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Two-dimensional proof-structures and the exchange rule

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In the paper Métayer (2001), Métayer transforms multiplicative proof-structures into orientable surfaces with boundaries. He investigates the link between the topological complexity and the number of exchanges in a sequentialisation. The theorem he achieves is about a particular rule of exchange (transpositions by blocks). We complete his approach by showing that the topological complexity does not provide any information in other cases (arbitrary exchange, upper bound of the number of exchanges). Then, we show that, on the other hand, the surface associated to a proof-structure is the surface of minimal complexity on which the proof can be drawn without crossing and respecting the local orientation.

1. Introduction

The article Métayer (2001) is an attempt to extract from multiplicative proof-nets some new information on the exchange rule using homology theory. To each proof, the author associates, in a natural way, an orientable surface with a boundary in which the proof is embedded. Such an object had already been defined in Bellin and Fleury (1998), although the authors did not concentrate on it.

A proof-structure is seen as a connected sum of tori with boundaries, the boundaries representing conclusions. The topological complexity of such a manifold can be measured by two invariants: the genus and the rank. One could hope to link this complexity to some measure of implicit exchange hidden in proof-nets. Actually, the main theorem of Métayer (2001) (Theorem 2) gives a lower bound of the number of exchanges in sequentialisations of a proof-net, as a function of the rank (see also Subsection 5.2). But the exchange rule is a particular one, allowing transpositions by blocks:

$$\mathbb{X}\ \frac{\vdash \Gamma, \Sigma, \Theta, \Delta}{\vdash \Gamma, \Theta, \Sigma, \Delta}$$

In this paper, we begin by completing this approach. Surfaces are manipulated by means of words that describe their planar representations (Subsection 3.1). This allows us to give a direct proof of the geometrical nature of complexes associated to proof-structures (Subsection 3.2). But we can also make explicit how a complex is modified when a cell is added, and formalise this in a decorated sequent calculus K(MLL) (Section 4), which shows itself useful when we deal with the link between topological complexity and the exchange rule (Section 5).

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Subsection 5.1 completes the correspondence between cut-free cyclicity and zero rank, given by Métayer, with a simple remark: a proof-net with cut can have a complex of zero rank, that is, a disk, and requires an exchange to be sequentialised. In Subsection 5.3, we show the impossibility of a theorem, analogous to Theorem 2 of Métayer (2001), for the rule of arbitrary exchange:

$$X\ \frac{\vdash A_1,\ldots,A_n}{\vdash A_{\sigma(1)},\ldots,A_{\sigma(n)}}\ \sigma \text{ any permutation.}$$

In Subsection 5.4, we reply in the negative to an open question from Métayer (2001), which was to find an upper bound of the least number of exchanges in sequentialisations of a proof-net.

The two trails have revealed themselves to be dead ends. However, the construction of complexes is of interest if we concentrate on embeddings of proof-structures. In Bellin and Fleury (1998), Bellin and Fleury consider an exchange link in such a way that every proof-structure is drawn on a disk. Here, a proof-structure is drawn, without crossing and respecting local orientation, on its associated complex. Theorem 6.1 shows that the complex has the minimal complexity to do such a drawing. A tubular neighbourhood of the proof-structure plays a part in the proof of this theorem, where we show that Métayer's complex is the amalgamated sum of this ribbon with some disks. Such a ribbon is defined by Melliès in Melliès (2001), where he gives topological correctness criteria for cyclic logic and multiplicative non-commutative logic.

2. Proof-structures in two dimensions

2.1. *Linear logic*

We simply recall here the two formalisms for representing proofs of the multiplicative fragment of linear logic (MLL).

