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A GEOMETRIC APPROACH TO POLYNOMIAL PROPERTY FOR PATTERN SUBGROUPS IN $U_{a,b,c,d}$

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Abstract

Let $U_n(\mathbb{F}_q)$ be the unitriangular group and $U_{a,b,c,d}(\mathbb{F}_q)$ the four-block unipotent radical of the standard parabolic subgroup of GL_n, where $a+b+c+d=n$. In this paper, we study the class of all pattern subgroups of $U_{a,b,c,d}(\mathbb{F}_q)$. We establish character-number formulae of degree q^e for all these pattern groups. For pattern subgroups $G_{\mathcal{D}_m}(\mathbb{F}_q)$ in this class, we provide an algebraic geometric approach to their polynomial properties, which verifies an analogue of Lehrer's conjecture for these pattern groups.

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1. Introduction

Let $U_n(\mathbb{F}_q)$ be the unitriangular group of degree n over a finite field $\mathbb{F}_q$ of order q. Let $\mathcal{D} \subset \Delta_n := \{(i,j) \mid 1 \le i < j \le n\}$ be a closed subset in the sense that $(i,j),(j,k) \in \mathcal{D}$ implies $(i,k) \in \mathcal{D}$. A pattern group $G_{\mathcal{D}}(\mathbb{F}_q)$ is a subgroup of $U_n(\mathbb{F}_q)$ consisting of matrices whose (i,j) th entry is nonzero only if $(i,j) \in \mathcal{D}$. In particular, $U_n(\mathbb{F}_q)$ is a pattern group associated with Δ_n. The classification of all irreducible representations of all pattern groups is a wild problem. Even counting the number of irreducible characters of such groups is not easy. In fact, the following conjecture given by Higman [7] in 1960 is still open.

CONJECTURE 1.1 (Higman's conjecture). *For every positive integer n, the number of conjugacy classes in $U_n(\mathbb{F}_q)$ is a polynomial in q with integral coefficients.*

It is known that the dimension of a complex irreducible representation of a pattern group is always a power of q (see [9]). As suggested by Isaacs in [10], one may ask whether the function that counts the number of degree q^e irreducible characters of

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a pattern group is a polynomial in q with integral coefficients. This version of the question about pattern groups can be traced back to Lehrer [13], who proved that the maximal degree of irreducible characters of $\mathrm{U}_n(\mathbb{F}_q)$ is $q^{\mu(n)}$, where

$$\begin{cases}\mu(n) = m^2 & \text{if } n = 2m+1,\\ \mu(n) = m(m-1) & \text{if } n = 2m,\end{cases}$$

and gave the following conjecture.

CONJECTURE 1.2 (Lehrer's conjecture). *For $0 \le e \le \mu(n)$, the number of irreducible characters of $\mathrm{U}_n(\mathbb{F}_q)$ with degree q^e is a polynomial in q with integral coefficients.*

This conjecture strengthens Higman's conjecture and is also longstanding up to now.

For a pattern group $G_{\mathcal{D}}(\mathbb{F}_q)$, we say it has the polynomial property if the analogue of Lehrer's conjecture holds for it. We caution that not all pattern groups have the polynomial property: Halasi and Pálfy [6] gave a counterexample of a pattern group whose number of conjugacy classes is not a polynomial in q. However, there are still lots of pattern groups having the polynomial property. For example, Nien proved the polynomial property for two families of pattern groups: unipotent radicals of 3-block standard parabolic groups and 2-layered Heisenberg groups [14, 15].

Let $\mathrm{U}_{a,b,c,d}(\mathbb{F}_q)$ be the unipotent radical of the 4-block standard parabolic subgroup in $\mathrm{GL}_n(\mathbb{F}_q)$ associated to a, b, c, d with $a+b+c+d = n$, which is a pattern group. In this paper, we study the family of pattern subgroups in $\mathrm{U}_{a,b,c,d}(\mathbb{F}_q)$. Each pattern group in this family has a canonical semidirect product decomposition into two of its abelian pattern subgroups. Let $\mathrm{G} \subset \mathrm{U}_{a,b,c,d}(\mathbb{F}_q)$ be such a one with semidirect product decomposition $\mathrm{G} = \mathrm{G}_{\mathcal{H}} \ltimes \mathrm{G}_{\mathcal{A}}$. We define a map π_c from $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$, the dual space of $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})$, to the set $\mathrm{Mat}_h(\mathbb{F}_q)$ consisting of matrices of size h over $\mathbb{F}_q$, where h satisfies $q^h = |\mathrm{G}_{\mathcal{H}}|$. We give a number formula of degree q^e irreducible characters for $e \in \mathbb{N}$. Roughly speaking, this formula is a q-power product with $|\tilde{\mathfrak{C}}(\mathrm{G}, e)|$, where $\tilde{\mathfrak{C}}(\mathrm{G}, e)$ is the preimage of π_c over a subset of $\mathrm{Mat}_h(\mathbb{F}_q)$. The problem of polynomiality of G then translates into the problem whether $|\tilde{\mathfrak{C}}(e, \mathrm{G})|$ is a polynomial in q with integral coefficients for $e \in \mathbb{N}$. As an application, we give another proof of the polynomiality of unipotent radicals of 3-block standard parabolic groups proved by Nien in [15]; see Example 4.7.

Let $\mathcal{D}_m = \{(i,j) \in \Delta_n \mid j - i \ge m\}$. The associated pattern group series

$$\mathrm{U}_n(\mathbb{F}_q) = G_{\mathcal{D}_1}(\mathbb{F}_q) \ge G_{\mathcal{D}_2}(\mathbb{F}_q) \ge \cdots \ge G_{\mathcal{D}_m}(\mathbb{F}_q) \ge \cdots \ge G_{\mathcal{D}_n}(\mathbb{F}_q) = 1$$

is the lower central series of $\mathrm{U}_n(\mathbb{F}_q)$. For $(n/3) \le m < (n/2)$, $G_{\mathcal{D}_m}(\mathbb{F}_q)$ is a pattern subgroup of $\mathrm{U}_{0,m,n-2m,m}(\mathbb{F}_q)$. We give character number formulae for $G_{\mathcal{D}_m}(\mathbb{F}_q)$. In this case, the map π_c is a fibration, and the cardinality of the fibre is a power of q. The problem of polynomiality of $G_{\mathcal{D}_m}(\mathbb{F}_q)$ then becomes whether the counting function $|\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|$ has the polynomial property. Using algebraic geometry, we show that $|\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|$ is actually a polynomial in q with integral coefficients. We show the connection between $\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)$ and one-sided mixed ladder determinantal varieties,

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and the polynomiality of $|\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|$ is a consequence of the polynomial-counting property of such varieties.

The paper is organized as follows. In Section 2, we give necessary preliminaries for pattern groups, one-sided mixed ladder determinantal varieties, Schubert varieties and varieties X_v^u, Y_v^u. In Section 3, we show that the varieties occurring in Section 2 are all polynomial-counting. In Section 4, we review the Clifford theory and coadjoint orbit theory. We establish character number formulae of a given degree for all pattern subgroups of $U_{a,b,c,d}$. In Section 5, we study a special class of pattern subgroups $G_{\mathcal{D}_m}$ of $U_{a,b,c,d}$. We show the connection between these groups and one-sided ladder determinantal varieties, and prove the analogue to Lehrer's conjecture for $G_{\mathcal{D}_m}$.

Throughout this paper, we denote the sets of nonnegative integers and positive integers by $\mathbb{N}$ and $\mathbb{Z}_+$, respectively. For a finite set S, we denote by $|S|$ its cardinality. For a real number r, we denote the maximal integer no more than r by $[r]$. For a pattern group A, we denote the Lie algebra of A by $\mathrm{Lie}(A)$ and always treat it as a subalgebra of the nilpotent algebra consisting of upper triangular matrices with 0 on their diagonals. For a matrix A, we denote its transposition by A^t.

We use the word *variety* for a separated reduced scheme of finite type over an algebraically closed field $\bar{k}$. For a commutative ring R and a ring homomorphism $R \to \bar{k}$, we say a variety X can be defined over R if there is an R-scheme X_0 such that $X_0 \otimes_R \bar{k} \cong X$. When we say the R-rational points of X (denoted by $X(R)$), it means the R-valued points of X_0. If X and Y can be defined over R, we say a morphism $f : X \to Y$ is defined over R if $f = f' \otimes_R \bar{k}$, where $f' : X_0 \to Y_0$ is a morphism between X_0 and Y_0.

2. Preliminaries

Let k be a field. Denote the set of $n \times n$ matrices over k by $\mathrm{Mat}_n(k)$. Let $e_{i,j}$ be the matrix unit in $\mathrm{Mat}_n(k)$ whose entries are 0 except the (i,j) th entry, which is 1.

