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Smoothness in Relative Geometry

by

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Abstract

In [TVa], Bertrand Toën and Michel Vaquié defined a scheme theory for a closed monoidal category $(\mathcal{C},\otimes,1)$. In this article, we define a notion of smoothness in this relative (and not necessarily additive) context which generalizes the notion of smoothness in the category of rings. This generalisation consists in replacing homological finiteness conditions by homotopical ones, using the Dold-Kan correspondence. To do this, we provide the category $s\mathcal{C}$ of simplicial objects in a monoidal category $\mathcal{C}$ and all the categories sA-mod, sA-alg ($A \in \mathrm{sComm}(\mathcal{C})$) with compatible model structures using the work of Rezk [R]. We then give a general notion of smoothness in $\mathrm{sComm}(\mathcal{C})$. We prove that this notion is a generalisation of the notion of smooth morphism in the category of rings and is stable under composition and homotopy pushouts. Finally we provide some examples of smooth morphisms, in particular in $\mathbb{N}$-alg and $\mathrm{Comm}(Set)$.

Key Words: Symmetric monoidal category, simplicial category, model category, relative geometry, smoothness, cohomology.

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Introduction

In [TVa], Bertrand Toën and Michel Vaquié defined a scheme theory for a closed monoidal category $(\mathcal{C},\otimes,1)$. In this article, we define a notion of smoothness in the relative context which generalizes the notion of smoothness in the category of rings. The motivation for this work is that interesting objects in the non additive contexts $\mathcal{C} = Set$ or $\mathbb{N}$-mod are not expected to be schemes, but rather stacks. The first step is to get a definition of smooth morphism. The following theorem gives a good definition of smoothness that can be generalised to the relative context :

Theorem 1 *(A proof can be found in [M1] (Théorème 4.1.15))*
Assume $\mathcal{C} = \mathbb{Z}$-mod. A morphism $A \to B$ of rings is smooth if and only if

i. The ring B is finitely presented in A-alg.

ii. The morphism $A \to B$ is flat.

iii. The ring B is a perfect complex of $B \otimes_A B$-modules.

The flatness of $A \to B$ is in fact equivalent to $\mathrm{Tordim}_A B = 0$, hence the two last conditions are homological finiteness conditions. The Dold-Kan correspondence will make the link between homological conditions and homotopical ones. Indeed, for a ring A, there is an equivalence of categories:

$$sA\text{-mod} \backsimeq Ch(A\text{-mod})^{\geq 0} \text{ and } \forall i\ \pi_i(Map(\mathbb{Z},X)) \backsimeq H_i(X)$$

which induces a correspondence between weak equivalences and quasi-isomorphisms.

The conditions (ii) and (iii) can then be translated into homotopical conditions. Finally, a result from [TV] asserts that B is a perfect complex in $Ch(B \otimes_A B)$ if and

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only if it is homotopically finitely presented (definition 1.8 (iii) below) in $sB \otimes_A B$-mod.

We then provide the category of simplicial objects in a monoidal category $s\mathcal{C}$ and all the categories of modules (sA-mod) and algebras (sA-alg) with model structures using the work of Rezk [R]. The classical functors between the categories A-mod and A-alg, $A \in \mathrm{sComm}(\mathcal{C})$ induce Quillen functors between the corresponding simplicial categories. We give a general definition of smoothness relative to a context $\mathcal{C}$. If there is no ambiguity regarding the context, relative smooth morphisms will just be called smooth morphisms.

Definition 2 A morphism $A \to B \in \mathrm{sComm}(\mathcal{C})$ is smooth if and only if

i. The simplical algebra B is homotopically finitely presented in sA-alg.

ii. The simplicial algebra B has Tor dimension 0 on A.

iii. The morphism $B \otimes^h_A B \to B$ is homotopically finitely presented in $sB \otimes^h_A B$-mod.

The first condition implies the first condition of theorem 1 ([TV], 2.2.2.4) and there is equivalence for smooth morphisms of rings. When A and B are rings, Tor dimension 0 implies that the derived tensor product is weakly equivalent to the tensor product. The equivalence with the previous theorem for rings is then clear.

We prove that relative smooth morphisms are stable under composition and pushouts of algebras and that a Zariski open immersion is smooth (1.28). We finally provide examples of smooth morphisms in the relative non-additive contexts $\mathcal{C} = \mathbb{N}$-mod and $\mathcal{C} = Set$. In particular the affine line $\mathbb{F}_1 \to \mathbb{N}$ and the morphism $\mathbb{F}_1 \to \mathbb{Z}$ (Corresponding intuitively to the scheme $\mathbb{G}_{m,F_1} \backsimeq Spec(\mathbb{Z})$) are smooth in $\mathrm{sComm}(Set)$. Similarly, the affine line $\mathbb{N} \to \mathbb{N}[X]$ and the morphism $\mathbb{N} \to \mathbb{N}(X)$ (corresponding intuitively to the scheme $\mathbb{G}_{m,\mathbb{N}}$) are smooth in $s\mathbb{N}$-mod.

Preliminaries

Let $(\mathcal{C}, \otimes, 1)$ be a complete and cocomplete closed symmetric monoidal category. In the category $\mathcal{C}$, there exists a notion of commutative monoid and for a given commutative monoid A, the notion of an A-module. Let $\mathrm{Comm}(\mathcal{C})$ denote the category of commutative monoids (with unit) in $\mathcal{C}$. For $A \in \mathrm{Comm}(\mathcal{C})$, let A-mod denote the category of A-modules. It is well known that the category A-mod is a closed monoidal tensored and cotensored category, which is complete and cocomplete. The category $\mathrm{Comm}(A\text{-mod})$ will be denoted by A-alg and is described

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by the equivalence $A/\mathrm{Comm}(\mathcal{C}) \backsimeq A\text{-alg}$. A pushout in A-alg is a tensor product[1] in the sense that for commutative monoids B and C in A-alg, $B \coprod_A C \backsimeq B \otimes_A C$.

Throughout this paper, $(\mathcal{C}, \otimes, 1)$ will be a locally finitely presentable monoidal category i.e. satisfies the property that the full subcategory of finitely presented objects[2], denoted by $\mathcal{C}_{fp}$, is essentially small, that the (Yoneda) functor from $\mathcal{C}$ to the category of presheaves on $\mathcal{C}_{fp}$, $i : \mathcal{C} \to \mathrm{Pr}(\mathcal{C}_{fp})$, is fully faithful, and that $\mathcal{C}_{fp}$ is stable under tensor product and contains the unit 1. The functor $\mathrm{Hom}_{\mathcal{C}}(1,-)$, denoted by $(-)_0 : X \to X_0$ is called the "underlying set functor". For $k \in \mathcal{C}_{fp}$, the functor $\mathrm{Hom}_{\mathcal{C}}(k,-)$, denoted by $(-)_k : X \to X_k$ is called the "weak underlying set functors". It is known that if $\mathcal{C}$ is a locally finitely presentable monoidal category, so are the categories of modules A-mod, where $A \in \mathrm{sComm}(\mathcal{C})$.

There are two fundamental adjunctions:

$$\mathcal{C} \underset{i}{\overset{(-\otimes A)}{\rightleftarrows}} A\text{-mod} \qquad\qquad \mathcal{C} \underset{i}{\overset{L}{\rightleftarrows}} \mathrm{Comm}(\mathcal{C})$$

where the forgetful functor i is a right adjoint and the functor "free associated monoid" L is defined by $L(X) := \coprod_{n\in\mathbb{N}} X^{\otimes n}/S_n$. In these adjunctions, $\mathcal{C}$ can be replaced by B-mod for $B \in \mathrm{Comm}(\mathcal{C})$, and L by L_B, where L_B is defined by $L_B(M) := \coprod_{n\in\mathbb{N}} M^{\otimes_B n}/S_n$. Let φ (resp φ_B) and ψ (resp ψ_B) denote these adjunctions for the category $\mathcal{C}$ (resp B-mod). For $X \in \mathcal{C}$ and $M \in A$-mod, the map $\varphi : \mathrm{Hom}_{\mathcal{C}}(X, M) \to \mathrm{Hom}_{A\text{-mod}}(X \otimes A, M)$ is easy to describe :

$$\varphi : f \to \mu_M \circ Id_A \otimes f$$

$$\varphi^{-1} : g \to g \circ (Id_X \otimes i_A) \circ r_X^{-1}.$$

Let $s\mathcal{C}$ denote the category of simplicial objects in $\mathcal{C}$. There is a functor "constant simplicial object" denoted by k from $\mathcal{C}$ to $s\mathcal{C}$, which is right adjoint to the functor π_0 from $s\mathcal{C}$ to $\mathcal{C}$ defined by $\pi_0(X) := \mathrm{Colim}(\ X[1] \rightleftarrows X[0]\ )$. The tensor product in $\mathcal{C}$ induces a tensor product on $s\mathcal{C}$, its unit element being $k(1)$. For A in $\mathrm{Comm}(\mathcal{C})$, sA-mod and sA-alg will denote respectively the simplicial categories $sk(A)$-mod and $sk(A)$-alg. As $\mathrm{sComm}(\mathcal{C}) \backsimeq \mathrm{Comm}(s\mathcal{C})$, we will always denote the simplicial category of commutative monoids in $s\mathcal{C}$ by $\mathrm{sComm}(\mathcal{C})$. The functor induced by L on simplical categories will be denoted by sL. The functor $i : \mathcal{C} \to \mathrm{Pr}(\mathcal{C}_{fp})$ induces a functor $si : s\mathcal{C} \to \mathrm{sPr}(s\mathcal{C}_{fp})$.

We need finally some hypotheses in order to endow $s\mathcal{C}$ and $\mathrm{sComm}(\mathcal{C})$, and for $A \in \mathrm{sComm}(\mathcal{C})$, sA-mod and sA-alg with compatible model structures. As $\mathcal{C}_{fp}$ is

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[1] The tensor product in A-mod is given by $M \otimes_A N := \mathrm{coker}(\ M \otimes A \otimes N \underset{\mu_N}{\overset{\mu_M}{\rightrightarrows}} M \otimes N\ )$ where the colimit is taken in $(\mathcal{C}, \otimes, 1)$

[2] An object k of $\mathcal{C}$ is finitely presented (or compact) if the functor $\mathrm{Hom}_{\mathcal{C}}(k,-)$ commutes with filtered colimits.

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essentially small, the family of isomorphism classes of its objects, denoted by J, is a set. A solution to the problem of constructing a proper model structure is given by a theorem of Rezk ([R]) under certain conditions. More precisely, the set J can be seen as a category (its only arrows are identities and its objects are the elements of the set J) and we will assume that the natural functors from $s\mathcal{C}$ and sComm($\mathcal{C}$) to $sSet^J := Fun(J, sSet)$ are monadic. The characterisation of monadic functors ([Bc]) implies that for any commutative simplicial monoid A, the induced functor from sA-mod to $sSet^J$ is also monadic.

