements. Monte Carlo simulation is described in details in the following section.

8. MONTE CARLO SIMULATION

Generally speaking, Monte Carlo simulation can be adopted to check the statistical quantities by some other methods, but is very costly since it requires tremendous repeated computations for all different samples, particularly when nonlinear problem is tackled. In the present study, the statistical dynamic responses obtained by the perturbation method are checked by Monte Carlo simulation. As stated before, the damping coefficient of the dashpot and the spring constants of the linear and nonlinear springs are considered as stochastic to characterize the random properties of the viscoelastic materials. All the stochastic parameters are assumed as Gaussian distributed. Therefore, we have to generate many different sample functions to describe the fluctuation of the random variable η, which will produce different damping coefficients of the dashpot and different linear and nonlinear spring constants. Then the nonlinear differential equation as expressed in Eq. 5 is solved many times due to many different sample functions. Eventually the statistical dynamic responses can be calculated from all these different dynamic responses. Here we use 1000 sample functions for the simulation process.

9. NUMERICAL COMPUTATIONS AND DISCUSSION

In the numerical computations, the following parameter values are used for describing the properties of the present model:

$\bar{L} = 10.0\text{N/cm}$, $\bar{N} = 10.0\text{N/cm/cm}$, $\bar{c} = 20.0\text{N-s/cm}$, $a = 1.0\text{cm/s}$, $E[\eta^2] = 0.01$.

All the stochastic dynamic responses are plotted with respect to time in the time duration of 2.0 second. In Figs. 6 ~ 8, the mean value, standard deviation and coefficient of variation (COV) of the displacement of the nonlinear model are presented with respect to time. As it can be seen from Fig. 6, the mean value of the displacement increases nonlinearly as time gets larger, which is very reasonable for the nonlinear model. In Fig. 7, the standard deviation of the displacement also increases nonlinearly with respect to time until it reaches the peak at $t = 1.2$sec, then it decreases nonlinearly to a local minimum at $t = 1.9$sec, soon after it increases again. It is noted that from Fig. 8 the coefficient of variation (COV) of the displacement fluctuates between 0.00 and 0.38. In Figs. 9 ~ 11, the mean value, standard deviation and coefficient of variation (COV) of the velocity of the nonlinear system are described with respect to time. As it can be found from Fig. 9, the mean value of the velocity increases nonlinearly as time gets larger, this is quite rational for the nonlinear model. In Fig. 10, it is fascinating to notice that the standard deviation of the velocity also increases nonlinearly with respect to time until it reaches the peak at $t = 0.6$sec, then it starts to decrease nonlinearly. In Fig. 11, the similar phenomena can be drawn as in Fig. 8 except the COV of the velocity is mostly lower than that of the displacement and fluctuates only between 0.00 and 0.19. Finally, in Figs. 12 ~ 14 we plot the mean value, standard deviation and coefficient of variation (COV) of the force of the nonlinear system with respect to time. In Fig. 12, the mean value of the force shows a similar trend as in Fig. 9, which is expected. In Fig. 13, it is quite interesting to notice that the standard deviation of the force increases nonlinearly with respect to time until it reaches the local maximum at $t = 0.25$sec, then it starts to decrease nonlinearly to $t = 0.5$sec, suddenly it increases all the

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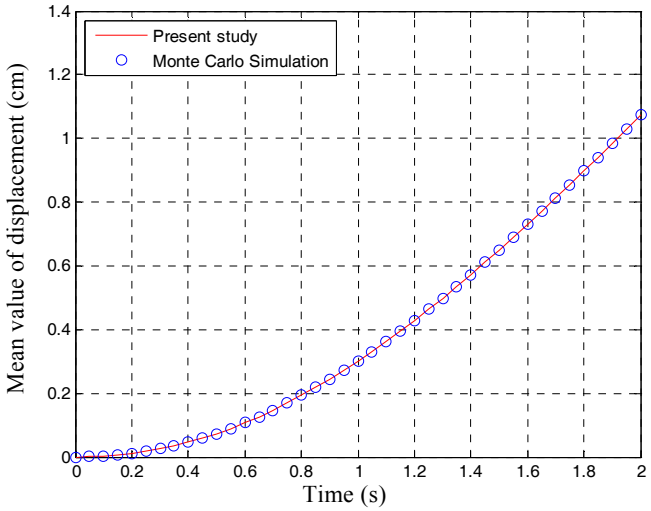