2.1. Pattern group. Let $\Delta_n = \{(i,j) \mid 1 \le i < j \le n\}$. A *closed* subset $\mathcal{D} \subset \Delta_n$ is a subset with the property that $\mathcal{D}$ contains (i,k) whenever it contains both (i,j) and (j,k). For any closed subset $\mathcal{D} \subset \Delta_n$, the *pattern group* $G_{\mathcal{D}}(k)$ is

$$G_{\mathcal{D}}(k) := \{1 + a_{ij}e_{i,j} \mid (i,j) \in \mathcal{D},\ a_{ij} \in k\}.$$

It is clear that

$$|G_{\mathcal{D}}(\mathbb{F}_q)| = q^{|\mathcal{D}|}.$$

Let $\mathrm{Lie}(G_{\mathcal{D}}(k))$ be the Lie algebra of $G_{\mathcal{D}}(k)$. We identify the dual space of $\mathrm{Lie}(G_{\mathcal{D}}(k))$ with $\mathrm{Lie}(G_{\mathcal{D}}(k))^t$ via the trace map

$$\mathrm{tr} : \mathrm{Mat}_n(k) \times \mathrm{Mat}_n(k) \to k, \quad (A,B) \mapsto \mathrm{tr}(A \cdot B).$$

2.2. Mixed ladder determinantal variety. We review some notation about mixed ladder determinantal varieties; for the details, see [4, 5].

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Let $X = (X_{ba})$, $1 \le b \le m$, $1 \le a \le n$, be an $m \times n$ matrix, whose entries X_{ba} are indeterminates. Given $1 = b_1 < \cdots < b_h < m$, $1 < a_1 < \cdots < a_h = n$, let

$$L = \{X_{ba} \mid \text{there exists } 1 \le i \le h \text{ such that } b_i \le b \le m, 1 \le a \le a_i\}$$

be a one-sided ladder in X. Set

$$L_i := \{X_{ba} \mid b_i \le b \le m, 1 \le a \le a_i\};$$

then $L = \bigcup_{i=1}^{h} L_i$. Here, we also regard L and L_i as submatrices of X.

Let $\bar{k}[L]$ be the polynomial ring $\bar{k}[X_{ba}, X_{ba} \in L]$ and $\mathbb{A}(L) := \mathbb{A}^{|L|}$ be the associated affine space. Fix $\underline{\mathbf{t}} = (t_1, \ldots, t_h) \in \mathbb{Z}_+^h$. Let I_i be the ideal of $\bar{k}[L]$ generated by the t_i-minors of X contained in L_i. Let $DV_{\underline{\mathbf{t}}}(L) \subset \mathbb{A}(L)$ be the variety defined by the ideal

$$I_{\underline{\mathbf{t}}}(L) := (I_1, \ldots, I_h) = \sum_{i=1}^{h} I_i.$$

We assume that the size of L_i is at least $t_i \times t_i$, that is, $1 \le t_i \le \min\{m - b_i + 1, a_i\}$. We also assume $I_i \not\subseteq I_j$ for $i \ne j$, and then we call $DV_{\underline{\mathbf{t}}}(L)$ a *one-sided mixed determinantal variety* associated with L and $\underline{\mathbf{t}}$. If $L = L_i$ and $\underline{\mathbf{t}} = (t_i)$, then $DV_{\underline{\mathbf{t}}}(L)$ becomes the *determinantal variety*. We simply denote it by $\mathrm{D}_{m,n}^{l}$, where L_i is of size $m \times n$ and $t_i = l$. It is clear that $DV_{\underline{\mathbf{t}}}(L)$ and $D_{m,n}^{l}$ can be defined over an arbitrary field.

2.3. Schubert variety and Bruhat cell. We give some preliminaries about Schubert varieties and Bruhat cells. For details, we refer to [1, 4, 8]. Let G be a simply connected simple algebraic group split over k. Let $T \subset G$ be a split maximal torus and B be a Borel subgroup of G defined over k that contains T. Let $W = N_G(T)/T$ be the Weyl group of G. Let S be the set of simple roots relative to (B, T). For a subset $S_Q \subset S$, denote by Q the standard parabolic subgroup associated to S_Q. Let W_Q be the Weyl group of Q, which is a parabolic subgroup of W. Let W^Q be the set of minimal length representatives of W/W_Q. For $w \in W$, we denote the coset wQ in G/Q by $e_{w,Q}$. We have *Bruhat decomposition*

$$G/Q = \bigsqcup_{w \in W^Q} Be_{w,Q}.$$

Each $Be_{w,Q}$ is called a *Bruhat cell*, which is isomorphic to an affine space. Let $X_Q(w)$ be the Zariski closure of $Be_{w,Q}$ in G/Q. Then, $X_Q(w)$ with the canonical reduced structure is called the *Schubert variety*. It is well known that the Schubert variety $X_Q(w)$ can be written as $X_Q(w) = \bigsqcup_{u \in W^Q, u \le w} Be_{u,Q}$, where $\le$ is the Bruhat order. Let B^- be the Borel subgroup of G opposite to B. We have opposite decomposition

$$G/Q = \bigsqcup_{w \in W^Q} B^- e_{w,Q}.$$

Here, $B^- e_{w,Q}$ is called an *opposite Bruhat cell*, which is also an affine space. Consider the canonical projection $\pi : G/B \to G/Q$; we have the following properties:

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(1) $\pi^{-1}(B^- e_{w,Q}) = \bigsqcup_{u \in wW_Q} B^- e_{u,B}$;
(2) for $w \in W^Q$, the restriction of π to $Be_{w,B}$ is an isomorphism onto $Be_{w,Q}$.

For $u, v \in W$, denote $X_u^v := Be_{u,B} \cap B^- e_{v,B}$ to be the intersection of $Be_{u,B}$ and $B^- e_{v,B}$; for $u, v \in W^Q$, define $Y_u^v := Be_{u,Q} \cap B^- e_{v,Q}$ to be the intersection of $Be_{u,Q}$ and $B^- e_{v,Q}$. Here, X_u^v and Y_u^v are nonempty if and only if $v \le u$. All varieties above can be defined over an arbitrary field.

3. Polynomial-counting variety

A variety X is called *polynomial-counting* if X can be defined over $\mathbb{Z}$ and for any finite field $\mathbb{F}_q$, $|X(\mathbb{F}_q)|$ is a polynomial in q with integral coefficients, where $X(\mathbb{F}_q)$ is the $\mathbb{F}_q$-rational points of X.

The following theorem was initially given by Kazhdan and Lusztig [11, Appendix, Lemma A4].

THEOREM 3.1. *The number of $\mathbb{F}_q$-rational points of X_u^v is determined by the R-polynomial $R_{v,u}(q) = |X_u^v(\mathbb{F}_q)|$. In particular, X_u^v is polynomial-counting.*

REMARK 3.2. Deodhar has given a finer decomposition of X_u^v. He proved that X_u^v is a disjoint union of varieties formed by $\mathbb{A}^n \times (\mathbb{A}^1 \setminus \{0\})^m$, where m, n are some nonnegative numbers. This decomposition is valid over $\mathbb{Z}$. Hence, $R_{v,u}(q)$ is actually a polynomial in $q-1$ with nonnegative integral coefficients. For the details, see [2, Theorem 1.1, 1.3 and Corollary 1.2].

COROLLARY 3.3. *Let $u, v \in W^Q$. Then, Y_u^v is polynomial-counting.*

PROOF. Fix a base field $\mathbb{F}_q$. Consider the varieties Y_u^v and X_u^v that are defined over $\mathbb{F}_q$. Noting that the isomorphism $\pi|_{Be_{u,B}} : Be_{u,B} \to Be_{u,Q}$ is defined over $\mathbb{F}_q$, we have

$$Y_u^v \simeq (\pi|_{Be_{u,B}})^{-1}(Y_u^v) = Be_{u,B} \cap (\pi^{-1}(B^- e_{v,Q})) = \bigsqcup_{w \in vW_Q, w \le u} X_u^w.$$

Hence,

$$|Y_u^v(\mathbb{F}_q)| = \sum_{w \in vW_Q, w \le u} |X_u^w(\mathbb{F}_q)|,$$

which is a polynomial in q with integral coefficients by Theorem 3.1. The proof is complete. □

Let $\bar{k}$ be the algebraic closure of k. Let $\mathrm{SL}(N, \bar{k})$ be the special linear group of size N over $\bar{k}$. Let T be the subgroup of $\mathrm{SL}(N, \bar{k})$ consisting of diagonal matrices and B be the subgroup of $\mathrm{SL}(N, \bar{k})$ consisting of upper triangular matrices. Then, T and B are a k-split maximal torus and a Borel subgroup containing T, respectively. Both T and B can be defined over k. Let W be the Weyl group of $\mathrm{SL}(N, \bar{k})$.