1. General Theory

1.1. Simplicial Categories and Simplicial Theories

Definition 1.1 A *simplicial theory* is a monad (on $sSet^J$) commuting with filtered colimits.

Theorem 1.2 *(Rezk)*[R] *Let T be a simplicial theory in $sSet^J$. Then T-alg admits a simplicial model structure. f is a weak equivalence or a fibration in T-alg if and only if its image in $sSet^J$ is one (for the projective model structure). Moreover, this model category is right proper.*

Proposition 1.3 *Model structures on simplicial categories.*

i. *Let $A = (A_p)$ be a commutative monoid in $s\mathcal{C}$. The monadic adjunctions A_p-mod $\rightleftarrows Set^J$ induce a monadic adjunction sA-mod $\rightleftarrows sSet^J$ i.e. there is an equivalence sA-mod $\backsimeq T_A$-alg where T_A is the monad induced by the adjunction. In particular $s\mathcal{C} \backsimeq T_1$-alg.*

ii. *Let $A = (A_p)$ be a commutative monoid in $s\mathcal{C}$. The monadic adjunctions A_p-alg $\rightleftarrows Set^J$ induce a monadic adjunction sA-alg $\rightleftarrows sSet^J$ i.e. there is an equivalence sA-alg $\backsimeq T_A^c$-alg where T_A^c is the monad induced by the adjunction. In particular sComm($\mathcal{C}$)$\backsimeq T_1^c$-alg.*

Proof: As explained in the preliminaries, this is due to the characterisation of monadic functors (see [Bc]). □

Remark 1.4 The right adjoint functors all commute with filtered colimits. So do the monads which are then simplicial theories on $sSet^J$.

Corollary 1.5 *Let A be a commutative monoid in $s\mathcal{C}$. The categories $s\mathcal{C}$, sA-mod and sComm($\mathcal{C}$) are model categories. Moreover, the functors $(A \otimes -)$ and sL are left Quillen and their adjoints preserve, by construction, weak equivalences and fibrations.*

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Theorem 1.6 *The category $s\mathcal{C}$ (resp sA-mod) is a monoidal model category (as defined in* [H]*).*

Proof: The proofs for $s\mathcal{C}$ and sA-mod are similar, so let us prove it for $s\mathcal{C}$. Let I, I' be respectively the sets of generating cofibrations and generating trivial cofibrations. I and I' are the images under the left adjoint functor of respectively the generating cofibrations and generating trivial cofibrations in $sSet^J$. Let $\square$ be the pushout product (definition 4.2.1 in [H]). We just have to prove (cf [H] 4.2.6) that $I \square I$ is a set of cofibrations and $I \square I'$ and $I' \square I$ are sets of trivial cofibrations. It is true for generating cofibrations and generating trivial cofibrations in $sSet^J$, which are all morphisms concentrated in one level. Moreover, it is easy to verify that the functor sK commutes with $\square$ of morphisms concentrated in one level. So it is true in $s\mathcal{C}$. The second axiom is clearly satisfied, as 1 is cofibrant. $\square$

Remark 1.7 In a symmetric monoidal model category in the sense of [H], the derived monoidal product will be denoted $\otimes^h$. Recall that, for $M, N \in \mathcal{C}$, $M \otimes^h N = qM \otimes qN$ where q is a cofibrant replacement in $\mathcal{C}$ ([H], proposition 4.3.1).

1.2. Compactly Generated Model categories

Definition 1.8 [H] Let $\mathcal{M}$ be a cofibrantly generated simplicial model category and let I be the set of generating cofibrations.

i. An object $X \in I$-cell is *strictly finite* if and only if there exists a finite sequence

$$\emptyset = X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n = X$$

and $\forall\, i$ a pushout diagram:

$$\begin{array}{ccc} X_i & \longrightarrow & X_{i+1} \\ \downarrow & & \downarrow \\ A & \xrightarrow[u_i]{} & B \end{array}$$

with $u_i \in I$.

ii. An object $X \in I$-cell is *finite* if and only if it is weakly equivalent to a strictly finite object.

iii. An object X is *homotopically finitely presented* (or ω-compact) if and only if for any filtered diagram Y_i, the morphism :

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$$\mathrm{Hocolim}_i \mathrm{Map}(X, Y_i) \to \mathrm{Map}(X, \mathrm{Hocolim}_i Y_i)$$

is an isomorphism in $\mathrm{Ho}(sSet)$

iv. A model category $\mathcal{M}$ is *compactly generated* if it is cellular, cofibrantly generated and if the domains and codomains of generating cofibrations and generating trivial cofibrations are cofibrant, ω-compact and ω-small (relative to $\mathcal{M}$).

Proposition 1.9 [TV]*Let $\mathcal{M}$ be a compactly generated model category.*

i. For any filtered diagram X_i, the natural morphism $\mathrm{Hocolim}_i X_i \to \mathrm{Colim}_i X_i$ *is an isomorphism in* $\mathrm{Ho}(\mathcal{M})$.

ii. Assume that filtered colimits are exact in $\mathcal{M}$. Then homotopically finitely presented objects in $\mathcal{M}$ are exactly objects equivalent to weak retracts of strictly finite I-cell objects.

Proposition 1.10 *i. The simplicial model category $sSet^J$ is compactly generated.*

ii. The categories of simplicial algebras over a simplicial theory are compactly generated.

Lemma 1.11 *Let A be in sComm(℮). Let u^j be the family of images under the left adjoint functor in sA-mod (resp sA-alg) of elements $*_j$ of $sSet^J$ defined by $*$ on level j and $\emptyset$ on other levels. Any codomain of a generating cofibration in sA-mod (resp sA-alg) is weakly equivalent to an object u^j. Any domain of a generating cofibration is weakly equivalent, for a given element j in J, to an object obtained from the initial object (denoted by $\emptyset$) and u^j by a finite number of homotopy pushouts.*

Proof: Generating cofibrations of sA-mod are images of generating cofibrations of $sSet^J$ under the left adjoint functor. Generating cofibrations of $sSet$ are morphisms $\delta\Delta^p \to \Delta^p$. Their codomains are contractible, thus so are the codomains of generating cofibrations of $sSet^J$ for the projective model structure, and their images under the left adjoint are weakly equivalent to the unit 1. For the domains, consider the relation $\delta\Delta^{p+1} \backsimeq \Delta^{p+1} \coprod^h_{\delta\Delta^p} \Delta^{p+1} \backsimeq * \coprod^h_{\delta\Delta^p} *$ and $\delta\Delta^0 = \emptyset$. Domains of generating cofibrations in $sSet^J$ for the projective model structure are objects $(\delta\Delta^{p,j})_{p\in N,\, j\in V}$ defined in level $i \neq j$ by $\emptyset$ and in level j by $\delta\Delta^p$ and satisfying the relation

$$(\delta\Delta^{p,j}) \backsimeq *_j \coprod^h_{(\delta\Delta^{p-1,j})} *_j.$$

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Clearly $\delta\Delta^{0,j} = \emptyset$ and $\delta\Delta^{1,j} = *_j$. Let $u^{p,j}$ denote the image of $\delta\Delta^{p,j}$. For all j, $u^{p,j}$ is obtained by a finite number of homotopy pushouts from $\emptyset$ and u^j. □

Corollary 1.12 *of proposition 1.10 and lemma 1.11.*

i. The Simplicial Model categories $s\mathcal{C}$, sA-mod ($A \in sComm(\mathcal{C})$), $sComm(\mathcal{C})$ and sA-alg ($A \in sComm(\mathcal{C})$) are compactly generated.

ii. Homotopically finitely presented objects of sA-mod (resp sA-alg) are exactly objects weakly equivalent to weak retracts of strictly finite I-cell objects.

iii. The subcategory of Ho(sA-mod) (resp Ho(sA-alg)) consisting of homotopically finitely presented objects and denoted by Ho(sA-mod)$_c$(resp Ho(sA-alg)$_c$) is the smallest full subcategory of Ho(sA-mod) (resp Ho(sA-alg)) containing the family $(u^j)_{j\in J}$ $(resp\ (u^j_A := L_A(u^j))_{j\in J})$, and stable under retracts and homotopy pushouts.

Proof: (of the assertion (iii)). Let $\mathcal{D}$ be the smallest full subcategory of Ho($s\mathcal{C}$) containing the family $(u^j)_{j\in J}$ $(resp\ (u^j_A)_{j\in J})$, the initial object $\emptyset$ and stable under retracts and homotopy pushouts. Clearly, by the assertion ii, as $(\emptyset \to u^j)_{j\in J}$ are generating cofibrations of $s\mathcal{C}$, the category Ho$(s\mathcal{C})_c$ contains the family $(u^j)_{j\in J}$, and is stable under retracts and homotopy pushouts. Thus $\mathcal{D} \subset$ Ho$(s\mathcal{C})_c$. Conversely, let X be an object of Ho$(s\mathcal{C})_c$. By the assertion ii, X is isomorphic to a weak retract of a strictly finite I-cell object. Therefore, there exists a finite sequence of maps

$$\emptyset = X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_n = X$$

such that, for $0 \leq k \leq n-1$, there exists a generating cofibration $K \to L$ and a pushout square

$$\begin{array}{ccc} X_k & \longrightarrow & X_{k+1} \\ \uparrow & & \uparrow \\ K & \longrightarrow & L\ . \end{array}$$

Now, as domains and codomains of generating cofibrations are in $\mathcal{D}$, X is in $\mathcal{D}$. □

1.3. Categories of Modules and Algebras

Proposition 1.13 *Let A be in $sComm(\mathcal{C})$ and B be a simplicial monoid in sA-alg, cofibrant in sA-mod. Then :*

i. The forgetful functor from sB-mod to sA-mod preserves cofibrations.

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ii. The forgetful functor from sA-alg to sA-mod preserves cofibrations whose domain is cofibrant in sA-mod. In particular, it preserves cofibrant objects.

Proof: In each case, we just have to prove the assertion for generating cofibrations and then generalize to any cofibration by the small object argument.

Proof of the assertion i: First, we choose a generating cofibration in sB-mod. As generating cofibrations of sB-mod are images of generating cofibrations of $sSet^J$, we just have to set $L \to M$, a generating cofibration in $sSet^J$. Let K_A and K_B denote the left adjoint functors (from $sSet^J$) for respectively sA-mod and sB-mod. The axiom of stability under $\square$ implies that the morphism $(\emptyset \to B)\square(K_A(L) \to K_A(M))$ is a cofibration in sA-mod. This morphism is in fact $K_B(L) = B \otimes_A K_A(L) \to B \otimes_A K_A(M) = K_B(M)$, hence generating cofibrations of sB-mod are cofibrations in sA-mod.