Formulas are built from propositional atoms $a_1, a_2, \ldots$ and $a_1^\perp, a_2^\perp, \ldots$, and binary connectives *tensor* ($\otimes$) and *par* ($\parr$). To avoid the possibility of hiding exchanges, we have to choose the non-commutative version of De Morgan's identities. Thus, *linear negation* is defined for all formulas by:

$$(A^\perp)^\perp = A \qquad (A \otimes B)^\perp = B^\perp \parr A^\perp \qquad (A \parr B)^\perp = B^\perp \otimes A^\perp.$$

In the same way, sequents must be finite sequences of formulas, instead of multisets. However, if the exchange rule is removed (which is possible because the cut-elimination procedure preserves the absence of exchange), then, in the resulting fragment, provability is closed under the rule of cyclic exchange (Abrusci 1997):

$$\frac{\vdash A_1,\ldots,A_n}{\vdash A_{\sigma(1)},\ldots,A_{\sigma(n)}}\ \sigma \text{ cyclic permutation of } \{1,\ldots,n\}.$$

Hence, for us, a *sequent* is an expression of the form

$$\vdash \Gamma$$

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where Γ is a finite sequence of formulas, up to a cyclic permutation. *Proofs* are defined inductively from the following set S of rules, possibly extended by an exchange rule:

$$\text{ax}\ \frac{}{\vdash A, A^{\perp}} \qquad \text{cut}\ \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

$$\otimes\ \frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad \parr\ \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \parr B, \Delta}$$

Two exchange rules will be considered:

$$X\ \frac{\vdash A_1, \ldots, A_n}{\vdash A_{\sigma(1)}, \ldots, A_{\sigma(n)}}\ \sigma \text{ any permutation} \qquad \mathbb{X}\ \frac{\vdash \Gamma, \Sigma, \Theta, \Delta}{\vdash \Gamma, \Theta, \Sigma, \Delta}$$

An *McyLL* (respectively, *MXLL* , *M*$\mathbb{X}$*LL*) *proof* is a proof using only rules of S (respectively, $S \cup \{\mathrm{X}\}$, $S \cup \{\mathbb{X}\}$).

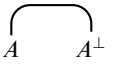
Remarks.

— From the point of view of provability, M$\mathbb{X}$LL and MXLL are clearly equivalent, since permutations by blocks allowed by $\mathbb{X}$ generate the symmetric group. However, the rules X and $\mathbb{X}$ have nothing like the same behaviour in the geometric translation given by Métayer, and one of the aims of this paper is to show this (Section 5).

— McyLL denotes the multiplicative cyclic linear logic (Yetter 1990).

A *proof-structure* is a graph built exclusively with the following schemata of links. Nodes are labelled by occurrences of formulas, each of them being the *conclusion* of exactly one link and the *premise* of at most one link.

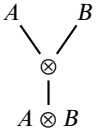
— Axiom link, with two conclusions and no premise:



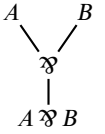
— Cut link, with two premises and no conclusion:



— $\otimes$-link, with two premises and one conclusion:



— $\parr$-link with two premises and one conclusion:



In each system of sequent calculus, a proof P can be translated as a proof-structure, which will be denoted P^*. A proof-structure π will be called *sequentialisable for McyLL*

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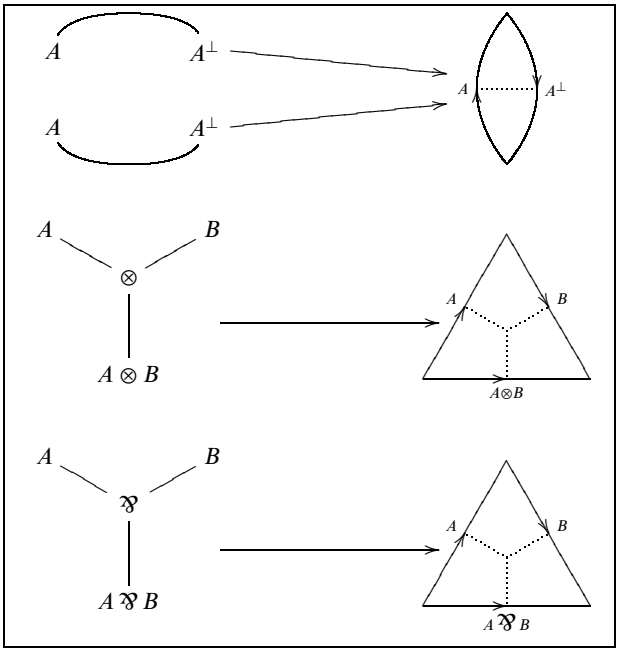


Fig. 1. Basic cells

(respectively, *MXLL, MⅨLL*) if there exists a proof P of McyLL (respectively, MXLL, MⅨLL) such that $P^* = \pi$. Evidently, a proof-structure is sequentialisable for MXLL if and only if it is for MⅨLL. Whereas sequentialisable proof-structures for McyLL are fewer. There are various correctness criteria defining *proof-nets* and guaranteeing that they are exactly the proof-structures that are sequentialisable. See, for example, Girard (1987), Danos and Régnier (1989)and Métayer (1994) for MXLL, and Abrusci and Ruet (2000) and Melliès (2001) for McyLL.