The following theorem given by Gonciulea and Miller gives a bridge between one-sided mixed ladder determinantal varieties and Schubert varieties in $\mathrm{SL}(N, \bar{k})/Q$.

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THEOREM 3.4 [4, Theorem 4.7.3]. *For a one-sided mixed ladder determinantal variety $DV_{\underline{\mathbf{t}}}(L)$ over $\bar{k}$, there exist integers N, r, a standard parabolic subgroup $Q \subset \mathrm{SL}(N, \bar{k})$ and a Schubert variety $X_Q(w)$ $(w \in W^Q)$ in $\mathrm{SL}(N, \bar{k})/Q$ such that $DV_{\underline{\mathbf{t}}}(L) \times \mathbb{A}^r$ is isomorphic to $B^- e_{id,Q} \cap X_Q(w)$.*

REMARK 3.5. The proof of the above theorem in [4] shows that the ideals of $B^- e_{id,Q} \cap X_Q(w)$ and $DV_{\underline{\mathbf{t}}}(L) \times \mathbb{A}^r$ share the same generators. Hence, the isomorphism in Theorem 3.4 is actually defined over k (and also defined over $\mathbb{Z}$).

COROLLARY 3.6. *Let $DV_{\underline{\mathbf{t}}}(L)$ be a one-sided mixed determinantal variety. Then, $DV_{\underline{\mathbf{t}}}(L)$ is polynomial-counting.*

PROOF. Note that $B^- e_{id,Q} \cap X_Q(w) = \bigsqcup_{u \le w, u \in W^Q} Y_u^{id}$. By Theorem 3.4 and Remark 3.5,

$$|DV_{\underline{\mathbf{t}}}(L)(\mathbb{F}_q)| \times q^r = \sum_{u \le W, u \in W^Q} |Y_u^{id}(\mathbb{F}_q)|. \tag{3-1}$$

By Corollary 3.3, the right-hand side of (3-1) is a polynomial in q with integral coefficients. Since $|DV_{\underline{\mathbf{t}}}(L)(\mathbb{F}_q)|$ is an integral number for any finite field $\mathbb{F}_q$, it is also a polynomial in q with integral coefficients, which shows that $DV_{\underline{\mathbf{t}}}(L)$ is polynomial-counting. □

4. Irreducible characters of pattern subgroups in $U_{a,b,c,d}$

4.1. Clifford theory and coadjoint orbits. Let us review Clifford theory and coadjoint orbit theory [12, 16]. Assume that the finite group $G = A \rtimes H$ is a semidirect product of A by H, where A is an abelian normal subgroup of G. Let $\widehat{A}$ be the set of all irreducible characters of A. Let $h \in H$ act on $\widehat{A}$ by $h \cdot \chi = \chi^h$, where $\chi^h \in \widehat{A}$ is given by

$$\chi^h(a) = \chi(h^{-1}ah) \quad \text{for } a \in A.$$

Let $H_\chi = \{h \in H \mid \chi^h = \chi\}$ be the stabilizer of χ in H. For $\chi \in \widehat{A}$, we identify it with a character of $A \rtimes H_\chi$ by $\chi(ah) := \chi(a)$, where $a \in A$ and $h \in H_\chi$. For $\tau \in \widehat{H_\chi}$, we identify it with a character of $A \rtimes H_\chi$ via the canonical projection $A \rtimes H_\chi \to H_\chi$. Then, the irreducible characters of G can be described in terms of induced characters of the following form.

THEOREM 4.1 [16, Section 8.2, Proposition 25]. *For $\chi, \chi' \in \widehat{A}$ and $\tau, \tau' \in \widehat{H_\chi}$, the following hold.*

(1) $\mathrm{Ind}_{A \rtimes H_\chi}^G \chi \otimes \tau$ *is an irreducible character of G.*
(2) $\mathrm{Ind}_{A \rtimes H_\chi}^G \chi \otimes \tau \cong \mathrm{Ind}_{A \rtimes H_{\chi'}}^G \chi' \otimes \tau'$ *if and only if χ and χ' are in the same H-orbit, and $\tau = \tau'$.*
(3) *Every irreducible character π of G is isomorphic to $\mathrm{Ind}_{A \rtimes H_\chi}^G \chi \otimes \tau$ for some $\chi \in \widehat{A}$ and $\tau \in \widehat{H_\chi}$.*

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Fix a nontrivial one-dimensional character ψ of k. Note that when $A = 1 + \mathrm{Lie}(A)$ is an abelian pattern group with subalgebra $\mathrm{Lie}(A) \subset \mathrm{Lie}(U_n(k))$; then

$$\widehat{A} = \{\psi_T \mid T \in \mathrm{Lie}(A)^t\},$$

where

$$\psi_T(x) := \psi(\mathrm{tr}Tx) \quad \text{for } x \in A.$$

DEFINITION 4.2. For a matrix $c = (c_{i,j}) \in \mathrm{Mat}_n(k)$, denote

$$\mathrm{Supp}(c) := \{(i,j) \in \Delta_n \mid c_{i,j} \neq 0\}.$$

For a subset $X \subset \mathrm{Mat}_n(k)$, denote

$$\mathrm{Supp}(X) := \bigcup_{c \in X} \mathrm{Supp}(c).$$

Let $\mathscr{P}$ be a pattern group such that $\mathscr{P} = A \rtimes H = (1 + \mathrm{Lie}(A)) \rtimes H$ with abelian pattern subgroups A and pattern subgroup H. We may identify the dual space of $\mathrm{Lie}(A)$ with $\mathrm{Lie}(A)^t$ via the trace map. The coadjoint action of $g \in H$ on $\alpha \in \mathrm{Lie}(A)^t$ in [12] is given by

$$(g \cdot \alpha)(a) = \mathrm{tr}(\alpha g^{-1} a g) \quad \text{for } a \in \mathrm{Lie}(A).$$

It can be identified with $\alpha \mapsto [g\alpha g^{-1}]_A$, where $[\cdot]_A$ is the projection from $\mathrm{Mat}_n(k)$ to $\mathrm{Lie}(A)^t$. More precisely, for $m = (m_{i,j}) \in \mathrm{Mat}_n(k)$,

$$([m]_A)_{i,j} = m_{i,j} \text{ if } (j,i) \in \mathrm{Supp}(\mathrm{Lie}(A)) \quad \text{and} \quad ([m]_A)_{i,j} = 0 \text{ if } (j,i) \notin \mathrm{Supp}(\mathrm{Lie}(A)).$$

Define the action of H on $\mathrm{Lie}(A)^t$ by

$$g \circ_A \alpha := [g\alpha g^{-1}]_A \quad \text{for } g \in H \text{ and } \alpha \in \mathrm{Lie}(A)^t.$$

Since $g \cdot \alpha(a) = \mathrm{tr}([g\alpha g^{-1}]_A a)$ for all $a \in \mathrm{Lie}(A)$, the action $\circ_A$ of H on $\mathrm{Lie}(A)^t$ coincides with the *coadjoint action*, whose orbits are called *coadjoint orbits*.

4.2. Pattern subgroups in $U_{a,b,c,d}$. Let

$$U_{a,b,c,d}(\mathbb{F}_q) := \left\{ \begin{pmatrix} \mathrm{Id}_a & A & B & C \\ 0 & \mathrm{Id}_b & D & E \\ 0 & 0 & \mathrm{Id}_c & F \\ 0 & 0 & 0 & \mathrm{Id}_d \end{pmatrix} \middle| \begin{array}{l} A \in \mathrm{Mat}_{a\times b}(\mathbb{F}_q), B \in \mathrm{Mat}_{a\times c}(\mathbb{F}_q), \\ C \in \mathrm{Mat}_{a\times d}(\mathbb{F}_q), D \in \mathrm{Mat}_{b\times c}(\mathbb{F}_q), \\ E \in \mathrm{Mat}_{b\times d}(\mathbb{F}_q), F \in \mathrm{Mat}_{c\times d}(\mathbb{F}_q) \end{array} \right\}$$

be the unipotent radical of a four-block standard parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$, where $a + b + c + d = n$. Let

$$U_{\mathcal{H}}(\mathbb{F}_q) := \left\{ \begin{pmatrix} \mathrm{Id}_a & A & 0 & 0 \\ 0 & \mathrm{Id}_b & 0 & 0 \\ 0 & 0 & \mathrm{Id}_c & F \\ 0 & 0 & 0 & \mathrm{Id}_d \end{pmatrix} \middle| A \in \mathrm{Mat}_{a\times b}(\mathbb{F}_q), F \in \mathrm{Mat}_{c\times d}(\mathbb{F}_q) \right\}$$