Proof of the assertion ii: As for i, let $N \to M$ be a generating cofibration in $sSet^J$. Let L^s denote the functor " free associated commutative monoid" of $sSet^J$. The functors L (resp sL_A in sA-mod) and K_A are defined by colimits and so commute up to isomorphisms. This means that $K_A \circ L^s \backsimeq sL_A \circ K_A$. So the generating cofibration in sA-alg corresponding to $N \to M$ is isomorphic to $K_A(L(N)) \to K_A(L(M))$. To prove that it is a cofibration in sA-mod, we have to prove that the morphism $L(N) \to L(M)$ is injective levelwise. But this is clear as any morphism $N^{\otimes n}/S_n \to M^{\otimes n}/S_n$ is injective. Thus any generating cofibration in sA-alg is a cofibration in sA-mod. In fact it is a generating cofibration in sA-mod. To use the small object argument (of sA-alg), we need to verify that it preserves cofibrations in sA-mod. In fact, we need to check that a homotopy pushout in sA-alg of a cofibration in sA-mod is still a cofibration in sA-mod. We leave it to the reader to verify that this is a consequence of the axiom of stability by $\square$. Finally, the forgetful functor preserves cofibrations and, as A is cofibrant in sA-mod, any cofibrant object of sA-alg is also cofibrant in sA-mod. $\square$

Lemma 1.14 *Let $A \to B \in sComm(\mathcal{C})$ be a trivial cofibration between cofibrant objects. Then the categories of modules are equivalent i.e.* Ho(sA-*mod*)$\backsimeq$ Ho(sB-*mod*).

Proof: We must prove that for X cofibrant in sA-mod and Y fibrant in sB-mod, the morphism $\varphi_A(f) : X \otimes_A B \to Y$ is a weak equivalence in sB-mod if and only if the morphism $f : X \to Y$ is one in sA-mod. By the previous proposition, $A \to B$ is a trivial cofibration in sA-mod. Thus as X is cofibrant, using the axiom of stability under $\square$, the morphism $g : X \to X \otimes_A B$ is a weak equivalence in sA-mod. By construction of the adjunction φ_A, the following diagram is commutative :

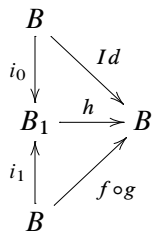
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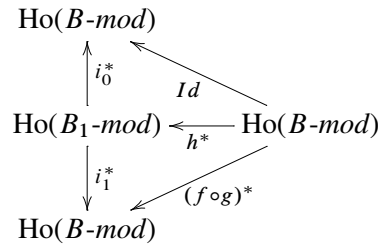
$$X \xrightarrow{g} X \otimes_A B \underset{f}{\overset{\varphi_A(f)}{\rightrightarrows}} Y \otimes_A B \to Y \ .$$

Thus $f = \varphi_A(f) \circ g$. Finally, $\varphi_A(f)$ is a weak equivalence in sA-mod if and only if it is so in sB-mod and the *two out of three* axiom ends the proof. □

Proposition 1.15 *Let $f : A \to B \in sComm(\mathcal{C})$ be a weak equivalence between cofibrant objects. Then the categories of modules are equivalent i.e.* Ho(sA-*mod*)$\backsimeq$ Ho(sB-*mod*).

Proof: Let r_c be the fibrant replacement of sComm($\mathcal{C}$). Then by the previous lemma, the homotopical categories of modules over A and r_cA (resp B and r_cB) are equivalent. Thus A and B can be taken fibrant and f is then a homotopy equivalence i.e. there exists g such that $f \circ g$ and $g \circ f$ are homotopic to the identity. The following diagrams are commutative:





where i_0 and i_1 are cofibrations and have the same right inverse p i.e. such that $p \circ i_1 = p \circ i_0 = Id_B$. The morphism h is a trivial fibration, thus i_0 is a weak equivalence. By the previous lemma, i_0^* is an equivalence of categories. Thus, so is p^*. As i_1^* and i_0^* are both inverses of p^*, they are isomorphic and i_1^* is also an equivalence. Finally, h^* is an equivalence and so is $(f \circ g)^*$. The same method proves that $(g \circ f)^*$ is an equivalence. □

1.4. Finiteness Conditions

Definition 1.16 Let q_c be a cofibrant replacement in sComm($\mathcal{C}$) and $f : A \to B$ a morphism in sComm($\mathcal{C}$).

- ◇ The morphism f is *homotopically finite* (denoted hf) if B is homotopically finitely presented in sq_cA-mod.
- ◇ The morphism f is *homotopically finitely presented* (denoted hfp) if B is homotopically finitely presented in sq_cA-alg.

Remark 1.17 The morphism $A \to B$ is hf (resp hfp) if and only if the morphism $q_cA \to q_cB$ is hf (resp hfp). The morphism $q_cB \to B$ is always hf.

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Lemma 1.18 *The hf (resp hfp) morphisms are stable under composition.*

Proof: The proofs for hf morphisms and hfp morphisms are analogous so let us prove it for hf morphisms. Let $A \to B \to C$ be the composition of two hf morphisms. There is a diagram

$$\begin{array}{ccccc} q_cA & \longrightarrow & q_cB & \longrightarrow & q_cC \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

and forgetful functors $F_1 : sq_cC\text{-mod} \to sq_cB\text{-mod}$ and $F_2 : sq_cB\text{-mod} \to sq_cA\text{-mod}$. The image $F_1(q_cC)$ of q_cC is homotopically finitely presented in sq_cB-mod, hence weakly equivalent to a retract of a finite homotopy colimit of q_cB in Ho(sq_cB-mod). The forgetful functor F_2 preserves retracts, equivalences, finite colimits, cofibrant objects and cofibrations whose domain is cofibrant. Thus it also preserves finite homotopy colimits and sends q_cC to a retract of a finite homotopy colimit of q_cB in Ho(sq_cA-mod). As q_cB is homotopically finitely presented in sq_cA-mod, and as homotopically finitely presented objects are stable under retracts, equivalences and finite homotopy colimits, C is sent by $F_2 \circ F_1$ in $sq_cA\text{-mod}_c$. Hence $A \to C$ is finite. □

Lemma 1.19 *The hf (resp hfp) morphims are stable under homotopy pushout of simplicial monoids.*

Proof: The proofs for hf morphisms and hfp morphisms are analogous so let us prove it for hf morphisms. Let $A \to B$ and $A \to C$ be in sComm(℮) such that the first is finite. Let q_cA be the cofibrant replacement of q_cA-alg. It is weakly equivalent to q_c and the object $q_{c_A}B$ is homotopically finitely presented in sq_cA-mod. Let us prove that the derived tensor product (1.7) $B \otimes^h_A C \backsimeq q_{cA}B \otimes_{q_cA} q_cC$ (in Ho(q_cA-mod)) is homotopically finitely presented in q_cC-mod. The forgetful functor $sq_cC\text{-mod} \to sq_cA$-mod preserves filtered colimits and weak equivalences, hence it preserves filtered homotopy colimits. Thus the derived functor $- \otimes_{q_cA} q_cC$ preserves homotopically finitely presented objects. So $B \otimes^h_A C$ is homotopically finitely presented in Ho(q_cC-mod). □

1.5. A Definition for Smoothness

Definition 1.20 A morphism $A \to B$ in sComm(℮) is *formally smooth* if the morphism $B \otimes^h_A B \to B$ is hf, where $\otimes^h$ is the derived monoidal product in the monoidal model category (1.7).

Remark 1.21 This definition does not generalise the definition of formal smoothness in the sense of rings. However, the corresponding notion of smoothness is a generalisation of the classical notion of smoothness, as will be proved later in this

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article.

Proposition 1.22 *Formally smooth morphisms are stable under composition.*

Proof: By the previous remarks, it can be assumed that A is cofibrant in sComm($\mathcal{C}$), B is cofibrant in sA-alg and C is cofibrant in sB-alg. Let $A \to B \to C$ be the composition of two formally smooth morphisms. The morphisms $B \coprod_A B \to B$ and $C \coprod_B C \to C$ are hf. The following diagram commutes and is clearly cocartesian :

$$\begin{array}{ccc} C & & \\ \uparrow & \nwarrow & \\ C \coprod_A C & \longrightarrow & C \coprod_B C \\ \uparrow & & \uparrow \\ B \coprod_A B & \longrightarrow & B \ . \end{array}$$

Thus, if it is cofibrant for the Reedy structure, it will be homotopically cocartesian. The morphisms $B \backsimeq B \coprod_A A \to B \coprod_A B$ and $B \coprod_A B \to C \coprod_A C$ are images under the left Quillen functor $colim$ of clear Reedy cofibrations (see [A] for a description of these cofibrations), thus are cofibrations. In particular $B \coprod_A B$ and $C \coprod_A C$ are cofibrant and the diagram considered is Reedy cofibrant. Finally, the morphism $C \coprod_A C \to C \coprod_B C$ is hf as a homotopy pushout of hf morphisms and $C \coprod_A C \to C$ is hf as a composition of hf morphisms. □

Proposition 1.23 *Formally smooth morphism are stable under homotopy pushout.*

Proof: Let $u : A \to B$ be a formally smooth morphism and C a commutative A-algebra. By the previous remarks it can be assumed that A and C are cofibrant in sComm($\mathcal{C}$) and that B is cofibrant in sA-alg. Let D denote the homotopy pushout $B \otimes^h_A C \backsimeq B \otimes_A C$ and f denote the morphism from B to D. Clearly:

$$D \otimes_C D \backsimeq B \otimes_A C \otimes_C B \otimes_A C \backsimeq B \otimes_A D.$$

Thus the following diagram commutes :

$$\begin{array}{ccccc} & & B \otimes_A B & \xrightarrow{m_B} & B \\ & \swarrow & \downarrow{\scriptstyle Id \otimes_A f} & & \downarrow{\scriptstyle f} \\ D \otimes_C D & \longrightarrow & B \otimes_A D & \xrightarrow[m_D]{} & D \ . \end{array}$$

And is cocartesian :

$$B \otimes_{B \otimes_A B} B \otimes_A B \otimes_A C \backsimeq B \otimes_A C \backsimeq D.$$

Moreover it is clearly cofibrant as $B \otimes_A -$ preserves cofibrations. Finally by stability of hf morphisms under homotopy pushout, the morphism $C \to D$ is formally smooth. □

Definition 1.24 Let A be in sComm($\mathcal{C}$) and let M be in sA-mod.

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i. The object M is *n-truncated* if $\mathrm{Map}_{s\mathcal{C}}(X,M)$ is n-truncated in $sSet$, $\forall\ X \in s\mathcal{C}$.

ii. The *Tor-Dimension* of B in sA-alg is defined by

$$\mathrm{Tordim}_A(B) = inf\{n \mid X \otimes^h_{q_c A\, q_c A} B\ is\ (n+p)\text{-truncated}\ \forall\ X \in sq_cA\text{-mod which are } p\text{-truncated}\}.$$

where q_c and q_cA are the cofibrant replacements respectively in sComm($\mathcal{C}$) and sA-alg.

iii. A morphism of monoids $A \to B$ has *Tor dimension n* if $\mathrm{Tordim}_A(B) = n$, where q_c is a cofibrant replacement in sComm($\mathcal{C}$).