2.2. *Métayer's complexes*

To a proof-structure π, Métayer associates a complex $K(\pi)$ by mapping links to cells as shown by Figure 1, and identifying edges representing the same formula occurrence. The dotted lines are only drawn to show effectively the embeddings of proof-structures into Métayer complexes.

Example. Figure 2 shows a proof-net whose complex is a torus with a hole. This geometrical form is general, as we shall see in Section 3.

In Métayer's paper, such a complex is associated to each proof in the sequent calculus. But two sequent proofs of the same net have the same complex. Actually, the geometrical rules of construction are local operations; there is here the same parallelism as for proof-nets. So Métayer's complexes must really be seen as a kind of 2-dimensional dual of proof-nets.

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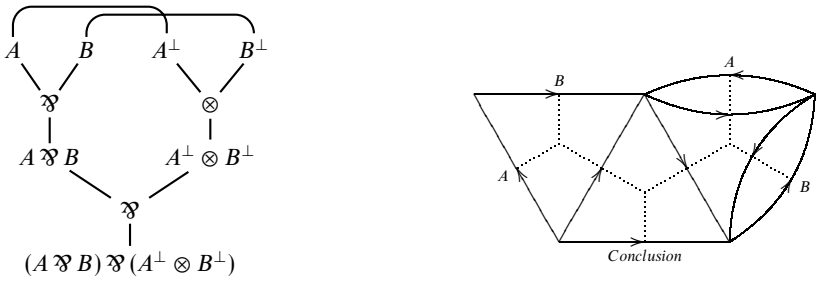


Fig. 2. A proof-net with its Métayer complex

Now it is time to open the geometrical toolbox.

3. Topological nature of complexes

To simplify, when we say *surface* we will mean a connected, compact and oriented topological manifold of dimension 2 with boundary. We shall use a standard notion of topology to reduce the handling of surfaces to a calculus of words. Some technical lemmas allow us to give another proof of the topological nature of complexes. But, above all, their proofs describe precisely what happens to the complex when a cell is added. This will be formalised in Section 4 and used in the following sections.

3.1. *Planar representations of surfaces*

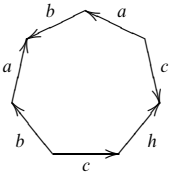
A *presentation* of a surface S is a finite set $\mathscr{W}$ of polygons whose boundaries are constituted by labelled and oriented edges such that:

1 No more than two edges can have the same label;
2 The quotient of $\mathscr{W}$, modulo the identification of oriented edges of same label, is the surface S.

Remarks.

— There always exists a presentation for a surface. For example, in Massey (1977), Massey defines what he calls a 'canonical form', which is a particular presentation with only one polygon.
— Each piece W of a presentation $\mathscr{W}$ is determined by its border ∂W, a cycle of oriented edges. So for us, a presentation of a surface will be a set of words, on an alphabet $\Sigma \cup \Sigma^{-1}$ (where $\Sigma = \{a, b, c \ldots A, B, C \ldots\}$), up to a circular permutation.

Example. The polygon below is presented by $aba^{-1}b^{-1}chc^{-1}$.



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Definition 3.1 (Operations on presentations). $\mathscr{W}$ denotes a set of words on $\Sigma \cup \Sigma^{-1}$, u, v, w are words on the same alphabet, and $a, a_1, \ldots, a_n$ are letters from Σ. We define elementary operations, denoted by $\rightsquigarrow$, to transform presentations.