Fig. 6 Mean value of displacement with respect to time

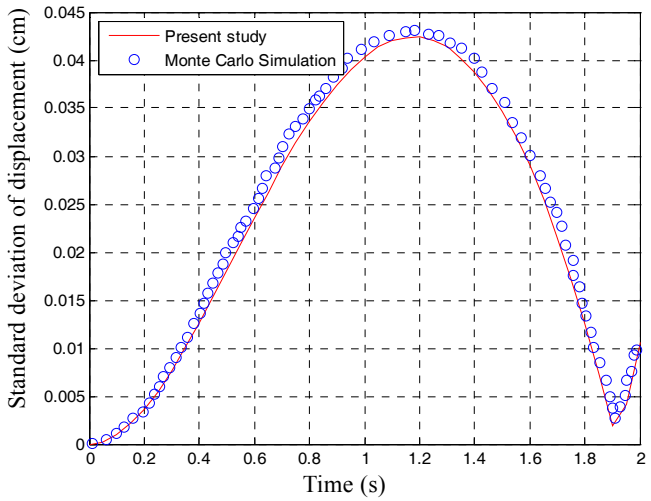


Fig. 7 Standard deviation of displacement with respect to time

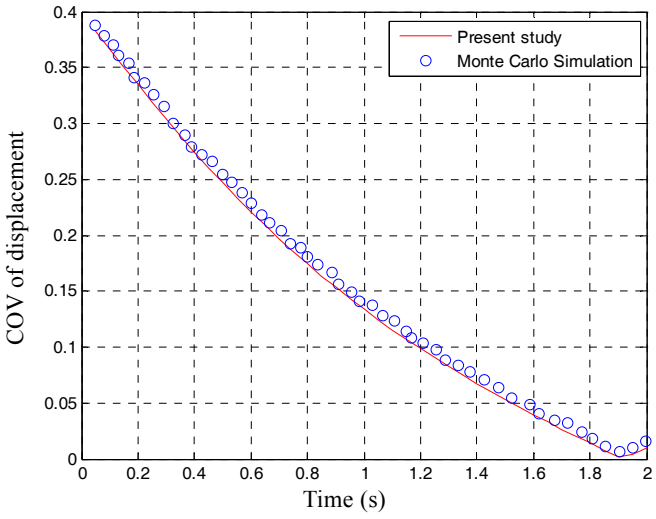


Fig. 8 COV of displacement with respect to time

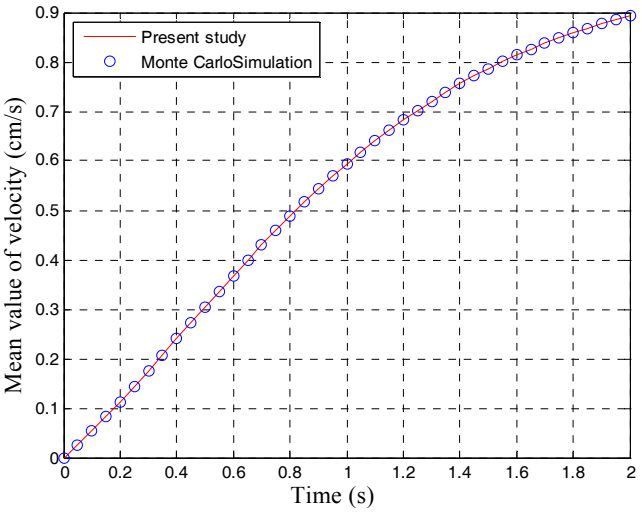