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and

$$\mathrm{U}_{\mathcal{A}}(\mathbb{F}_q) := \left\{ \begin{pmatrix} \mathrm{Id}_a & 0 & B & C \\ 0 & \mathrm{Id}_b & D & E \\ 0 & 0 & \mathrm{Id}_c & 0 \\ 0 & 0 & 0 & \mathrm{Id}_d \end{pmatrix} \middle| \begin{array}{l} B \in \mathrm{Mat}_{a\times c}(\mathbb{F}_q), C \in \mathrm{Mat}_{a\times d}(\mathbb{F}_q) \\ D \in \mathrm{Mat}_{b\times c}(\mathbb{F}_q), E \in \mathrm{Mat}_{b\times d}(\mathbb{F}_q) \end{array} \right\}$$

be two abelian pattern subgroups of $\mathrm{U}_{a,b,c,d}(\mathbb{F}_q)$. We have semidirect product decomposition:

$$\mathrm{U}_{a,b,c,d}(\mathbb{F}_q) \simeq \mathrm{U}_{\mathcal{A}}(\mathbb{F}_q) \rtimes \mathrm{U}_{\mathcal{H}}(\mathbb{F}_q).$$

Let G be a pattern subgroup in $\mathrm{U}_{a,b,c,d}(\mathbb{F}_q)$. Then, $\mathrm{G}_{\mathcal{H}} := \mathrm{G} \cap \mathrm{U}_{\mathcal{H}}(\mathbb{F}_q)$ and $\mathrm{G}_{\mathcal{A}} := \mathrm{G} \cap \mathrm{U}_{\mathcal{A}}(\mathbb{F}_q)$ are abelian pattern subgroups of G, and we also have semidirect product decomposition:

$$\mathrm{G} \simeq \mathrm{G}_{\mathcal{A}} \rtimes \mathrm{G}_{\mathcal{H}}.$$

4.3. Character-number formula of G. We simply write $U(X, Y, Z, W)$ for

$$\begin{pmatrix} 0_{a\times a} & 0 & 0 & 0 \\ 0 & 0_{b\times b} & 0 & 0 \\ X & Z & 0_{c\times c} & 0 \\ Y & W & 0 & 0_{d\times d} \end{pmatrix}.$$

Let

$$V(\mathbf{h}^1, \mathbf{h}^2) := \begin{pmatrix} \mathrm{Id}_a & \mathbf{h}^1 & 0 & 0 \\ 0 & \mathrm{Id}_b & 0 & 0 \\ 0 & 0 & \mathrm{Id}_c & \mathbf{h}^2 \\ 0 & 0 & 0 & \mathrm{Id}_d \end{pmatrix} \in \mathrm{G}_{\mathcal{H}},$$

where $\mathbf{h}^1 = (\mathbf{h}^1_{i,j})_{a\times b}$ and $\mathbf{h}^2 = (\mathbf{h}^2_{i,j})_{c\times d}$ are two matrices that take the values in $\mathbb{F}_q$ and satisfy $\mathbf{h}^k_{i,j} = 0$ $(k = 1, 2)$ if $(i, j) \notin \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{H}}))$. Let $U(X, Y, Z, W) \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$ and consider the coadjoint action of $V(\mathbf{h}^1, \mathbf{h}^2)$ on $U(X, Y, Z, W)$. To be more precise, that is,

$$\begin{aligned} &[V(\mathbf{h}^1, \mathbf{h}^2) \cdot U(X, Y, Z, W) \cdot V(-\mathbf{h}^1, -\mathbf{h}^2)]_{\mathrm{G}_{\mathcal{A}}} \\ &\quad = U(X, Y, Z, W) + [U(\mathbf{h}^2 Y, 0, \mathbf{h}^2 W - X\mathbf{h}^1 - \mathbf{h}^2 Y\mathbf{h}^1, -Y\mathbf{h}^1)]_{\mathrm{G}_{\mathcal{A}}}. \end{aligned}$$

Now, we want to determine the stabilizer of $U(X, Y, Z, W)$ in $\mathrm{G}_{\mathcal{H}}$. To do this, we consider the following equation:

$$[U(\mathbf{h}^2 Y, 0, \mathbf{h}^2 W - X\mathbf{h}^1 - \mathbf{h}^2 Y\mathbf{h}^1, -Y\mathbf{h}^1)]_{\mathrm{G}_{\mathcal{A}}} = 0. \tag{4-1}$$

It is clear that (4-1) gives *algebraic* equations in variables $\mathbf{h}^k_{i,j}$ $(k = 1, 2)$, where $(i, j) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{H}}))$. Then, the solutions of (4-1) give the stabilizer of $U(X, Y, Z, W)$ in $\mathrm{G}_{\mathcal{H}}$ under the coadjoint action, that is,

$$\{V(\mathbf{h}^1, \mathbf{h}^2) \mid (\mathbf{h}^1, \mathbf{h}^2) \text{ is a solution of (4-1)}\}.$$

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Also, we consider another equation as follows:

$$[U(\mathbf{h}^2Y, 0, \mathbf{h}^2W - X\mathbf{h}^1, -Y\mathbf{h}^1)]_{\mathrm{G}_{\mathcal{A}}} = 0. \tag{4-2}$$

Equation (4-2) forms a system of *linear* equations in the same variables as (4-1).

LEMMA 4.3. *Equations (4-1) and (4-2) share the same solutions over* $\mathbb{F}_q$.

PROOF. Note that the equation $[U(A, B, C, D)]_{\mathrm{G}_{\mathcal{A}}} = 0$ equals the system of equations

$$\begin{cases} [U(A,0,0,0)]_{\mathrm{G}_{\mathcal{A}}} = 0, \\ [U(0,B,0,0)]_{\mathrm{G}_{\mathcal{A}}} = 0, \\ [U(0,0,C,0)]_{\mathrm{G}_{\mathcal{A}}} = 0, \\ [U(0,0,0,D)]_{\mathrm{G}_{\mathcal{A}}} = 0, \end{cases}$$

due to the linearity of the projection $[-]_{\mathrm{G}_{\mathcal{A}}}$. It is enough to show that

$$[U(\mathbf{h}^2Y, 0, \mathbf{h}^2W - X\mathbf{h}^1 - \mathbf{h}^2Y\mathbf{h}^1, 0)]_{\mathrm{G}_{\mathcal{A}}} = 0 \tag{4-3}$$

and

$$[U(\mathbf{h}^2Y, 0, \mathbf{h}^2W - X\mathbf{h}^1, 0)]_{\mathrm{G}_{\mathcal{A}}} = 0 \tag{4-4}$$

share the same solutions over $\mathbb{F}_q$. We have

$$(4\text{-}3) \Leftrightarrow \begin{cases} [U(\mathbf{h}^2Y,0,0,0)]_{\mathrm{G}_{\mathcal{A}}} = 0, \\ [U(0,0,\mathbf{h}^2W - X\mathbf{h}^1 - \mathbf{h}^2Y\mathbf{h}^1, 0)]_{\mathrm{G}_{\mathcal{A}}} = 0, \end{cases}$$

and

$$(4\text{-}4) \Leftrightarrow \begin{cases} [U(\mathbf{h}^2Y,0,0,0)]_{\mathrm{G}_{\mathcal{A}}} = 0, \\ [U(0,0,\mathbf{h}^2W - X\mathbf{h}^1, 0)]_{\mathrm{G}_{\mathcal{A}}} = 0. \end{cases}$$

A single equation in $[U(0,0,\mathbf{h}^2W - X\mathbf{h}^1 - \mathbf{h}^2Y\mathbf{h}^1, 0)]_{\mathrm{G}_{\mathcal{A}}} = 0$, which is indexed by $(i,l) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t)$, is formed by

$$\sum_j \mathbf{h}^2_{i,j}w_{j,l} - \sum_j x_{i,j}\mathbf{h}^1_{j,l} - \sum_{k,(k,l)\in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{H}}))} \Big(\sum_j \mathbf{h}^2_{i,j}y_{j,k}\Big)\mathbf{h}^1_{k,l} = 0.$$

Now, we show that $\sum_j \mathbf{h}^2_{i,j}y_{j,k} = 0$ $((k,l) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{H}})))$ appears as a single equation in $[U(\mathbf{h}^2Y,0,0,0)]_{\mathrm{G}_{\mathcal{A}}} = 0$. It is enough to show $(i,k) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t)$. Note that

$$(i,l) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t) \quad \text{implies} \quad (l,i) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{A}}).$$

Additionally, we also have $(k,l) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{H}}))$, which implies $(k,i) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{A}}))$ since $\mathrm{G}_{\mathcal{A}}$ is normal. Hence, $(i,k) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t)$. Let $\mathbb{F}_q[\mathbf{h}^k_{i,j}]$ be the polynomial ring with indeterminates $\mathbf{h}^k_{i,j}$ over $\mathbb{F}_q$, where $(i,j) \in \mathrm{Supp}(\mathrm{Lie}(\mathrm{G}_{\mathcal{H}}))$. Now, consider the two ideals of $\mathbb{F}_q[\mathbf{h}^k_{i,j}]$ determined by (4-3) and (4-4). The arguments above show that these two ideals are actually the same, which finishes the proof. □