— *Inverse*:

$$\mathscr{W}, a_1^{\varepsilon_1} \ldots a_n^{\varepsilon_n} \rightsquigarrow \mathscr{W}, a_n^{-\varepsilon_n} \ldots a_1^{-\varepsilon_1} \quad \text{where } \varepsilon_i = \pm 1.$$

— *Cutting out*:

$$\mathscr{W}, vw \rightsquigarrow \mathscr{W}, va, a^{-1}w \quad \text{if } a \text{ does not appear in } \mathscr{W}, vw.$$

— *Gluing of two pieces*:

$$\mathscr{W}, va, a^{-1}w \rightsquigarrow \mathscr{W}, vw.$$

— *Gluing on a piece*:

$$\mathscr{W}, waa^{-1} \rightsquigarrow \mathscr{W}, w \quad \text{if } w \text{ is not empty.}$$

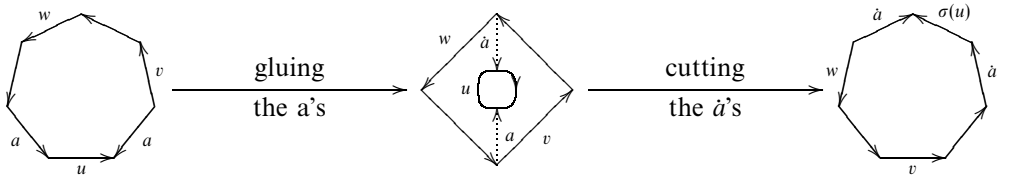
— *Shifting*:

$$\mathscr{W}, waua^{-1}v \rightsquigarrow \mathscr{W}, wva\sigma(u)a^{-1} \quad \sigma \text{ a circular permutation.}$$

Notation. $\sim$ is the equivalence relation induced by these operations. The set of rewriting rules given above is not the minimal one since the gluing operation is generated by the cutting out operation, but this is the one we will need in the following.

Lemma 3.1. If $\mathscr{W}$ is a presentation of a surface S and $\mathscr{W} \sim \mathscr{W}'$, then $\mathscr{W}'$ is also a presentation of S.

Proof. The point here is that the order of quotients does not affect the topology (Massey 1977; Maunder 1970). Geometrically, the inverse operation amounts to turning over a piece of the presentation. In an operation of cutting out, we cut a piece along a new edge, while we identify two edges in the gluing operations. Finally, the figure below explains what occurs when an operation of shifting is applied:



□

Remark. The converse of Lemma 3.1 also holds. Each presentation of a surface can be reduced to a singleton by inverse and gluing operations since a surface is connected. Then one can follow the procedure described by Massey (Massey 1977), among others, to obtain what he calls a canonical form. This canonical form is uniquely determined by the surface. Thus two presentations of a given surface are equivalent.

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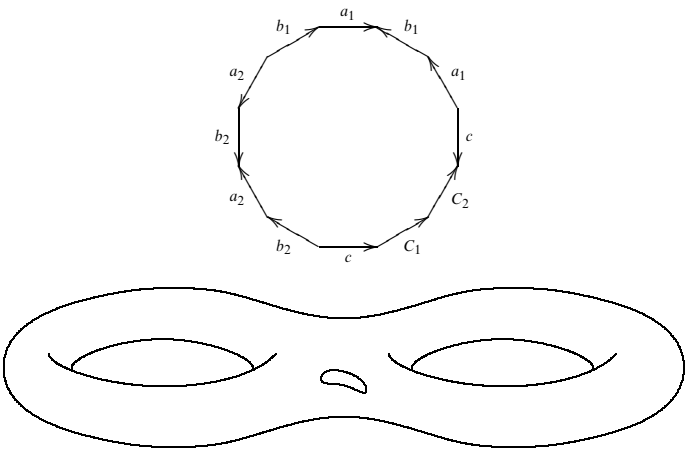


Fig. 3. A surface and one of its presentations

3.2. *Nature of complexes*

Notation. w_p is defined by

$$\begin{cases} aa^{-1} & \text{if } p = 0 \\ a_1 b_1 a_1^{-1} b_1^{-1} \dots a_p b_p a_p^{-1} b_p^{-1} & \text{if } p > 0. \end{cases}$$

If π is a connected proof-structure, $\mathcal{N}(\pi)$ will denote the following property:

$K(\pi)$ *has a presentation* $W = w_p c_1 T_1 c_1^{-1} \dots c_q T_q c_q^{-1}$ *for* $(p, q) \in \mathbb{N} \times \mathbb{N}^*$, *where* T_i *are non-empty words whose letters are conclusions of* π, *with no exponent* -1, *each conclusion appearing only once in* W.