Fig. 9 Mean value of velocity with respect to time

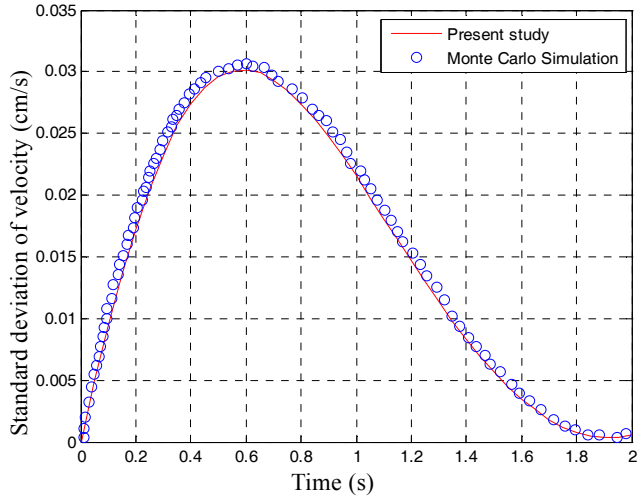


Fig. 10 Standard deviation of velocity with respect to time

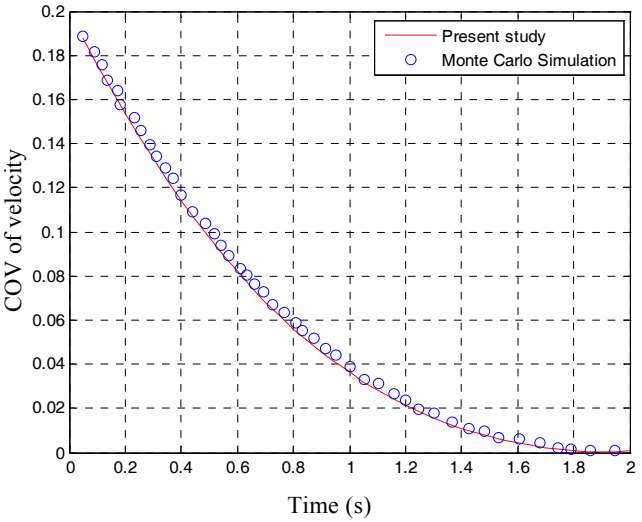


Fig. 11 COV of velocity with respect to time

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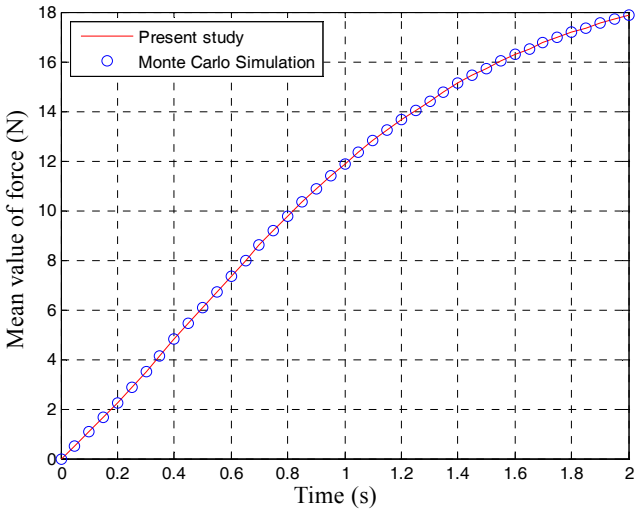