Lemma 1.25 *Tor dimension zero morphisms are stable under composition and homotopy pushout.*

Proof: Let $A \to B \to C$ be the composition of two Tor dimension zero morphisms. Considering the definition of Tor dimension, we just have to prove it for A cofibrant in sComm($\mathcal{C}$), B cofibrant in sA-alg and C cofibrant in sB-alg. Let M be a p-truncated A-module and q denote the cofibrant replacement in sA-mod,

$$M \otimes^h_A C \backsimeq qM \otimes_A C \backsimeq qM \otimes_A B \otimes_B C \backsimeq M \otimes^h_A B \otimes^h_B C.$$

As $\mathrm{Tordim}_A(B) = 0$, the B-module $M \otimes^h_A B$ is p-truncated. As $\mathrm{Tordim}_B(C) = 0$, the C-module $(M \otimes^h_A B) \otimes^h_B C$ is p-truncated. Thus $\mathrm{Tordim}_A(C) = 0$.

Let $A \to B$ be a Tor dimension zero morphism and $A \to C$ a morphism in Comm($\mathcal{C}$). Considering the definition of Tor dimension, we just have to prove it for A,C cofibrant in sComm($\mathcal{C}$), B cofibrant in sA-alg. Let M be in C-mod and let D denote the homotopy pushout $B \otimes^h_A C \backsimeq B \otimes_A C$. We have

$$M \otimes^h_C D \backsimeq qM \otimes_C D \backsimeq qM \otimes_A B \backsimeq M \otimes^h_A B.$$

Thus, $\mathrm{Tordim}_A(B) = 0$ implies $\mathrm{Tordim}_C(D) = 0$. □

Definition 1.26 A morphism $A \to B$ in Comm($\mathcal{C}$) is *smooth* if it is formally smooth, hfp and has Tor-Dimension zero. A morphism of affine schemes is *smooth* if the corresponding morphism of monoids is smooth. We say that an affine scheme X is *smooth* if the morphism $X \to \mathrm{Spec}(1)$ is smooth.

Theorem 1.27 *Smooth morphisms are stable under composition and homotopy pushout.*

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Proof: This a corollary of 1.18, 1.19, 1.22, 1.23, and 1.25 . □

Theorem 1.28 *Let $\mathcal{C}$ be a relative context in the sense of* [M] *and let $A \to B$ be a Zariski open immersion in Comm($\mathcal{C}$), where A is cofibrant in sComm($\mathcal{C}$) and B is cofibrant in sA-alg. Then the morphism $A \to B$ is smooth.*

Proof: A Zariski open immersion is always formally smooth and its diagonal is even an isomorphism. Thus we will need to prove that it is hfp and of Tor dimension zero. First, if there exists $f \in A_0$, an object of the underlying set of A, such that $B \backsimeq A_f$, the result is clear. Indeed, A_f is given by a filtered colimit of A thus is of Tor dimension zero. Let us prove that it is hfp. It is clear that $A \to A[X]$ is homotopically finitely presented. Thus as everything is cofibrant, we can write A_f as a finite colimit of $A[X]$ ([M] proposition 1.20) which is in fact a finite homotopy colimit. Thus finally $A \to A_f$ is hfp.

Now if B defines a Zariski open object of A, we can write B as a cokernel of products of A_f. As the functors π_i commute with products in $sSet^J$, the products preserve weak equivalences for the model structure (1.2). Thus it is clear that $A \to B$ is hfp. For Tor dimension, recall that there is a finite family of functors reflecting isomorphisms B-mod$\to A_f$-mod. Let M be a p-truncated A-module. This family sends $M \otimes^L_A B$ and its n-truncations, $n > p$ to the same module qM_f (where q is the cofibrant replacement of sA-mod). Thus clearly $M \otimes^L_A B$ is p-truncated and $\mathrm{TorDim}_A(B) = 0$. □

2. Simplicial Presheaf Cohomology

In the article [T1], B. Toën defines a cohomology for connected and pointed simplicial presheaves. We will define here a cohomology for general simplicial presheaves. This theory will be used to find examples of smooth morphisms of commutative monoids (in sets). The references cited in this section are [T1], [GJ] and [J].

2.1. Definitions

In this section, $\mathcal{D}$ is a category and sPr($\mathcal{D}$) is the category of simplicial presheaves over $\mathcal{D}$.

Definition 2.1 [GJ]$VI.3$ Consider $F \in \mathrm{sPr}(\mathcal{D})$. The *tower of n-truncations of F* is the Postnikov tower :

$$\ldots \longrightarrow \tau_{\leq n} F \longrightarrow \tau_{\leq n-1} F \longrightarrow \ldots \longrightarrow \tau_{\leq 1} F \longrightarrow \tau_{\leq 0} F \ .$$

Definition 2.2 Let F be a simplicial presheaf.

◇ The functor $\pi_0(F) : \mathcal{D} \to Set$ is defined by $\pi_0(F) := X \to \pi_0(F(X))$.

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◇ The category $(\mathcal{D}/F)_0$ is the full subcategory of $\mathrm{sPr}(\mathcal{D})/F$ whose objects are in $\mathcal{D}$.

◇ The functor $\pi_n(F) : (\mathcal{D}/F)_0 \to Set$ is defined by

$$\pi_n(F)(X,u) := \pi_n(F(X),u).$$

Definition 2.3 Let G be a simplicial group.

◇ The bisimplicial set $E(G,1)$ is defined as the nerve of the morphism $G \to *$. In particular ([T1], paragraph 1.3), we have $E(G,1)_{p,q} = G_p^q$.

◇ The *classifying space* of G, denoted by $K(G,1)$, is given by the diagonal of the bisimplicial set $E(G,1)/G$. More precisely $K(G,1)_n = G_n^n/G_n$. It is abelian if G is abelian.

◇ The endofunctor of abelian groups $K(G,1)^{\circ n}$ is denoted by $K(G,n)$.

Remarks 2.4 As the diagonal of $E(G,1)$ is pointed (by the identity), the simplicial set $K(G,1)$ is also pointed. In particular, $\pi_n(K(G,1),*) \backsimeq \pi_{n-1}(G,e_g)$. This construction is functorial (in G) and extends to presheaves of simplicial groups.

2.2. Simplicial Presheaf Cohomology

It is necessary to work in the proper category to construct a cohomology for a simplicial presheaf F which is not connected or pointed. In fact the 1-truncation of F is the nerve NG of a groupoid G and in the category $\mathrm{sPr}(\mathcal{D})/NG$, F becomes connected and pointed. But in this category, there is no clear construction for classifying spaces. The solution of this problem is given by a Quillen equivalence with the category $\mathrm{sPr}(\mathcal{D}/G)$, for a well chosen category $\mathcal{D}/G$. We choose now a simplicial presheaf F.

The Category of Presheaves

The left adjoint functor $\widetilde{(-)}$

Definition 2.5 The category $\mathcal{D}/G$ is the Grothendieck construction associated to G i.e. is the category whose objects are couples (X,x), $x : X \to NG$, and whose morphisms from (X,x) to (Y,y) are couples (f,u) where $f : X \to Y$ and $u : y \circ f \backsimeq x$ in $G(X) \backsimeq \pi_1 F(X)$.

The first step is to construct a functor $\widetilde{(-)} : \mathcal{D}/G \to \mathrm{sPr}(\mathcal{D})/NG$.

Definition 2.6 Let (X,x) be in $\mathcal{D}/G$. Let us define the presheaf of groupoids $G_{X,x}$ on $\mathcal{D}$. The image of $S \in \mathcal{D}$ under $G_{X,x}$ is the groupoid described as follow:

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◇ The objects are triples (u,y,h), $u : S \to X$, $y \in G(S)$, and $h : x \circ u \backsimeq y \in G(S)$.

◇ A morphism from (u,y,h) to (u',y',h') is an endomorphism k of S such that $k^*(h : x \circ u \to y) = h' : x' \circ u' \to y'$.

Let $\check{X}$ denote the nerve of this groupoid.

Remark 2.7 There is a commutative diagram of presheaves of groupoids

$$\begin{array}{ccc} X & \xrightarrow{\quad x \quad} & G \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle l} & \\ & G_{X,x} & \end{array}$$

where l is the projection on G and j is given for $S \in \mathcal{D}$ by $j(S) : u \in \mathrm{Hom}_{\mathcal{D}}(S,X) \to (u, x \circ u, Id) \in G_{X,x}$.

Applying the nerve functor, one gets a morphism $\check{x} := Nl : \check{X} \to NG$. We define a functor

$$\begin{aligned} (\check{-}) : \mathcal{D}/G &\to \mathrm{sPr}(\mathcal{D})/NG \\ (X,x) &\to (\check{X},\check{x}). \end{aligned}$$

Definition 2.8 The functor $\widetilde{(-)} : \mathcal{D}/G \to \mathrm{sPr}(\mathcal{D})/NG$ is defined by

$$\widetilde{(-)} : (X,x) \to (\widetilde{X},\widetilde{x}) := Q(\check{X},\check{x})$$

where Q is a cofibrant replacement in $\mathrm{sPr}(\mathcal{D})/NG$.

Remarks 2.9 This functor has a Kan extension to $\mathrm{sPr}(\mathcal{D}/G)$, still denoted by

$$\widetilde{(-)} : \mathrm{sPr}(\mathcal{D}/G) \to \mathrm{sPr}(\mathcal{D})/NG.$$

In fact, the category $\mathrm{sPr}(\mathcal{D}/G)$ is equivalent to the category $\mathrm{sPr}(\mathcal{D})^{NG}$ defined in [J] and the equivalence of categories we are constructing is constructed in a different way and in a more general situation in [J].

The right adjoint functor $(-)_1$ We construct now the (right) adjoint of $\widetilde{(-)}$, denoted by $(-)_1$.

Definition 2.10 The functor $(-)_1 : \mathrm{sPr}(\mathcal{D})/NG \to \mathrm{sPr}(\mathcal{D}/G)$ is defined by

$$(-)_1 : (H,u) \to H_1 := (X,x) \to \underline{\mathrm{Hom}}^{\Delta}_{\mathrm{sPr}(\mathcal{D})/NG}((\widetilde{X},\widetilde{x}),(H,u))$$

where $\underline{\mathrm{Hom}}^{\Delta}$ is the simplicial Hom. As $(\widetilde{X},\widetilde{x})$ is constructed cofibrant, the functor $(-)_1$ is a right Quillen functor and its adjoint is a left Quillen functor. Let us prove that $R(-)_1$ commutes with homotopy colimits. We need to recall first some properties.