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We denote the coefficient matrix of (4-2) by $\mathscr{C}(X,Y,Z,W)$ under a fixed total order of $\mathbf{h}_{i,j}^k$ $(k=1,2)$ and observe that $\mathscr{C}(X,Y,Z,W)$ does not depend on Z. Define

$$\mathfrak{C}(\mathrm{G}) := \{\mathscr{C}(X,Y,Z,W) \mid U(X,Y,Z,W) \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t\}.$$

Let $\pi_c : \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t \to \mathfrak{C}(\mathrm{G})$ be the canonical map via $U(X,Y,Z,W) \mapsto \mathscr{C}(X,Y,Z,W)$. For $e \in \mathbb{N}$, define

$$\mathfrak{C}(\mathrm{G},e) := \{\mathscr{C}(X,Y,Z,W) \in \mathfrak{C}(\mathrm{G}) \mid \mathrm{rank}(\mathscr{C}(X,Y,Z,W)) = e\}.$$

Then, $\mathfrak{C}(\mathrm{G}) = \bigcup_{e=0}^{\infty} \mathfrak{C}(\mathrm{G},e)$ is a finite partition of $\mathfrak{C}(\mathrm{G})$. Denote the fibre of π_c over $\mathfrak{C}(\mathrm{G},e)$ by $\tilde{\mathfrak{C}}(\mathrm{G},e)$.

PROPOSITION 4.4 (Characters number formula). *Let* G *be a pattern subgroup of* $\mathrm{U}_{a,b,c,d}(\mathbb{F}_q)$. *Then, the number of irreducible characters of degree* q^e *is*

$$q^{-2e} \cdot |\tilde{\mathfrak{C}}(\mathrm{G},e)| \cdot |\mathrm{G}_{\mathcal{H}}|.$$

Furthermore, the set of integers that occur as degrees of irreducible characters of G *is precisely* $\{q^e \mid \mathfrak{C}(\mathrm{G},e) \neq \emptyset\}$.

PROOF. Fix a nontrivial one-dimensional character ψ of $\mathbb{F}_q$. Then we have that $\hat{\mathrm{G}}_{\mathcal{A}} = \{\psi_a \mid a \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t\}$, where

$$\psi_a(x) := \psi(\mathrm{tr}\, ax) \quad \text{for } x \in \mathrm{G}_{\mathcal{A}},$$

which identify $\hat{\mathrm{G}}_{\mathcal{A}}$ with $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$ via $\Psi : \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t \to \hat{\mathrm{G}}_{\mathcal{A}}, a \mapsto \psi_a$. Moreover, Ψ is compatible with the action of $\mathrm{G}_{\mathcal{H}}$ on $\hat{\mathrm{G}}_{\mathcal{A}}$ and the coadjoint action of $\mathrm{G}_{\mathcal{H}}$ on $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$. To be more precise, we have a commutative diagram as follows:

$$\begin{array}{ccc} \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t \times \mathrm{G}_{\mathcal{H}} & \xrightarrow{\Psi\times\mathrm{id}} & \hat{\mathrm{G}}_{\mathcal{A}} \times \mathrm{G}_{\mathcal{H}} \\ \Big\downarrow{\scriptstyle \text{coadjoint action}} & & \Big\downarrow \\ \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t & \xrightarrow{\Psi} & \hat{\mathrm{G}}_{\mathcal{A}} \end{array}$$

Below, we use Theorem 4.1 to construct the irreducible characters of G. Under the identification of $\hat{\mathrm{G}}_{\mathcal{A}}$ and $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$, we no longer distinguish between the coadjoint orbit $\mathbb{O}_a$ in $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$ and its corresponding orbit $\mathbb{O}_{\psi_a}$ in $\hat{\mathrm{G}}_{\mathcal{A}}$, as well as the stabilizer subgroups of $a \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$ and $\psi_a \in \hat{\mathrm{G}}_{\mathcal{A}}$. For a coadjoint orbit $\mathbb{O}_a$ with $a \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$, let $\mathrm{G}_{\mathcal{H}a}$ be the stabilizer of a in $\mathrm{G}_{\mathcal{H}}$. Clearly, $\mathrm{G}_{\mathcal{H}a}$ is an abelian group (but usually not a pattern group) and $\hat{\mathrm{G}}_{\mathcal{H}a}$ consists of $|\mathrm{G}_{\mathcal{H}a}|$ one-dimensional irreducible characters. Let τ be an irreducible character of $\mathrm{G}_{\mathcal{H}a}$, which is one-dimensional. Theorem 4.1(1) tells us, $\mathrm{Ind}_{\mathrm{G}_{\mathcal{A}}\rtimes\mathrm{G}_{\mathcal{H}a}}^{\mathrm{G}} \psi_a \otimes \tau$ is an irreducible character of G of degree $[\mathrm{G}_{\mathcal{H}} : \mathrm{G}_{\mathcal{H}a}]$, where $[\mathrm{G}_{\mathcal{H}} : \mathrm{G}_{\mathcal{H}a}]$ is the index of $\mathrm{G}_{\mathcal{H}a}$ in $\mathrm{G}_{\mathcal{H}}$. Theorem 4.1(2) tells us that we can construct $|\mathrm{G}_{\mathcal{H}a}|$ pairwise nonisomorphic irreducible characters associated with $\mathbb{O}_a$, that is, $\{\mathrm{Ind}_{\mathrm{G}_{\mathcal{A}}\rtimes\mathrm{G}_{\mathcal{H}a}}^{\mathrm{G}} \psi_a \otimes \tau \mid \tau \in \hat{\mathrm{G}}_{\mathcal{H}a}\}$. By Theorem 4.1(3), we can construct all of the irreducible characters of G if we consider all of the coadjoint orbits and the stabilizers.

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Now, we calculate the number of degree q^e irreducible characters. We write

$$h = 1 + \begin{pmatrix} 0_{a\times a} & h^1 & 0 & 0 \\ 0 & 0_{b\times b} & 0 & 0 \\ 0 & 0 & 0_{c\times c} & h^2 \\ 0 & 0 & 0 & 0_{d\times d} \end{pmatrix} \in G_{\mathcal{H}},$$

where $h^k = (h^k_{i,j})$ $(k = 1, 2)$, $h^k_{i,j} = 0$ if $(i, j) \notin \mathrm{Supp}(\mathrm{Lie}(G_{\mathcal{H}}))$. By Lemma 4.3,

$$G_{\mathcal{H}a} = \{h \in G_{\mathcal{H}} \mid (h^k_{i,j}) \text{ is a solution of (4-2)}, k = 1, 2\}.$$

We write $a = U(X, Y, Z, W)$; then $|G_{\mathcal{H}a}| = q^{-\mathrm{rank}(\mathscr{C}(X,Y,Z,W))} \cdot |G_{\mathcal{S}}|$, and the degree of the irreducible characters associated to $\mathbb{O}_a$ is $q^{\mathrm{rank}(\mathscr{C}(X,Y,Z,W))}$. We denote the number of $G_{\mathcal{H}}$-coadjoint orbits with stabilizer (of a representative element) of index q^e by N_e, then the number of irreducible characters of G of degree q^e is

$$N_e \cdot q^{-e} \cdot |G_{\mathcal{H}}|.$$

Let

$$\mathfrak{O}_e := \bigcup_{[G_{\mathcal{H}}:G_{\mathcal{H}a}]=q^e} \mathbb{O}_a$$

be the set of unions of all coadjoint orbits whose stabilizer in $G_{\mathcal{H}}$ has index q^e. Then,

$$\mathfrak{O}_e = \{U(X, Y, Z, W) \in \mathrm{Lie}(G_{\mathcal{A}})^t \mid \mathrm{rank}(\mathscr{C}(X, Y, Z, W)) = e\},$$

which is just the set $\tilde{\mathfrak{C}}(G, e)$. Since the cardinality of a $G_{\mathcal{H}}$-orbit equals the index of its stabilizer, the coadjoint orbits in $\mathfrak{O}_e$ have the same cardinality, which is q^e. Hence, $N_e = q^{-e} \cdot |\tilde{\mathfrak{C}}(G, e)|$ and the number of irreducible characters of G of degree q^e is

$$q^{-2e} \cdot |\tilde{\mathfrak{C}}(G, e)| \cdot |G_{\mathcal{H}}|.$$

The remaining part is clear, since the map π_c is surjective and all the irreducible characters can be given in the above way. □

REMARK 4.5. Since $G_{\mathcal{H}}$ is also an abelian group, by Theorem 4.1 and the proof of Proposition 4.4 above, we have $e \le \log_q(|G_{\mathcal{H}}|)$ if q^e is the degree of an irreducible character of G. This upper bound is sharp if there exists $U(X, Y, Z, W) \in \mathrm{Lie}(G_{\mathcal{A}})^t$ such that $\mathscr{C}(X, Y, Z, W)$ has full rank. In some cases, this upper bound is sharp. For example, consider $G = U_{a,a,a,a}$ and $X = Z = W = 0$, $Y = I_a$, then $\mathscr{C}(0, I_a, 0, 0)$ is of full rank. However, this upper bound is not sharp for general G; see Theorem 5.4. We do not know the sharp upper bound for the general case.