Fig. 12 Mean value of force with respect to time

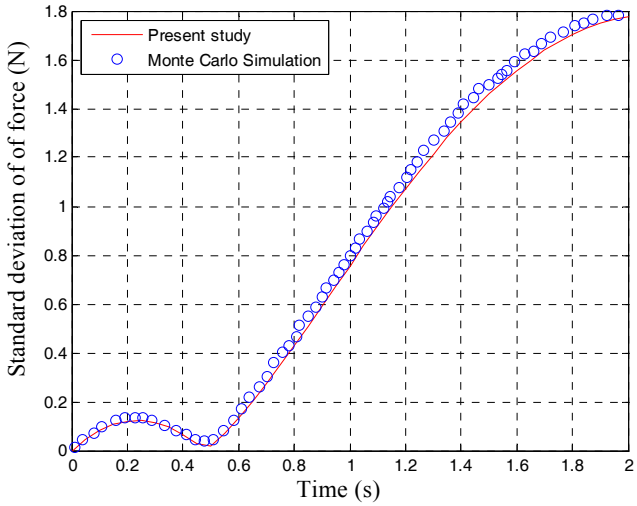


Fig. 13 Standard deviation of force with respect to time

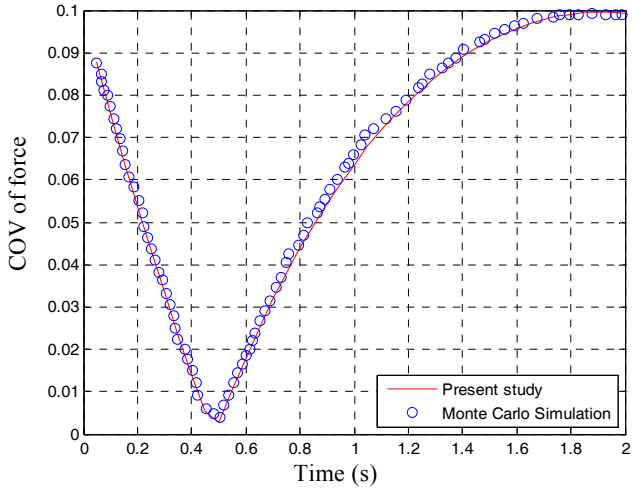


Fig. 14 COV of force with respect to time

way to $t = 2.0$sec. In Fig. 14, the COV of the force decreases from $t = 0.0$sec, then it reaches a local minimum at $t = 0.5$sec, soon later it starts to increase all the way to $t = 2.0$sec. The value of COV of force oscillates between 0.004 and 0.10.

All the stochastic dynamic responses based on the proposed method are checked by Monte Carlo simulation, they show very good agreements as detected from Figs. 6 ~ 14.

It is noticed from Figs. 6 ~ 14 that the mean values of the dynamic responses by the proposed method perfectly match those by Monte Carlo simulation; while the standard deviation and COV of the dynamic responses estimated by the proposed method are slightly lower than those by Monte Carlo simulation.

In order to understand the effect of the stochastic nature of damping coefficient on the stochastic dynamic responses such as displacement, velocity and force, Figs. 15 ~ 17 present the variation of COV of dynamic responses for different COV of damping coefficient. As it can be detected from Fig. 15, the COV of