Definition 2.11 Let (H,h) be in $\mathrm{sPr}(\mathcal{D})/NG$ and let (X,x) be in $\mathcal{D}/G$. Define an object (H_X,h_x) by the homotopy pullback diagram

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$$\begin{array}{ccc} H_X & \longrightarrow & H \\ \downarrow & & \downarrow h \\ X & \xrightarrow[x]{} & NG\ . \end{array}$$

Lemma 2.12 *Let (H,h) be a homotopy colimit, $H \backsimeq \mathrm{Hocolim}(H_i)$, in $\mathrm{sPr}(\mathcal{D})/NG$ and let (X,x) be in $\mathcal{D}/G$. Then :*

- ◇ *There is an isomorphism $H_X \backsimeq \mathrm{Hocolim}\,(H_i)_X$ in $\mathrm{Ho}(\mathrm{sPr}(\mathcal{D})/NG)$.*
- ◇ *There is an isomorphism $RH_1(X) \backsimeq \mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/X}((X,Id),(H_X,h_x))$.*

Corollary 2.13 *The functor $R(-)_1$ commutes with homotopy colimits.*

Proof: Let $H \backsimeq \mathrm{Hocolim}(H_i)$ be a homotopy colimit and let (X,x) be in $\mathcal{D}/G$.

$$\begin{array}{rcl} RH_1(X) & \backsimeq & \mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/X}((X,Id),([\mathrm{Hocolim}(H_i)]_X,[\mathrm{Hocolim}(h_i)]_x)) \\ & \backsimeq & \mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/X}((X,Id),(\mathrm{Hocolim}[(H_i)_X],\mathrm{Hocolim}[(h_i)_x])) \\ & \backsimeq & \mathrm{Hocolim}(R(H_i)_1(X)). \end{array}$$

□

The Equivalence

Proposition 2.14 *The Quillen functors $\widetilde{(-)}$: $sPr(\mathcal{D}/G) \to sPr(\mathcal{D})/NG$ (2.9) and $(-)_1$: $sPr(\mathcal{D})/NG \to sPr(\mathcal{D}/G)$ (2.10) define a Quillen equivalence.*

Proof: The functor $\widetilde{(-)}$ commutes with homotopy colimits and as any object in $\mathrm{sPr}(\mathcal{D}/G)$ is a homotopy colimit of representable objects $H \backsimeq \mathrm{Hocolim}(X_i)$, its image can be computed in terms of representable objects, i.e. $\widetilde{H} \backsimeq \mathrm{Hocolim}(\widetilde{X_i})$. The short exact sequence $H_1 \to H \to \tau_{\leq 1}H$ proves that $(-)_1$ preserves weak equivalences. Then

$$(\widetilde{H})_1 \backsimeq (\mathrm{Hocolim}(\widetilde{X_i}))_1 \backsimeq \mathrm{Hocolim}(X_i) \backsimeq H.$$

If H is cofibrant in $\mathrm{sPr}(\mathcal{D}/G)$ and H' is fibrant in $\mathrm{sPr}(\mathcal{D})/NG$, we consider a morphism between short exact sequences

$$\begin{array}{ccccc} H & \longrightarrow & \widetilde{H} & \longrightarrow & NG \\ \downarrow & & \downarrow & & \downarrow Id \\ H_1' & \longrightarrow & H' & \longrightarrow & NG\ . \end{array}$$

Applying the functors π_i, it is clear that if $\widetilde{H} \to H'$ is an equivalence, so is $H \to H_1'$. Conversely, if $H \to H_1'$ is an equivalence, the homotopy fibers of $\widetilde{H} \to H'$ over NG are equivalences, thus so is $\widetilde{H} \to H'$. □

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THE COHOMOLOGY

Definition 2.15 Let F be in $\mathrm{sPr}(\mathcal{D})$. Then a *local system* on F is a presheaf of abelian groups on $\mathcal{D}/G$, where G satisfies $NG \backsimeq \tau_{\leq 1}F$. A *morphism of local systems* is a morphism of presheaves of abelian groups. The category of local systems on F will be denoted by $\mathrm{sysloc}(F)$. The n-th classifying space of M is denoted by $K(M,n)$ and its image under $\widetilde{\mathrm{L}(-)}$ is denoted by $\mathbf{L}\tilde{K}(M,n)$.

Remark 2.16 The object $\mathbf{L}\tilde{K}(M,n)$ is characterized up to equivalence by the isomorphisms

$$\pi_n(\mathbf{L}\tilde{K}(M,n)) \backsimeq M \qquad \pi_1(\mathbf{L}\tilde{K}(M,n)) \backsimeq \pi_1(F)$$
$$\pi_0(\mathbf{L}\tilde{K}(M,n)) \backsimeq \pi_0(F) \quad \pi_k(\mathbf{L}\tilde{K}(M,n)) \backsimeq *, k \neq 0,1,n.$$

Definition 2.17 Let F be in $\mathrm{sPr}(\mathcal{D})$ and let M be a local system on F. Then the *n-th cohomology group of F with coefficients in M* is

$$\mathrm{H}^n(F,M) := \pi_0 \mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/NG}(F, \mathbf{L}\tilde{K}(M,n)).$$

The standard example of a local system is π_n. Indeed, it has been defined on $(\mathcal{D}/F)_0$ but it clearly lifts to $\mathcal{D}/G$.

Here is the important theorem.

Theorem 2.18 *Let G be a groupoid. Then the functor*

$$\mathrm{H}^m(NG,-) : \mathrm{Sysloc}(NG) \to Ab$$
$$M \to \mathrm{H}^m(NG,M)$$

is isomorphic to the m-th derived functor of the functor $\mathrm{H}^0(NG,-)$.

Proof: There is an equivalence between the category of simplicial abelian group presheaves on $\mathcal{D}/G$, denoted by $\mathrm{sAb}(\mathcal{D}/G)$, and the category of complexes of abelian group presheaves concentrated in degrees ≤ 0, denoted by $C^-(\mathcal{D}/G, Ab)$. This is a generalisation of Dold-Kan correspondence. There is a correspondence between quasi-isomorphisms of complexes and weak equivalences of simplicial presheaves, and an induced equivalence between the homotopy categories :

$$\Gamma : D^-(\mathcal{D}/G, Ab) \backsimeq \mathrm{Ho}(\mathrm{sAb}(\mathcal{D}/G)).$$

The derived functors of H^0 are given by

$$\mathrm{H}^m_{der}(\mathcal{D}/G, M) \backsimeq \mathrm{Hom}_{D^-(\mathcal{D}/G,Ab)}(\mathbb{Z}, M[m])$$

where $\mathbb{Z}$ is regarded as a complex concentrated in degree zero and $M[m]$ is concentrated in degree $-m$, with value M. The presheaf $\Gamma(\mathbb{Z})$ is the constant presheaf with fiber $\mathbb{Z}$ (still denoted by $\mathbb{Z}$) and the presheaf $\Gamma(M[m])$ is equivalent to $K(M,m)$. The functor Γ then induces an isomorphism :

$$\mathrm{Hom}_{D^-(\mathcal{D}/G,Ab)}(\mathbb{Z}, M[m]) \backsimeq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{sAb}(\mathcal{D}/G))}(\mathbb{Z}, K(M,m)).$$

Finally, the adjunction between the abelianization functor, denoted by $\mathbb{Z}(-)$, from $\mathrm{sPr}(\mathcal{D}/G)$ to $\mathrm{sAb}(\mathcal{D}/G)$ and the forgetful functor gives

$$\mathrm{Hom}_{D^-(\mathcal{D}/G,Ab)}(\mathbb{Z}, M[m]) \backsimeq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{sPr}(\mathcal{D}/G))}(*, K(M,m)) \backsimeq \mathrm{H}^m(NG,M).$$

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□

THE OBSTRUCTION THEORY

There is a homotopy pullback diagram in $\mathrm{sPr}(\mathcal{D}/G)$:

$$\begin{array}{ccc} \tau_{\leq n}F_1 & \longrightarrow & * \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}F_1 & \longrightarrow & K(\pi_n(F),n+1)\ . \end{array}$$

As F_1 is 1-connex, this pullback diagram is a (functorial) generalisation to presheaves of the diagram given by Proposition 5.1 of [GJ]. By the Quillen equivalence $((-)_1,\widetilde{(-)})$, there is a homotopy pullback diagram:

$$\begin{array}{ccc} \tau_{\leq n}F & \longrightarrow & NG \\ \downarrow & & \downarrow \\ \tau_{\leq n-1}F & \longrightarrow & \mathbf{L}\widetilde{K}(\pi_n(F),n+1)\ . \end{array}$$

If $H \to \tau_{\leq n-1}F$ is a morphism in $\mathrm{Ho}(\mathrm{sPr}(\mathcal{D})/NG)$, then it has a lift to $\tau_{\leq n}F$ if and only if it vanishes in the group

$$\pi_0\mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/NG}(H,L\widetilde{K}(\pi_n(F),n+1)).$$

This group can be described in terms of cohomology. Indeed, let G' be a groupoid such that $NG' \backsimeq \tau_{\leq 1}H$ and let u denote the morphism $u : NG' \to NG$. There is a Quillen adjunction:

$$\mathrm{sPr}(\mathcal{D})/NG' \underset{-\times_{NG}NG'}{\overset{u^*}{\rightleftarrows}} \mathcal{D}/NG\ .$$

This adjunction induces an isomorphism:

$$\begin{array}{c} \mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/NG}(H,\mathbf{L}\widetilde{K}(\pi_n(F),n+1)) \\ \backsimeq \\ \mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/NG'}(H,\mathbf{L}\widetilde{K}(\pi_n(F),n+1)\times_{NG}NG') \end{array}$$

where, to simplify the notations, we still write H for what we should call u^*H. There is a clear weak equivalence

$$\mathbf{L}\widetilde{K}(\pi_n(F),n+1)\times_{NG}NG' \backsimeq \mathbf{L}\widetilde{K}(\pi_n(F)\circ u^*,n+1).$$

Thus

$$\pi_0\mathrm{Map}_{\mathrm{sPr}(\mathcal{D})/NG}(H,\mathbf{L}\widetilde{K}(\pi_n(F),n+1)) \backsimeq \mathrm{H}^{n+1}(H,\pi_n(F)\circ u^*).$$

2.3. Simplicial Module Cohomology

Let B be a commutative monoid in $(Set,\times,\mathbb{F}_1)$. Then it is well known that there is an equivalence

$$\mathrm{sPr}(\mathcal{B}B)\backsimeq sB\text{-mod}$$

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where $\mathcal{B}B$ is the category with one object, the set of endomorphisms of which is isomorphic to B. We will identify these two categories in this part. Let A be a commutative monoid in sets and let $B \to A$ be a morphism of commutative monoids. We are in a particular case of the previous section, the category $\mathcal{D}$ is $\mathcal{B}B$ and the presheaf of groupoids G is just A. Let M be a local system on $\mathcal{B}B$. Then the n-th cohomology group obtained from definition 2.17 is

$$\mathrm{H}^n(A,M) := \pi_0 \mathrm{Map}_{sB\text{-mod}/A}(A, \mathbf{L}\widetilde{K}(M, n+1)).$$

Let Z denote the abelianization functor from B-mod$/A$ to the category of abelian group objects in B-mod$/A$ (denoted by $\mathrm{Ab}(B\text{-mod}/A)$). There is an equivalence between $\mathrm{Ab}(B\text{-mod}/A)$ and the category of A-graded $Z(B)$-modules, denoted by $Z(B)\text{-mod}^{A\text{-grad}}$. The following functor realizes this equivalence and its inverse is the forgetful functor.

$$\Theta : (\ M \xrightarrow{f} A\ ) \in \mathrm{Ab}(sB\text{-mod}/A) \to \oplus_{m \in A} f^{-1}(m) \in Z(B)\text{-mod}^{A\text{-grad}}.$$

This equivalence lifts to simplicial categories and it is easy to see that

$$\mathrm{H}^{n+1}(A,M) \backsimeq \pi_0 \mathrm{Map}_{Z(B)\text{-mod}^{A\text{-grad}}}(Z(A), \mathbf{L}\widetilde{K}(M, n+1)). \tag{1}$$

Here is the proposition that interests us.