REMARK 4.6. For the degenerate case $a = 0$, that is, G is a pattern subgroup in the unipotent radical of a 3-block parabolic, one may give another proof for the character-number formula above by applying the formulae established for supercharacters given in [3].

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EXAMPLE 4.7. Let $\mathrm{G} = \mathrm{U}_{0,b,c,d}(\mathbb{F}_q)$ be the unipotent radical of a 3-block parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$. In this case, each $a \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$ is of the form $U(0,0,Z,W)$ and

$$\mathscr{C}(0,0,Z,W) = \mathrm{diag}(\overbrace{W,\ldots,W,\ldots,W}^{c \text{ times}}).$$

Then, $\mathrm{rank}(\mathscr{C}(0,0,Z,W)) = c \cdot \mathrm{rank}(W)$. Let m_k be the number of matrices of rank k in $\mathrm{Mat}_{d\times b}(\mathbb{F}_q)$. We have $|\tilde{\mathfrak{C}}(\mathrm{G}, ck)| = q^{cb} m_k$ and the number of degree q^{ck} irreducible characters is

$$q^{(b+d-2k)c} m_k,$$

which gives another proof of the character number part of [15, Theorem 2.3].

REMARK 4.8. To obtain the irreducible characters of G explicitly, we need to choose a set of complete representatives of coadjoint orbits. Then, the irreducible characters are in bijection with equivalence classes of pairs $(\mathbb{O}_a, \tau_a)$, where τ_a is the one-dimensional character of $\mathrm{G}_{\mathcal{H}a}$.

REMARK 4.9. Nien uses a different approach to obtain the classification theorem of a 3-part nilpotent radical of a standard parabolic subgroup of GL_n (compare with Theorem 4.1). She considers all coadjoint orbits rather than the coadjont orbits only for $\mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$. She then obtains a beautiful classification theorem in this special case: irreducible characters of $\mathrm{G} = \mathrm{U}_{n_1,n_2,n_3,0}$ are in bijection with the coadjoint orbits. Here, we use $\mathrm{U}_{n_1,n_2,n_3,0}$ rather than $\mathrm{U}_{0,n_1,n_2,n_3}$. We now compare these two methods briefly: Nien's coadjoint orbit containing $s(A,E,F) \in \mathrm{Lie}(\mathrm{U}_{n_1,n_2,n_3,0})^t$ corresponds to our pair $(\mathbb{O}_a, \tau_E)$, where $a = U(A,0,F,0)$ and $\tau_E \in \hat{\mathrm{G}}_{\mathcal{H}a}$ (τ_E is determined by $s(0,E,0) \in \mathrm{Lie}(\mathrm{G}_{\mathcal{H}})^t$, that is, $\tau_E(h) := \psi(\mathrm{tr}(h \cdot s(0,E,0)))$, $h \in \mathrm{G}_{\mathcal{H}a}$); Nien's subgroup $N_A(\mathbb{F}_q)$ corresponds to our $\mathrm{G}_{\mathcal{A}} \rtimes \mathrm{G}_{\mathcal{H}a}$; Nien's character $\chi_{s(A,E,F)}$ (respectively $\mathrm{Ind}_{N_A(\mathbb{F}_q)}^{N_{n_1,n_2,n_3}(\mathbb{F}_q)} \chi_{s(A,E,F)}$) corresponds to our $\psi_a \otimes \tau_E$ (respectively $\mathrm{Ind}_{\mathrm{G}_{\mathcal{A}}\rtimes\mathrm{G}_{\mathcal{H}a}}^{\mathrm{G}} \psi_a \otimes \tau_E$); [15, Lemma 2.1] corresponds to Theorem 4.1(2).

5. Geometric approach to the polynomial property of G

5.1. Polynomial property for pattern groups. We say a pattern group $G_{\mathcal{D}}(\mathbb{F}_q)$ has the *polynomial property* if the number of its characters of degree q^e is a polynomial in q with integral coefficients for any $e \in \mathbb{N}$.

By Proposition 4.4, we reduce the problem of the polynomial property of G to the polynomial property of $\tilde{\mathfrak{C}}(\mathrm{G}, e)$. In other words, G has the polynomial property if and only if $|\tilde{\mathfrak{C}}(\mathrm{G}, e)|$ is a polynomial in q with integral coefficients.

EXAMPLE 5.1. Let us focus on the case $\mathrm{G} = \mathrm{U}_{0,b,c,d}(\mathbb{F}_q)$ again. We have $|\tilde{\mathfrak{C}}(\mathrm{G}, e)| \neq 0$ only if $c \mid e$. For $e = ck$,

$$\tilde{\mathfrak{C}}(\mathrm{G}, ck) = q^{(b+d-2k)c} |\mathfrak{C}(\mathrm{G}, ck)|.$$

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We can identify $\mathfrak{C}(G, ck)$ with $\{T \in \mathrm{Mat}_{d\times b}(\mathbb{F}_q) \mid \mathrm{rank}(T) = k\}$. The latter can be viewed as $D^{k+1}_{d,b}(\mathbb{F}_q)\backslash D^{k}_{d,b}(\mathbb{F}_q)$. The polynomial property of G is a consequence of the fact that the determinantal varieties are polynomial-counting.

Let

$$\mathcal{D}_m = \{(i,j) \in \Delta_n \mid \ j - i \geq m\}.$$

We have

$$U_n(\mathbb{F}_q) = G_{\mathcal{D}_1}(\mathbb{F}_q) \geq G_{\mathcal{D}_2}(\mathbb{F}_q) \geq \cdots \geq G_{\mathcal{D}_m}(\mathbb{F}_q) \geq \cdots \geq G_{\mathcal{D}_n}(\mathbb{F}_q) = 1,$$

which becomes the lower central series of $U_n(\mathbb{F}_q)$.

We assume $(n/3) \leq m < (n/2)$ $(n \geq 5)$. Under this assumption, $G_{\mathcal{D}_m}(\mathbb{F}_q)$ is not abelian and is a pattern subgroup of some $U_{0,b,c,d}(\mathbb{F}_q)$.

LEMMA 5.2. *For $(n/3) \leq m < (n/2)$ $(n \geq 5)$, $G_{\mathcal{D}_m}(\mathbb{F}_q)$ is a pattern subgroup in $U_{0,m,n-2m,m}(\mathbb{F}_q)$.*

PROOF. It is enough to show $\mathrm{Supp}(\mathrm{Lie}(G_{\mathcal{D}_m}(\mathbb{F}_q))) \subset \mathrm{Supp}(\mathrm{Lie}(U_{0,m,n-2m,m}(\mathbb{F}_q)))$. Indeed,

$$\begin{aligned}&\mathrm{Supp}(\mathrm{Lie}(U_{0,m,n-2m,m}(\mathbb{F}_q)))\\ &\quad = \{(i,j) \mid 1 \leq i \leq n-m,\ m+1 \leq j \leq n\}\backslash\{(i,j) \mid m+1 \leq i,j \leq n-m\}\end{aligned}$$

and $\mathrm{Supp}(\mathrm{Lie}(G_{\mathcal{D}_m}(\mathbb{F}_q))) = \{(i,j) \in \Delta_n \mid j - i \geq m\}$. One can verify the inclusion easily, which completes the proof. □

Set $s := n - 2m$. Denote by $\mathrm{Tri}_s(\mathbb{F}_q)$ the set of all lower triangular matrices of size $s \times s$ over $\mathbb{F}_q$. Let $\mathbf{a} = (a_{i,j})_{1\leq j\leq i\leq s} \in \mathrm{Tri}_s(\mathbb{F}_q)$ and define

$$f(\mathbf{a}) := \begin{pmatrix} a_{1,1} & & & & & & & & \\ \vdots & & & & & & & & \\ a_{s,1} & & & & & & & & \\ & \ddots & & & & & & & \\ & & a_{i,1} & \dots & a_{i,i} & & & & \\ & & \vdots & \ddots & \vdots & & & & \\ & & a_{s,1} & \dots & a_{s,i} & & & & \\ & & & & & \ddots & & & \\ & & & & & & a_{s,1} & \dots & a_{s,s} \end{pmatrix} \in \mathrm{Tri}_{(1/2)s(s+1)}(\mathbb{F}_q),$$

which is a block matrix with i th block

$$f(\mathbf{a})_i := \begin{pmatrix} a_{i,1} & \dots & a_{i,i} \\ \vdots & \ddots & \vdots \\ a_{s,1} & \dots & a_{s,i} \end{pmatrix}, \quad i = 1, \dots, s.$$

It is clear that f is an injective map from $\mathrm{Tri}_s(\mathbb{F}_q)$ to $\mathrm{Tri}_{(1/2)s(s+1)}(\mathbb{F}_q)$.