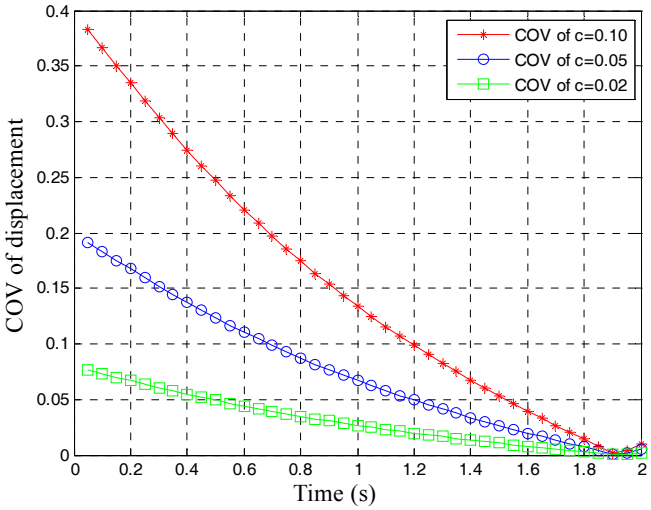


Fig. 15 Variation of COV of displacement for different values of COV of c

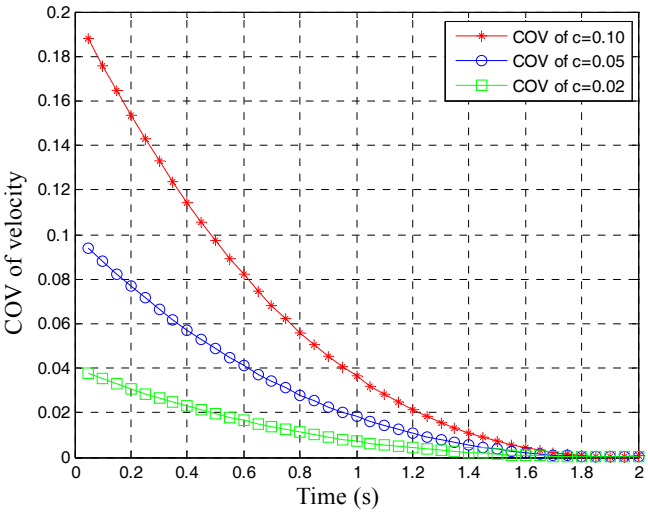


Fig. 16 Variation of COV of velocity for different values of COV of c

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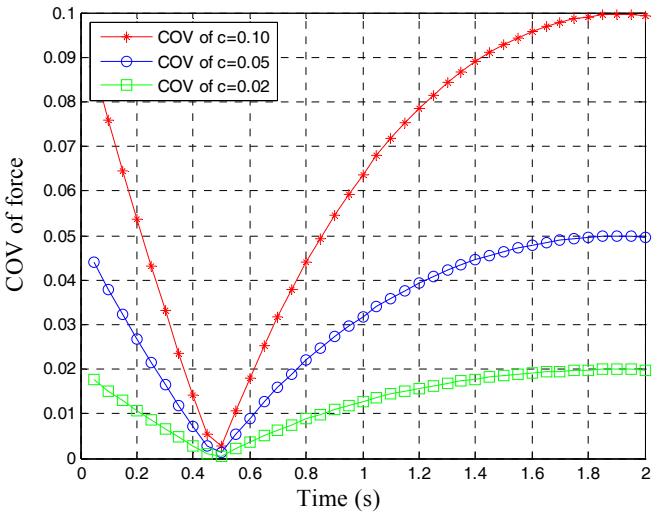


Fig. 17 Variation of COV of force for different values of COV of c

displacement gets larger as the COV of damping coefficient increases, which is quite reasonable. In Figs. 16 ~ 17, similar phenomena can be drawn for COV of velocity and force as the COV of damping coefficient gets larger.

10. CONCLUSIONS

In the present study, we propose a simplified nonlinear fluid model to characterize the complex nonlinear response of some viscoelastic materials. In the first place, we use the standard Maxwell fluid model to get the softening behavior of the material, then we add a nonlinear spring in parallel to the linear spring to produce a simplified nonlinear fluid model which is capable of simulating the properties of nonlinear viscoelastic materials in engineering. Occasionally it is almost impossible to evaluate the constant parameters in the model in deterministic sense, therefore, the damping coefficient of the dashpot and the spring constants of the linear and nonlinear springs are considered as stochastic to present the stochastic properties of the viscoelastic materials. After some transformations, the closed-form solution can be obtained for the mean value of the displacement of the simplified nonlinear fluid model, subjected to constant rate of displacement. Based on the closed-form solution, the proposed method generates the statistical responses of the simplified nonlinear fluid model. The calculated statistical responses are very useful in estimating the reliability of the simplified nonlinear fluid model.

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