Proposition 2.19 *Let $B \to A$ be a morphism of commutative monoids in sets. Then the morphism $B \to A$ is hf if and only if*

- *$Z(A)$ is homotopically finitely presented in $Z(B)$-mod$^{A\text{-grad}}$.*
- *A is homotopically finitely presented for the 1-truncated model structure i.e. in the category B-Gpd.*

Proof: Let us prove first the easiest part. Let A be an homotopically finitely presented object in sB-mod. Let $sB\text{-mod}^{\leq 1}$ denote the category sB-mod endowed with its 1-truncated model structure. In the adjunctions

$$sB\text{-mod} \underset{Id}{\overset{Id}{\rightleftarrows}} sB\text{-mod}^{\leq 1}$$

$$sB\text{-mod} \underset{i}{\overset{-\times A}{\rightleftarrows}} sZ(B)\text{-mod}/A$$

$$sB\text{-mod}/A \underset{i}{\overset{Z}{\rightleftarrows}} sZ(B)\text{-mod}^{A\text{-grad}}$$

the left adjoint functors preserve weak equivalences and cofibrations thus the right adjoints preserve homotopically finitely presentable objects.

Let us now prove the hardest part. We start with this lemma:

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Lemma 2.20 *There exists $m_0 \in \mathbb{N}$ such that for any local system M and all $n \geq m_0$*
$$\mathrm{H}^n(A, M) \backsimeq *.$$

Proof: The isomorphism 1 proves that the cohomology of A is isomorphic to the Ext functors of $Z(A)$ in $sZ(B)$-mod$^{A\text{-grad}}$. Moreover, there is an equivalence of abelian categories
$$sZ(B)\text{-mod}^{A\text{-grad}} \backsimeq C^-(\mathcal{B}B/A, Ab)$$
which induces by 2.18 an equivalence with the derived functors of H^0. In particular as $Z(A)$ is homotopically finitely presented, the derived functors of H^0 vanish after a set rank denoted by m_0. □

Remark 2.21 Two corollaries comes next. They are consequences of this lemma and the short exact sequence below. Let C be in A/sB-mod, then :
$$\begin{array}{ccc} \mathrm{Map}_{sB\text{-mod}/\tau_{\leq n-1}C}(A, \tau_{\leq n}C) & \longrightarrow & \mathrm{Map}_{sB\text{-mod}}(A, \tau_{\leq n}C) \\ & & \downarrow \\ \mathrm{Map}_{sB\text{-mod}/NG}(A, \mathbf{L}\widetilde{K}(\pi_n(C), n+1)) & \longleftarrow & \mathrm{Map}_{sB\text{-mod}}(A, \tau_{\leq n-1}C)\ . \end{array}$$

Corollary 2.22 *Let $A \xrightarrow{v} C$ be in A/sB-mod. For all $i \geq 1$, for all $n \geq n_i = n_0 + i + 1$*
$$\pi_0 \mathrm{Map}_{sB\text{-}mod}(A, \tau_{\leq n-1}C) \backsimeq \pi_0 \mathrm{Map}_{sB\text{-}mod}(A, \tau_{\leq n}C)$$
$$\pi_i(\mathrm{Map}_{sB\text{-}mod}(A, \tau_{\leq n-1}C), v) \backsimeq \pi_i(\mathrm{Map}_{sB\text{-}mod}(A, \tau_{\leq n}C), v).$$

Proof: We first prove that the simplicial set $\mathrm{Map}_{sB\text{-mod}/\tau_{\leq n-1}C}(A, \tau_{\leq n}C)$ is not empty. There are pushout squares :
$$\begin{array}{ccccc} A \times^h_{\mathbf{L}\widetilde{K}(\pi_n(C), n+1)} NG & \longrightarrow & \tau_{\leq n}C & \longrightarrow & NG \\ \downarrow & & \downarrow & & \downarrow s \\ A & \longrightarrow & \tau_{\tau_{\leq n-1}C} & \longrightarrow & \mathbf{L}\widetilde{K}(\pi_n(C), n+1) \end{array} \qquad p$$
where $p \circ s = Id$. Then there are equivalences
$$\begin{aligned} \mathrm{Map}_{sB\text{-mod}/\tau_{\leq n-1}F}(A, \tau_{\leq n}C) &\backsimeq \mathrm{Map}_{sB\text{-mod}/\mathbf{L}\widetilde{K}(\pi_n(C), n+1)}(A, NG) \\ &\backsimeq \mathrm{Map}_{sB\text{-mod}/A}(A, A \times^h_{\mathbf{L}\widetilde{K}(\pi_n(C), n+1)} NG). \end{aligned}$$

Let f be the morphism from A to $\mathbf{L}\widetilde{K}(\pi_n(C), n+1)$. Then there is a morphism $p \circ f : A \to NG$. As the cohomology of A vanishes for $n \geq n_0$, the elements $s \circ p \circ f$ and f of the cohomology group are equals and thus
$$p \circ f \in \pi_0 \mathrm{Map}_{sB\text{-mod}/\mathbf{L}\widetilde{K}(\pi_n(C), n+1)}(A, NG).$$
Then, for $i = 0$, the corollary is a clear consequence of lemma 2.20 and the short exact sequence of remark 2.21.

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Now, let us study the case $i > 0$. As

$$NG \times^h_{\mathbf{L}\widetilde{K}(\pi_n(C),n+1)} NG \backsimeq \mathbf{L}\widetilde{K}(\pi_n(C),n+1),$$

we obtain

$$A \times^h_{\mathbf{L}\widetilde{K}(\pi_n(C),n+1)} NG \backsimeq \mathbf{L}\widetilde{K}(\pi_n(C)\circ v^*,n).$$

Thus

$$\begin{aligned}\pi_i(\mathrm{Map}_{sB\text{-mod}/\tau_{\leq n-1}C}(A,\tau_{\leq n}C),v) &\backsimeq \pi_i((\mathrm{Map}_{sB\text{-mod}/A}(A,\mathbf{L}\widetilde{K}(\pi_n(C)\circ v^*,n)),q)\\ &\backsimeq \mathrm{H}^{n-i}(A,\pi_n(C))\end{aligned}$$

where q is the natural morphism from A to $\mathbf{L}\widetilde{K}(\pi_n(C)\circ v^*,n)$. We then deduce the result from Lemma 2.20 and the short exact sequence of remark 2.21. □

Corollary 2.23 *Let* $A \xrightarrow{v} C$ *be in* A/sB*-mod. Then the pointed tower of fibrations*

$$(\mathrm{Map}_{sB\text{-}mod}(A,\tau_{\leq n}C),v)$$

converges completly in the sense of [GJ].

Proof: It can be checked with the corollary 2.21 of the complete convergence lemma of [GJ]. □

Corollary 2.24 *For all* $i \geq 0$*, all* $n \geq n_i$ *and all* $A \xrightarrow{v} C$ *in* A/sB*-mod, there are isomorphisms*

$$\begin{aligned}\pi_i(Map_{sB\text{-}mod}(A,C),v) &\backsimeq lim_{n\in\mathbb{N}}\,\pi_i(Map_{sB\text{-}mod}(A,\tau_{\leq n}C),v)\\ &\backsimeq \pi_i(Map_{sB\text{-}mod}(A,\tau_{\leq n_i}C),v).\end{aligned}$$

Proof: The first isomorphism is a consequence of Milnor exact sequence ([GJ], 2.15) and the vanishing of the lim^1 induced by the complete convergence. The second isomorphism is a consequence of Corollary 2.22. □

Let us now recall a well known lemma with which we will prove the last technical lemma necessary for the proof of 2.19.

Lemma 2.25 *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow[g]{} & T \end{array}$$

be a commutative square in sSet where g *is a weak equivalence. Then the morphism* f *is a weak equivalence if and only if for all* $z \in Z$*, the homotopy fibers* X_z *and* $Y_{g(z)}$ *are simultaneously empty and are equivalent when not empty.*

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Here is the last technical lemma :

Lemma 2.26 *Let $C \backsimeq \mathrm{Hocolim}_{\alpha\in\Theta}(C_\alpha)$ be a filtered homotopy colimit. Then there is a weak equivalence in $sSet$*

$$\mathrm{Map}_{sB\text{-}mod}(A,C) \backsimeq \mathrm{Hocolim}(\mathrm{Map}_{sB\text{-}mod}(A,C_\alpha)).$$

Proof: The proof proceeds by induction on the truncation level n of C. The statement for $n=1$ is a hypothesis of 2.19. Let us assume that it is true for $n-1$. Let C be a n-truncated object in sB-mod and let $\bar{u}$ be in $\mathrm{Hocolim}(\mathrm{Map}_{sB\text{-mod}}(A,\tau_{n-1}C_\alpha))$, represented by $u \in \mathrm{Map}_{sB\text{-mod}}(A,\tau_{n-1}C_{\alpha_0})$. Let $\tilde{u}$ denote its image in $\mathrm{Map}_{sB\text{-mod}}(A,C)$. Then the filtered homotopy colimit along Θ is weak equivalent to the homotopy colimit along α_0/Θ. We will use the previous lemma, computing the fibers along u in the following diagram :

$$\begin{array}{ccc}
\mathrm{Hocolim}_{\alpha_0/\Theta}\mathrm{Map}_{sB\text{-mod}/\tau_{n-1}C_\alpha}((A,u_\alpha),C_\alpha) & \longrightarrow & \mathrm{Map}_{sB\text{-mod}/\tau_{n-1}C}((A,\tilde{u}_{n-1}),C) \\
\downarrow & & \downarrow \\
\mathrm{Hocolim}_{\alpha_0/\Theta}\mathrm{Map}_{sB\text{-mod}}(A,C_\alpha) & \longrightarrow & \mathrm{Map}_{sB\text{-mod}}(A,C) \\
\downarrow & & \downarrow \\
\mathrm{Hocolim}_{\alpha_0/\Theta}\mathrm{Map}_{sB\text{-mod}}(A,\tau_{n-1}C_\alpha) & \longrightarrow & \mathrm{Map}_{sB\text{-mod}}(A,\tau_{n-1}C)
\end{array}$$

where $u_\alpha : A \xrightarrow{u} C_{\alpha_0} \longrightarrow C_\alpha$ and $\tilde{u}_{n-1} : A \xrightarrow{\tilde{u}} C \longrightarrow \tau_{n-1}C$.