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EXAMPLE 5.3. For $s = 3$, $\mathbf{a} = \begin{pmatrix} a_{1,1} & & \\ a_{2,1} & a_{2,2} & \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$, we have

$$f(\mathbf{a}) = \begin{pmatrix} a_{1,1} & & & & & \\ a_{2,1} & & & & & \\ a_{3,1} & & & & & \\ & a_{2,1} & a_{2,2} & & & \\ & a_{3,1} & a_{3,2} & & & \\ & & & a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix},$$

which has three blocks.

Let $\mathrm{G} = G_{\mathcal{D}_m}(\mathbb{F}_q)$ $((n/3) \le m < (n/2),\ n \ge 5)$, which is a pattern subgroup in $\mathrm{U}_{0,m,n-2m,m}(\mathbb{F}_q)$ by Lemma 5.2. For $U(0,0,Z,W) \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$, $W = (w_{i,j})_{m\times m}$, define

$$\mathscr{M}(0,0,Z,W) := \begin{pmatrix} w_{m-s+1,1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ w_{m,1} & \cdots & w_{m,s} \end{pmatrix} \in \mathrm{Tri}_s(\mathbb{F}_q)$$

to be the lower triangular matrix of size $s \times s$, which is in the left corner of W.

Denote the number of degree q^e irreducible characters of $G_{\mathcal{D}_m}(\mathbb{F}_q)$ by $N_{G_{\mathcal{D}_m}(\mathbb{F}_q),\, e}$.

THEOREM 5.4. *For $(n/3) \le m < (n/2)$ with $n \ge 5$ and $e \in \mathbb{N}$,*

$$N_{G_{\mathcal{D}_m}(\mathbb{F}_q),\, e} = q^{(s+m)(s+m+1)/2-s(s+1)/2-2e} \cdot |\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|.$$

Moreover, the set of integers that occur as degrees of irreducible characters of $G_{\mathcal{D}_m}(\mathbb{F}_q)$ is precisely $\{q^e \mid 0 \le e \le r_{\max}(s),\ e \in \mathbb{N}\}$, where $s = n - 2m$ and

$$r_{\max}(s) := \begin{cases} \dfrac{s^2+2s}{4} & \text{if } n \text{ is even,} \\ \dfrac{(s+1)^2}{4} & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. By Proposition 4.4,

$$N_{G_{\mathcal{D}_m}(\mathbb{F}_q),\, e} = q^{s(s+1)/2-2e} \cdot |\tilde{\mathfrak{C}}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|.$$

For $U(0,0,Z,W) \in \mathrm{Lie}(\mathrm{G}_{\mathcal{A}})^t$,

$$\mathscr{C}(0,0,Z,W) = f(\mathscr{M}(0,0,Z,W)).$$

Noting that f is injective, the map π_c in this case is a fibration with the cardinality of the fibre equal to $q^{(s+m)(s+m+1)/2-s(s+1)}$. We have

$$|\tilde{\mathfrak{C}}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)| = q^{(s+m)(s+m+1)/2-s(s+1)} \cdot |\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|$$

and hence,

$$N_{G_{\mathcal{D}_m}(\mathbb{F}_q),\, e} = q^{(s+m)(s+m+1)/2-s(s+1)/2-2e} \cdot |\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|.$$

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For the remaining part, it is enough to prove the claim as follows. For any $n \in \mathbb{Z}_+$ and $0 \le e \le r_{\max}(n)$, there exists a $\mathbf{t} \in \mathrm{Tri}_n(\mathbb{F}_q)$ such that $\mathrm{rank}(f(\mathbf{t})) = e$. Furthermore, $\mathrm{rank}(f(\mathbf{t})) \le r_{\max}(n)$ for any $\mathbf{t} \in \mathrm{Tri}_n(\mathbb{F}_q)$.

We prove our claim by induction on n. For $n = 1$, it is clear. Now, we assume it holds for $n = n_0 (n_0 \ge 1)$. By induction, for $0 \le e \le r_{\max}(n_0)$, we can find

$$\mathbf{t} = \begin{pmatrix} t_{1,1} & & \\ \vdots & \ddots & \\ t_{n_0,1} & \cdots & t_{n_0,n_0} \end{pmatrix} \in \mathrm{Tri}_{n_0}(\mathbb{F}_q),$$

such that $\mathrm{rank}(f(\mathbf{t})) = e$. Let

$$\mathbf{t}' = \begin{pmatrix} t_{1,1} & & & \\ \vdots & \ddots & & \\ t_{n_0,1} & \cdots & t_{n_0,n_0} & \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathrm{Tri}_{n_0+1}(\mathbb{F}_q).$$

We have $\mathrm{rank}(f(\mathbf{t}')) = \mathrm{rank}(f(\mathbf{t})) = e$. Consider the following matrices:

$$\tilde{\mathbf{t}}_{n_0+1,m} = \begin{pmatrix} \tilde{t}_{1,1} & & & \\ \vdots & \ddots & & \\ \tilde{t}_{n_0,1} & \cdots & \tilde{t}_{n_0,n_0} & \\ \delta_{m,n_0+1} & \cdots & \cdots & \delta_{m,1} \end{pmatrix} \in \mathrm{Tri}_{n_0+1}(\mathbb{F}_q), \quad 1 \le m \le \left[\frac{n_0+2}{2}\right],$$

where $\tilde{t}_{i,j} = 1$ for $0 \le i - j \le n_0 - [(n_0+1)/2]$ and $\tilde{t}_{i,j} = 0$ whenever $i - j \ge n_0 - [(n_0+1)/2] + 1$. One can verify that $\mathrm{rank}(f(\tilde{\mathbf{t}}_{n_0+1,m})) = r_{\max}(n_0) + m$ for $1 \le m \le [(n_0+2)/2]$. In particular, $\mathrm{rank}(f(\tilde{\mathbf{t}}_{n_0+1,[(n_0+2)/2]})) = r_{\max}(n_0) + [(n_0+2)/2] = r_{\max}(n_0+1)$. Finally,

$$\mathrm{rank}(f(\mathbf{t})) \le \sum_{i=1}^{n} \min\{i, n-i+1\} = r_{\max}(n)$$

for any $\mathbf{t} \in \mathrm{Tri}_n(\mathbb{F}_q)$. □

5.2. A partition of $\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)$. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathbb{Z}^s$. If we have $0 \le \lambda_i \le \min\{i, s-i+1\}$ for $i = 1, \ldots, s$, we call $\underline{\lambda}$ *admissible*. For $e \in \mathbb{N}$, define

$$\Lambda(s, e) := \left\{ \underline{\lambda} \mid \underline{\lambda} \text{ is admissible with } \sum_{i=1}^{s} \lambda_i = e \right\}.$$

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For $\mathbf{t} \in \mathrm{Tri}_s(\mathbb{F}_q)$, define $\mathbf{t}_i := \begin{pmatrix} t_{i,1} & \cdots & t_{i,i} \\ \vdots & \ddots & \vdots \\ t_{s,1} & \cdots & t_{s,i} \end{pmatrix}$ to be the ith block of $f(\mathbf{t})$ for $1 \le i \le s$. We call $\mathbf{t}_i$ the *ith submatrix* of $\mathbf{t}$. Clearly,

$$\mathrm{rank}(f(\mathbf{t})) = \sum_{i=1}^{s} \mathrm{rank}(\mathbf{t}_i).$$

For an admissible $\underline{\lambda}$, we define

$$V_q(s, \underline{\lambda}) := \{\mathbf{t} \in \mathrm{Tri}_s(\mathbb{F}_q) \mid \mathrm{rank}(\mathbf{t}_i) = \lambda_i, i = 1, \ldots, s\},$$
$$\overline{V}_q(s, \underline{\lambda}) := \{\mathbf{t} \in \mathrm{Tri}_s(\mathbb{F}_q) \mid 0 \le \mathrm{rank}(\mathbf{t}_i) \le \lambda_i, i = 1, \ldots, s\}.$$

