On examination of the non-singular model E', with γ_1', γ_2' denoting the proper transforms of γ_1, γ_2, one finds that $2(\gamma_1' - \gamma_2')$ is equivalent on E' to a linear combination of surfaces which 'arise out of' the eight new quadruple points of E, this linear combination having coefficients ± 1. Since this does not indicate torsion on E', there seems no further reason to suppose that E' is a counter-example to Serre's theorem. It still remains, of course, to decide whether the threefold is rational or not.

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A CRYSTAL DISLOCATION PROBLEM

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The problem considered in this note arises in the theory of perfect crystal dislocations and I am indebted to Prof. F. R. N. Nabarro for drawing my attention to it.

A vector $\mathbf{p}$ of a three-dimensional lattice Λ will be called a *slip vector* if $\mathbf{t}.(\mathbf{p}-\mathbf{t}) \leqslant 0$ for all $\mathbf{t} \in \Lambda$. Following Frank and Nicholls (1), we shall call a slip vector $\mathbf{p}$ *stable* if $\mathbf{t}.(\mathbf{p}-\mathbf{t}) < 0$ for all $\mathbf{t} \in \Lambda$ other than $\mathbf{0}$ and $\mathbf{p}$. Frank and Nicholls have enumerated the slip vectors for the simple cubic, face-centred cubic, diamond, hexagonal close-packed and body-centred cubic lattices, and have shown that there are only a finite number of slip vectors in each of these cases. In this note we show that, for any lattice, there are only a finite number of slip vectors and that they can be found by examining those points of the lattice that lie inside or on a certain sphere. More precisely, we prove the

Theorem. *Let the lattice* Λ *be generated by the vectors* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ *and let* $\mathbf{p}$ *be a point of* Λ. *If* $\mathbf{p}^2 > \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$, *there exists a vector* $\mathbf{t} \in \Lambda$ *for which* $\mathbf{t}.(\mathbf{p}-\mathbf{t}) > 0$; *if* $\mathbf{p}^2 = \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$, *there exists a vector* $\mathbf{t} \in \Lambda$, *with* $\mathbf{t} \neq \mathbf{0}, \mathbf{p}$, *for which* $\mathbf{t}.(\mathbf{p}-\mathbf{t}) \geqslant 0$.

It follows that, if $\mathbf{p}$ *is a slip vector, then* $\mathbf{p}^2 \leqslant \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$; *further, if* $\mathbf{p}$ *is stable then* $\mathbf{p}^2 < \mathbf{a}_1^2+\mathbf{a}_2^2+\mathbf{a}_3^2$.

Proof. Suppose that $\mathbf{p} = n_1\mathbf{a}_1+n_2\mathbf{a}_2+n_3\mathbf{a}_3$, where n_1, n_2 and n_3 are integers. For $\nu = 1, 2, 3$ take ϵ_ν to be 0 or 1 according as n_ν is even or odd. Write

$$\mathbf{t}_0 = \tfrac{1}{2}\{(n_1+\epsilon_1)\,\mathbf{a}_1+(n_2+\epsilon_2)\,\mathbf{a}_2+(n_3+\epsilon_3)\,\mathbf{a}_3\}.$$

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Then $\mathbf{t}_0$ belongs to Λ, and so do the vectors $\mathbf{t}_\nu = \mathbf{t}_0 - \epsilon_\nu \mathbf{a}_\nu$ ($\nu = 1, 2, 3$). But

$$\begin{aligned}\sum_{\nu=0}^{3} (\mathbf{t}_\nu - \tfrac{1}{2}\mathbf{p})^2 &= \tfrac{1}{4}\{(\epsilon_1\mathbf{a}_1 + \epsilon_2\mathbf{a}_2 + \epsilon_3\mathbf{a}_3)^2 + (-\epsilon_1\mathbf{a}_1 + \epsilon_2\mathbf{a}_2 + \epsilon_3\mathbf{a}_3)^2 \\ &\quad + (\epsilon_1\mathbf{a}_1 - \epsilon_2\mathbf{a}_2 + \epsilon_3\mathbf{a}_3)^2 + (\epsilon_1\mathbf{a}_1 + \epsilon_2\mathbf{a}_2 - \epsilon_3\mathbf{a}_3)^2\} \\ &= \epsilon_1^2\mathbf{a}_1^2 + \epsilon_2^2\mathbf{a}_2^2 + \epsilon_3^2\mathbf{a}_2^2,\end{aligned}$$

so that, for at least one value of ν ($\nu = 0, 1, 2, 3$),

$$(\mathbf{t}_\nu - \tfrac{1}{2}\mathbf{p})^2 \leqslant \tfrac{1}{4}\{\epsilon_1^2\mathbf{a}_1^2 + \epsilon_2^2\mathbf{a}_2^2 + \epsilon_3^2\mathbf{a}_3^2\},$$

i.e.
$$\mathbf{t}_\nu . (\mathbf{p} - \mathbf{t}_\nu) \geqslant \tfrac{1}{4}\{\mathbf{p}^2 - \epsilon_1^2\mathbf{a}_1^2 - \epsilon_2^2\mathbf{a}_2^2 - \epsilon_3^2\mathbf{a}_3^2\}.$$

The right-hand side is positive if $\mathbf{p}^2 \geqslant \mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_3^2$, except when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and $\mathbf{p}^2 = \mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_3^2$, so that the theorem is proved except in this case, which we now consider. Then

$$\sum_{\nu=0}^{3} (\mathbf{t}_\nu - \tfrac{1}{2}\mathbf{p})^2 = \mathbf{p}^2,$$

and it follows, as before, that either $\mathbf{t}_\nu = \mathbf{0}$ or $\mathbf{p}$ for each value of ν, or else $\mathbf{t}_\nu . (\mathbf{p} - \mathbf{t}_\nu) \geqslant 0$ for some value of ν for which $\mathbf{t}_\nu \neq \mathbf{0}, \mathbf{p}$. The first possibility cannot arise, since $\mathbf{t}_0 - \mathbf{t}_\nu = \mathbf{a}_\nu$, and $\mathbf{a}_\nu \neq \mathbf{0}, \mathbf{p}$; for n_1, n_2 and n_3 are all odd. This completes the proof of the theorem.

The theorem is best possible in the sense that slip vectors $\mathbf{p}$ for which $\mathbf{p}^2 = \mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_3^2$ can exist. For if Λ is the simple cubic lattice generated by $\mathbf{a}_1 = (1, 0, 0)$, $\mathbf{a}_2 = (0, 1, 0)$, $\mathbf{a}_3 = (0, 0, 1)$, and $\mathbf{p} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, then there is no lattice point inside the sphere on $\mathbf{p}$ as diameter, so that $\mathbf{t} . (\mathbf{p} - \mathbf{t}) \leqslant 0$ for all $\mathbf{t} \in \Lambda$. We note also that the theorem generalizes in an obvious way to n dimensions ($n \geqslant 2$).

For applications it is clearly advantageous to choose the generating vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ in such a way that $\mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_3^2$ is as small as possible. One way to make $\mathbf{a}_1^2 + \mathbf{a}_2^2 + \mathbf{a}_3^2$ small is to take $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ to be the three shortest linearly independent vectors in Λ. (In $n \geqslant 5$ dimensions it may not be possible to choose n shortest linearly independent vectors which generate Λ.)

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