Let us show first that the fibers are simultaneously empty. The natural morphism

$$\begin{array}{ccc}
\mathrm{Hocolim}_{\alpha_0/\Theta}\mathrm{Map}_{sB\text{-mod}}(A,\tau_{n-1}C_\alpha) & \to & \mathrm{Map}_{sB\text{-mod}}(A,\tau_{n-1}C) \\
\bar{u} & \to & \tilde{u}
\end{array}$$

induces the natural morphism on cohomology groups

$$\mathrm{Hocolim}_{\alpha_0/\Theta}\mathrm{H}^{n+1}(A,\pi_nC_\alpha) \to \mathrm{H}^{n+1}(A,\pi_nC)$$

which is a weak equivalence. Indeed, the H^n functors are isomorphic to the Ext functors in $sZ(B)$-mod$^{A\text{-grad}}$ which commute with filtered homotopy colimits by the first hypothesis of 2.19. Thus, the images of $\bar{u}$ and $\tilde{u}$ in the cohomology groups vanish simultaneously, and the fibers are simultaneously empty.

Let us assume now that the fibers are nonempty and prove that they are equivalent. The functors π_i commute with filtered homotopy colimits. Applying them on the fibers, we get the natural morphism :

$$\mathrm{colim}_{\alpha_0/\Theta}\pi_i\mathrm{Map}_{sB\text{-mod}/\tau_{n-1}C_\alpha}((A,u_\alpha),C_\alpha) \to \pi_i\mathrm{Map}_{sB\text{-mod}/\tau_{n-1}C}((A,\tilde{u}_{n-1}),C).$$

As the functors here are in fact isomorphic to the functors H^{n-1}, these morphisms are isomorphisms. By 2.25, this ends the proof of the lemma. □

Let us now finish the proof of 2.19. Let $v : A \to C$ be in A/sB-mod such that $C \backsimeq \mathrm{Hocolim}(C_\alpha)$. Let us prove that the morphism

$$\mathrm{Hocolim}(\mathrm{Map}_{sB\text{-mod}}(A,C_\alpha)) \to \mathrm{Map}_{sB\text{-mod}}(A,C)$$

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is a weak equivalence. Let i be a positive integer. To check if the image of this morphism under π_i is an isomorphism, we can just consider the case when C is n-truncated by 2.24. As the truncation commutes with filtered homotopy colimits, this is a consequence of 2.26. This ends the proof of 2.19. □

3. Examples

3.1. The Category ($\mathbb{Z}$-*mod*,$\otimes_{\mathbb{Z}}$,$\mathbb{Z}$)

In classical algebraic geometry, the notion of (projective) resolution is obtained using chain complex of modules or rings. In fact, considering the correspondence of Dold-Kan this method is equivalent to taking cofibrant resolutions in the simplical category (cf [Q]).

Remark 3.1 Let A be a ring. The set of morphisms levelwise equal to $\{0\} \to A$ or Id_A is a set of generating cofibrations in the monoidal model category Ch(A-mod)$^{\geq 0}$.

Remark 3.2 Let $\mathrm{ProjDim}_A(M)$ and $\mathrm{TorDim}_A(M)$ denote Projective and Tor dimension of an A-module M. The functor of Dold-Kan correspondence is a strong monoidal functor, as a consequence Tor dimension can be computed with the homotopy groups π_i.

Lemma 3.3 *Let X be in Ho($sSet$) and M in $s\mathbb{Z}$-mod (resp sA-mod, for A a ring)*

- ◇ *The object X is n-truncated if and only if* $\mathrm{Map}(*,X) \backsimeq \mathrm{Map}(S^i,X)\ \forall\ i > n$ *in* Ho($sSet$).
- ◇ *The object M is n-truncated if and only if* $\mathrm{Map}_{s\mathbb{Z}\text{-}mod}(\mathbb{Z},M)$ (*resp* $\mathrm{Map}_{sA\text{-}mod}(A,M)$) *is n-truncated in* Ho($sSet$).

Proof: For the first statement, by 2.25, we can consider equivalently the homotopy fibers of this morphism over $\mathrm{Map}(*,X)$. The fiber of $\mathrm{Map}(*,X)$ is a point and the fiber of $\mathrm{Map}(S^i,X)$ is $\mathrm{Map}_{sSet/*}(S^i,X)$. As $\pi_j \mathrm{Map}_{sSet/*}(S^i,X) \backsimeq \pi_{i+j}(X)$, the equivalence is clear.

For the second statement, any object in $s\mathbb{Z}$-mod is a homotopy colimit of free objects, i.e. $\forall\ N \in s\mathbb{Z}$-mod there exists a family of sets $(\lambda_i)_{i\in I}$ such that $qN \backsimeq \mathrm{hocolim}_I \coprod_{\lambda_i} \mathbb{Z}$ in Ho($s\mathbb{Z}$-mod). Assume that $\mathrm{Map}_{s\mathbb{Z}\text{-mod}}(\mathbb{Z},M)$ is n-truncated. Since $\mathrm{Map}_{s\mathbb{Z}\text{-mod}}(N,M) \backsimeq \mathrm{holim}_I \prod_{\lambda_i} (\mathrm{Map}_{s\mathbb{Z}\text{-mod}}(\mathbb{Z},M))$, it is a homotopy limit of n-truncated objects. By the first statement, n-truncated objects in $sSet$ are clearly stable under homotopy limits. □

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Lemma 3.4 (cf [TV]) *Let $u : A \to B$ be in $s\mathbb{Z}$-alg. Then the morphism u is flat if and only if*

i. The natural morphism $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$ is an isomorphism.

ii. The morphism $\pi_0(u)$ is flat.

In particular, if A is cofibrant and n-truncated, u flat implies B n-truncated.

Remark 3.5 [TV] Let $f : A \to B$ be in $\mathbb{Z}$-alg. The flatness of the morphism f (in the sense of lemma 3.4) is clearly equivalent to $\mathrm{TorDim}_A(B) = 0$. As a consequence this notion of flatness for simplicial algebras recovers the usual notion of flatness.

We give now the lemmas necessary for the theorem of comparison of the notions of smoothness in rings and relative smoothness.

Lemma 3.6 *Let $A \to B$ be a smooth morphism of rings. Then there exists a pushout square*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

such that $A' \to B'$ is a smooth morphism of noetherian rings.

Proof: This is the affine case in the corollary 17.7.9(b) of [EGAIV]. □

Lemma 3.7 *Let $A \to B$ and $A \to C$ be two morphisms in $\mathbb{Z}$-alg. Then, if B is a perfect complex of $B \otimes_A B$-modules then $D := B \otimes_A C$ is a perfect complex of $D \otimes_C D$ modules.*

Proof: Perfect complexes are clearly stable under base change. As $D \otimes_C D \backsimeq B \otimes_A D$, the natural morphism $D \otimes_C D \to D$ is a pushout of $B \otimes_A B \to B$, hence D is a perfect complex. □

Lemma 3.8 *Assume that A is a noetherian ring and consider the morphism $A \to B \in \mathbb{Z}$-alg, where B is of finite type. There is an equivalence between*

i. The ring B is of finite Tor-dimension over A.

ii. The ring B is of finite projective dimension over A.

Lemma 3.9 *Let $u : A \to B$ be in rings. Assume that A is an algebraically closed field, then there is an equivalence between*

- *The morphism u is formally smooth in the sense of rings.*
- *Any morphism $x : B \to A$ in rings provides A with a structure of B-module of finite projective dimension over B.*

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Lemma 3.10 *Let $u : A \to B$ be a finitely presented flat morphism in rings. Then the morphism u is smooth if and only if for all algebraically closed fields K under A, the morphism $K \to K \otimes_A B$ is smooth.*

Theorem 3.11 *A morphism $A \to B$ in $\mathbb{Z}$-alg is smooth in the sense of rings if and only if*

i. The ring B is finitely presented in A-alg.

ii. The morphism $A \to B$ is flat.

iii. The ring B is a perfect complex of $B \otimes_A B$-modules.

Proof: Let us now prove the first part of the theorem. Assume that $A \to B$ is smooth. Then the assertions i and ii are clear.

Let us prove the assertion iii. By 3.6, as this assertion is stable under pushout, we just have to prove it for A and B noetherian. Let us prove first that the morphism $B \otimes_A B \to B$ is of finite Tor dimension (hence of finite projective dimension by 3.8).

Let L be an algebraically closed field in A-alg. Set $B_L := B \otimes_A L$. Clearly

$$B \otimes_{B \otimes_A B} L \backsimeq B_L \otimes_{B_L \otimes_L B_L} L,$$

hence computing the Tor dimension of B over $B \otimes_A B$ is equivalent to compute the Tor dimension of B_L over $B_L \otimes_L B_L$. The morphism $L \to B_L \to B_L \otimes_L B_L$ is smooth, by composition of smooth morphisms, over an algebraically closed field. The ring $B_L \otimes_L B_L$ is smooth over a field, hence regular. Now, B_L is a module of finite type over this regular ring thus it is a perfect complex. In particular, it is of finite projective dimension, hence of finite Tor dimension. Finally, B is of finite Tor dimension, hence of finite projective dimension over $B \otimes_A B$. As previously, B is a module of finite type over $B \otimes_A B$. As these rings are noetherian, B is a perfect complex. Indeed, B has a finite projective resolution (P_i). Each P_i is of finite Tor dimension, hence of finite projective dimension.

Let us prove the second part of the theorem. Let $A \to B$ be a morphism of rings satisfying the assertions i, ii and iii. Let K be an algebraically closed field under A. We will use 3.10 and 3.9. Let $x : B \to K$ be in $\mathbb{Z}$-alg. Then the following commutative diagram is a homotopy pushout:

$$\begin{array}{ccc} B \otimes_K B & \xrightarrow{Id \otimes_K x} & B \otimes_K K \backsimeq B \\ \downarrow & & \downarrow x \\ B & \xrightarrow[x]{} & K \,. \end{array}$$

Thus K has finite projective dimension in B-mod. Finally, by 3.10, the morphism $K \to B$ is smooth in the sense of rings. As it is true for any field K, by 3.9, the morphism $A \to B$ is smooth in the sense of rings. □

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Here is the comparison theorem.