Since $\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q)) = f(\mathrm{Tri}_s(\mathbb{F}_q))$ and f is injective, we identify $\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q))$ with $\mathrm{Tri}_s(\mathbb{F}_q)$ via f. Then,

$$\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e) = \bigsqcup_{\underline{\lambda} \in \Lambda(n,e)} V_q(s, \underline{\lambda}). \tag{5-1}$$

PROPOSITION 5.5. *The number $|\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q), e)|$ is a linear combination of $|\overline{V}_q(s, \underline{\lambda})|$ with integral coefficients for some admissible $\underline{\lambda}$.*

PROOF. For an admissible $\underline{\lambda}$ and $1 \le i_1 < \cdots < i_t \le s$, define

$$\underline{\lambda^{i_1,\ldots,i_t}} := (\lambda_1^{i_1,\ldots,i_t}, \ldots, \lambda_s^{i_1,\ldots,i_t}),$$

where

$$\lambda_j^{i_1,\ldots,i_t} = \begin{cases} \lambda_j - 1, & j \in \{i_1, \ldots, i_t\}. \\ \lambda_j & \text{otherwise.} \end{cases}$$

Then,

$$\overline{V}_q(s, \underline{\lambda^{i_1,\ldots,i_t}}) = \bigcap_{r \in \{i_1,\ldots,i_t\}} \overline{V}_q(s, \underline{\lambda^r}). \tag{5-2}$$

If $\underline{\lambda^{i_1,\ldots,i_t}}$ is not admissible, take $\overline{V}_q(s, \underline{\lambda^{i_1,\ldots,i_t}}) = \emptyset$. Note that

$$\overline{V}_q(s, \underline{\lambda}) \setminus \bigcup_{k=1}^{s} \overline{V}_q(s, \underline{\lambda^k}) = V_q(s, \underline{\lambda})$$

and $\overline{V}_q(s, \underline{\lambda^k}) \subset \overline{V}_q(s, \underline{\lambda}), (k = 1, \ldots, s)$. We have

$$|V_q(s, \underline{\lambda})| = |\overline{V}_q(s, \underline{\lambda})| - \left| \bigcup_{k=1}^{s} \overline{V}_q(s, \underline{\lambda^k}) \right|.$$

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By the principle of inclusion-exclusion and (5-2),

$$|V_q(s,\underline{\lambda})| = |\overline{V}_q(s,\underline{\lambda})| + \sum_{t=1}^{s}(-1)^t \sum_{1\le i_1<\cdots<i_t\le s} |\overline{V}_q(s,\underline{\lambda^{i_1,\ldots,i_s}})|. \tag{5-3}$$

Then, by (5-1) and (5-3),

$$|\mathfrak{C}(G_{\mathcal{D}_m}(\mathbb{F}_q),e)| = \sum_{\underline{\lambda}\in\Lambda(n,e)} |\overline{V}_q(s,\underline{\lambda})| + \sum_{\underline{\lambda}\in\Lambda(n,e)} \sum_{t=1}^{s}(-1)^t \sum_{1\le i_1<\cdots<i_t\le s} |\overline{V}_q(s,\underline{\lambda^{i_1,\ldots,i_s}})|,$$

which is a linear combination of $|\overline{V}_q(s,\underline{\lambda})|$ with integral coefficients for some admissible $\underline{\lambda}$. □

PROPOSITION 5.6. *For admissible $\underline{\lambda} = (\lambda_1,\ldots,\lambda_s) \in \mathbb{Z}^s$, if $\underline{\lambda} \neq (\min\{i, s-i+1\})_{i=1}^s$, then $\overline{V}_q(s,\underline{\lambda})$ equals the set of $\mathbb{F}_q$-rational points of the product of a one-sided mixed ladder determinantal variety and an affine space.*

PROOF. Let $T = (T_{ba})_{s\times s}$ be an $s\times s$ lower triangular matrix with indeterminates T_{ba}, $1\le a\le b\le s$. For $1\le i\le s$, let T_i be the ith submatrix of T. For i such that $\lambda_i < \min\{i, s-i+1\}$, let I_i be the ideal of $\overline{\mathbb{F}}_q[T] := \overline{\mathbb{F}}_q[T_{ba}, T_{ba}\in T]$ generated by all $(\lambda_i+1)\times(\lambda_i+1)$ minors of T_i; for $\lambda_i = \min\{i, s-i+1\}$, we let $I_i = (0)$. Let $\Im = \{I_i \neq 0 \mid i = 1,\ldots,s\}$. Then, $\Im$ becomes a poset if we take the inclusion relationship between ideals as the partial order. Let

$$M = \{i \mid I_i \textit{ is maximal in } \Im\} = \{i_1 < \cdots < i_{|M|}\}.$$

Since $\underline{\lambda} \neq (\min\{i, s-i+1\})_{i=1}^s$, we have $M \neq \emptyset$. Let $T' = \bigcup_{i\in M} T_i$ and

$$\underline{\lambda}' = (\lambda_{i_1}+1,\ldots,\lambda_{i_{|M|}}+1).$$

We have a one-sided mixed ladder determinantal variety $DV_{\underline{\lambda}'}(T')$. We claim that $\overline{V}_q(s,\underline{\lambda})$ equals the set of $\mathbb{F}_q$-rational points of $DV_{\underline{\lambda}'}(T')\times\mathbb{A}^{|T\setminus T'|}$.

Note that $DV_{\underline{\lambda}'}(T')\times\mathbb{A}^{|T\setminus T'|} \cong \operatorname{Spec}\ \overline{\mathbb{F}}_q[T]/(I_1,\ldots,I_s)$. It remains to show that $\overline{V}_q(s,\underline{\lambda})$ equals the set of common zeros of $I_1,\ldots,I_s$ over $\mathbb{F}_q$. We identify $\mathrm{Tri}_s(\mathbb{F}_q)$ with the set of $\mathbb{F}_q$-valued points of $\operatorname{Spec}\mathbb{F}_q[T]$. Let $\mathbf{t} = (t_{i,j})_{s\times s} \in \mathrm{Tri}_s(\mathbb{F}_q)$ be a common zero of $I_1,\ldots,I_s$, and let $\mathbf{t}_i$ be the i th submatrix of $\mathbf{t}$, $i = 1,\ldots,s$.

However, if $\lambda_i = \min\{i, s-i+1\}$, since $\mathbf{t}_i$ is of size $i\times(s-i+1)$, we have $\operatorname{rank}(\mathbf{t}_i)\le\lambda_i$; if $\lambda_i < \min\{i, s-i+1\}$, since $(t_{i,j})_{s\times s}$ is a zero of I_i and I_i is generated by the $(\lambda_i+1)\times(\lambda_i+1)$ minors of T_i, then all $(\lambda_i+1)\times(\lambda_i+1)$ minors of $\mathbf{t}_i$ are equal to zero, which force that $\operatorname{rank}(\mathbf{t}_i)\le\lambda_i$. Hence, $DV_{\underline{\lambda}'}(T')\times\mathbb{A}^{|T\setminus T'|}(\mathbb{F}_q) \subset \overline{V}_q(s,\underline{\lambda})$. However, letting $\mathbf{t}' = (t'_{i,j})_{s\times s} \in \overline{V}_q(s,\underline{\lambda})$, we show it is a common zero of $I_1,\ldots,I_s$: for $I_i \neq 0$, it is generated by the $(\lambda_i+1)\times(\lambda_i+1)$ minors of T_i and $\operatorname{rank}(\mathbf{t}'_i)\le\lambda_i$, so $(t'_{i,j})_{s\times s}$ is a zero of I_i; for $I_i = 0$, it holds clearly. The proposition follows. □

5.3. Lehrer conjecture for $G_{\mathcal{D}_m}$. Now, we can prove the analogue to Lehrer's conjecture for finite pattern group $G_{\mathcal{D}_m}(\mathbb{F}_q)$ ($(n/3)\le m < (n/2)$, $n\ge 5$).

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THEOREM 5.7. *For* $(n/3) \le m < (n/2)$ $(n \ge 5)$*, the pattern group* $G_{\mathcal{D}_m}(\mathbb{F}_q)$ *has the polynomial property, that is, the number of irreducible characters of degree* q^e *is a polynomial in* q *with integral coefficients for* $e \in \mathbb{N}$.

PROOF. By Theorem 5.4 and Proposition 5.5, it remains to show that $|\overline{V}_q(s, \underline{\lambda})|$ is a polynomial in q with integral coefficients for any admissible $\underline{\lambda}$. For admissible $\underline{\lambda} \ne (\min\{i, s-i+1\})_{i=1}^{s}$, this follows by Proposition 5.6 and Corollary 3.6; for $\underline{\lambda} = (\min\{i, s-i+1\})_{i=1}^{s}$, $|\overline{V}_q(s, \underline{\lambda})| = |\mathrm{Tri}_s(\mathbb{F}_q)| = q^{s(s+1)/2}$, which is clearly a polynomial in q with integral coefficients. □

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