Theorem 3.12 *Let $A \to B$ be a morphism of rings. Then, it is smooth if and only if it is smooth in the sense of rings.*

Proof: The two following lemmas, and remark 3.5 prove the theorem.

Lemma 3.13 [TV] *Let $A \to B$ be a morphism in $\mathbb{Z}$-alg.*

i. if $A \to B$ is hfp then it is finitely presented in $\mathbb{Z}$-alg.

ii. if $A \to B$ is smooth and finitely presented then it is hfp.

Lemma 3.14 [TV] *Let $A \to B$ be a morphism of rings. Then the ring B is a perfect complex of A-modules if and only if $A \to B$ is hf.*

□

3.2. The Category Set

The most difficult problem consists in finding examples of formally smooth morphisms. The proposition 2.19 gives us a characterisation of these morphisms in the relative context $\mathcal{C} = Set$.

The nerve functor and the "fundamental groupoid" functor define a Quillen equivalence between the category sB-mod endowed with its 1-truncated model structure and the category B-Gpd. Moreover, this last category is compactly generated and thus its filtered Hocolimits can be computed as filtered colimits. Here is the formula to do this:

Lemma 3.15 *Let $\mathcal{I}$ be a filtered diagram and let F be a functor from I to Gpd. Then the colimit of F consists of*

◇ *On objects :*

$$(\mathrm{Colim} F)_0 := \mathrm{Colim}(u \circ F)$$

where u is the forgetful functor from Gpd to Set.

◇ *On morphisms : let $\bar{x}$ and $\bar{y}$ be in* $\mathrm{Colim}(u \circ F)$ *represented by $x \in F(i)$ and $y \in F(i')$. Then there exists k under i and i' such that*

$$\mathrm{Hom}_{\mathrm{Hocolim}(F)}(\bar{x}, \bar{y}) := \mathrm{Colim}_{k/\mathcal{I}}(\mathrm{Hom}_{F(j)}((l_{i,j})_*)(x), (l_{i',j})_*)(y))$$

where $l_{i,j} : i \to j$ and $l_{i',j} : i' \to j$.

We also need to describe the derived enriched Homs.

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Lemma 3.16 *Let B be a monoid in Set. Then there is an equivalence of categories between* Ho(B-Gpd) *and the category* [B-Gpd] *whose objects are B-groupoids and morphisms are isomorphism classes of functors. In particular, for two B-groupoids G and G', $\mathbf{R}\underline{\mathrm{Hom}}^{\Delta\leq 1}_{B\text{-}gpd}(G,G') \backsimeq \underline{\mathrm{Hom}}^{\Delta\leq 1}_{[B\text{-}gpd]}(G,G')$ in* Ho(Gpd), *where the exponent $\Delta \leq 1$ means that the Homs are enriched on groupoids.*

Lemma 3.17 *The commutative monoid $\mathbb{N}$ is homotopically finitely presented for the* 1*-truncated model structure i.e. in the category* ($\mathbb{N}\times\mathbb{N}$)*-Gpd.*

Proof: Let $\mathbb{N}^2$ denote $\mathbb{N}\times\mathbb{N}$. Let $F : \mathcal{J} \to Gpd$ be a functor from a filtered diagram $\mathcal{J}$ to Gpd. We have to prove

$$\mathrm{Hocolim}(\underline{\mathrm{Hom}}^{\Delta\leq 1}_{[\mathbb{N}^2\text{-Gpd}]}(\mathbb{N}, F(-))) \backsimeq \underline{\mathrm{Hom}}^{\Delta\leq 1}_{[\mathbb{N}\text{-Gpd}]}(\mathbb{N}, \mathrm{Hocolim}(F)).$$

We leave it to the reader to verify that the following functor, denoted by φ, define an equivalence of groupoids.

Let $\bar{H}$ be in $\mathrm{Hocolim}(\underline{\mathrm{Hom}}^{\Delta\leq 1}_{[\mathbb{N}^2\text{-Gpd}]}(\mathbb{N}, F(-)))$ represented by $H \in \underline{\mathrm{Hom}}_{[\mathbb{N}^2\text{-Gpd}]}(\mathbb{N}, F(j))$. We define the functor φ on objects by

$$\varphi : \bar{H} \to \hat{H} := n \to \bar{H(n)}.$$

Now, by construction, any morphism $\bar{\eta}$ in $\mathrm{Hocolim}(\underline{\mathrm{Hom}}^{\Delta\leq 1}_{[\mathbb{N}^2\text{-Gpd}]}(\mathbb{N}, F(-)))$ has a representative

$$\eta : G \to G' \in \mathrm{Hom}_{\underline{\mathrm{Hom}}_{[\mathbb{N}^2\text{-Gpd}]}(\mathbb{N}, F(j))}(G, G').$$

We define the functor φ on morphisms by

$$\varphi : \bar{\eta} \to \hat{\eta} := n \to \bar{\eta_n}.$$

□

Lemma 3.18 *The commutative group $\mathbb{Z}$ is homotopically finitely presented for the* 1*-truncated model structure i.e. in the category* ($\mathbb{Z}\times\mathbb{Z}$)*-Gpd.*

Proof: This is the same proof as the previous lemma, replacing $\mathbb{N}$ by $\mathbb{Z}$. □

Corollary 3.19 *The morphisms $\mathbb{F}_1 \to \mathbb{N}$ and $\mathbb{F}_1 \to \mathbb{Z}$ are smooth. In particular, the morphism $\mathbb{F}_1 \to \mathbb{Z}$ corresponds intuitively to the affine scheme $Gl_{1,\mathbb{F}_1} \backsimeq Spec(\mathbb{Z})$, denoted by $\mathbb{G}_{m,\mathbb{F}_1}$ in* [TVa]*, whose smoothness was an expected result.*

Proof: They are clearly hfp and of Tor dimension zero. Their diagonal is hf for the 1-truncated model structure, thus, we just have to check that the diagonal of their abelianization is hf in the simplicial graded category given in 2.19. The abelianization of $\mathbb{N}$ is $\mathbb{Z}[X]$ and the abelianization of $\mathbb{Z}$ is $\mathbb{Z}(X)$, and the morphisms $\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[X] \to \mathbb{Z}[X]$ and $\mathbb{Z}(X)\otimes_{\mathbb{Z}}\mathbb{Z}(X) \to \mathbb{Z}(X)$ are hf in respectively $s(\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[X])$-Mod$^{\mathbb{N}\text{-grad}}$ and $s(\mathbb{Z}(X)\otimes_{\mathbb{Z}}\mathbb{Z}(X))$-Mod$^{\mathbb{Z}\text{-grad}}$. □

Remark 3.20 The scheme $Gl_{n,\mathbb{F}_1}$ is not affine but, according to its description in [TVa] and the last corollary, it will be possible to call it smooth as soon as a proper definition of smooth morphisms for relative schemes (or schemes over F_1)

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is written.

3.3. Some other Examples

The first general example is given by theorem 1.28 : an open immersion in a relative context is a smooth morphism. We will here study two other classic examples : the affine line $1 \to 1[X]$ and the morphism $1 \to 1(X)$ (corresponding intuitively to the scheme $\mathbb{G}_m$).

Remark 3.21 If $(\mathcal{C},\otimes,1)$ is a symmetric monoidal category as described in the preliminaries, its associated category of simplicial objects has simplicial Homs, denoted by $\underline{\mathrm{Hom}}^{\Delta}$, and there is an adjunction

$$s\mathcal{C} \underset{sK_0}{\overset{\underline{\mathrm{Hom}}^{\Delta}(1,-)}{\rightleftarrows}} sSet$$

where $sK_0((X_n)_{n\in\mathbb{N}}) = (\coprod_{X_n} 1)_{n\in\mathbb{N}}$. One verifies easily that, as 1 is cofibrant, finitely presentable, and as $\underline{\mathrm{Hom}}^{\Delta}(1,-)$ preserves weak equivalences (by construction of the model structure on $\mathcal{C}$), the functor sK_0 preserves homotopically finitely presentable objects. In particular, sK_0 preserves hf morphisms and formally smooth morphisms. Restricting the adjunction to the categories of algebra, where weak equivalences and filtered homotopy colimits are obtained with the forgetful functor, it is also clear that sK_0 preserves hfp morphisms.

Proposition 3.22 *Let $u : A \to B$ be a smooth morphism in* $\mathrm{Comm}(Set)$. *Then $sK_0(u)$ is smooth if and only if $sK_0(B)$ is of finite Tor dimension over $sK_0(A)$.*

This gives particular examples : the affine line, corresponding to the morphism $1 \to 1[X] := \coprod_{\mathbb{N}} 1$, and the morphism $1 \to 1(X) := \coprod_{\mathbb{Z}} 1$.

Theorem 3.23 *The morphisms $1 \to 1[X] := \coprod_{\mathbb{N}} 1$ and $1 \to 1(X) := \coprod_{\mathbb{Z}} 1$ are smooth in any context where respectively $1[X]$ and $1(X)$ are of finite Tor dimension over* 1.

This theorem can be applied in particular to the context $\mathbb{N}$-mod. The following lemma provides us, in this context, some examples of morphisms of Tor-dimension 0.

Lemma 3.24 *Let $A \to B$ be in* $\mathrm{Comm}(\mathbb{N}\text{-}mod)$ *such that B is free over A. Then the monoid B has Tor-dimension* 0 *over A.*

Proof: Let $M \in A$-mod be a n-truncated module. Then there exists a set λ such that $B \backsimeq \coprod_{\lambda} A$. Thus $B \otimes^L_A M' \backsimeq \coprod_{\lambda} QM$ in Q_cA-mod where Q and Q_c are cofibrant replacement in respectively Q_cA-mod and Comm($\mathbb{N}$-mod). Thus, as this coproduct is a product in sets, we get

$$B \otimes^L_A M' \backsimeq Colim_{\lambda' fini\subset\lambda} \prod_{\lambda'} QM.$$

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As the functors π_i commute with products in sets and filtered colimits, the Tor dimenson of B over A is zero. □

Theorem 3.25 *Examples in $\mathbb{N}$-mod.*

- ◇ *The morphism $\mathbb{N} \to \mathbb{N}[X] \in \mathbb{N}$-mod is smooth (corresponding intuitively to the affine line).*
- ◇ *The morphism $\mathbb{N} \to \mathbb{N}(X) \in \mathbb{N}$-mod is smooth (corresponding intuitively to the scheme $\mathbb{G}_m$).*

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