In other words, $\mathcal{N}(\pi)$ means that $K(\pi)$ is the connected sum of p tori with q holes – that is, the connected components of the boundary – composed of all edges of conclusions of π. The pair (p, q) is called the *genus of the surface*. The classification of surfaces and the homology theory allow one to show that the genus characterises homeomorphism classes.

For a surface S with non-empty boundary, the only non-trivial homology group (in reduced homology) is in dimension 1:

$$\mathrm{H}_1(S) = \mathbb{Z}^{2p+q-1}$$

if (p, q) is the genus of S. $2p + q - 1$ is called the *rank of* S and is another topological invariant.

For simplicity, for π a proof-structure, we shall just say the *genus and the rank of* π when we actually mean the genus and the rank of $K(\pi)$.

Lemma 3.2 (non-splitting triangular cell). Let π be a connected proof-structure verifying $\mathcal{N}(\pi)$ and having formulas A and B among its conclusions. Let π' be the same proof-structure where A and B have been linked by a binary operation $*$. Then $\mathcal{N}(\pi')$ is true.

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Proof. Let $\{W\}$ be a presentation of $K(\pi)$ and (p,q) its genus. A presentation of $K(\pi')$ is then given by $\{W,(A*B)B^{-1}A^{-1}\}$. There are three possibilities for the respective positions of conclusions A and B on W. We can always suppose that the hole(s) we are interested in is (are) the last of the word, and that the formulas we consider are the last of the hole, thanks to the shifting operation.

1 $K(\pi)$ is presented by

$$W = w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma A B c_q^{-1}.$$

$$\begin{aligned} W,(A*B)B^{-1}A^{-1} &\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma A A^{-1}(A*B)c_q^{-1} && \text{by gluing the } B\text{'s} \\ &\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma (A*B) c_q^{-1} && \text{by gluing the } A\text{'s.} \end{aligned}$$

Thus $\mathcal{N}(\pi')$ is true and π' has (p,q) for genus.

2 $K(\pi)$ is presented by

$$W = w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma A \Delta B c_q^{-1}$$

where Δ is not empty.

$$\begin{aligned} W,(A*B)B^{-1}A^{-1} &\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma A \Delta A^{-1}(A*B)c_q^{-1} && \text{by gluing the } B\text{'s} \\ &\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma (A*B) c_q^{-1} A \Delta A^{-1} && \text{by shifting.} \end{aligned}$$

Thus π' verifies $\mathcal{N}(\pi')$, its genus is $(p,q+1)$ since Δ forms a new hole.

3 $K(\pi)$ is presented by

$$W = w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma A c_{q-1}^{-1} c_q B \Delta c_q^{-1}.$$

$$\begin{aligned} &W,(A*B)B^{-1}A^{-1} \\ &\quad\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma (A*B) B^{-1} c_{q-1}^{-1} c_q B \Delta c_q^{-1} && \text{by gluing} \\ &\quad\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma (A*B) B^{-1} c_{q-1}^{-1} a,\ a^{-1} c_q B \Delta c_q^{-1} && \text{by cutting out} \\ &\quad\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma (A*B) \Delta c_q^{-1} a^{-1} c_q c_{q-1}^{-1} a && \text{by gluing the } B\text{'s} \\ &\quad\rightsquigarrow w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma (A*B) \Delta c_{q-1}^{-1} a c_q^{-1} a^{-1} c_q && \text{by shifting} \\ &\quad= a c_q^{-1} a^{-1} c_q w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma (A*B) \Delta c_{q-1}^{-1}. \end{aligned}$$

Here, apart from renaming, $a c_q^{-1} a^{-1} c_q w_p$ is w_{p+1}. Thus, $\mathcal{N}(\pi')$ is true and π' has $(p+1,q-1)$ for genus. □

Lemma 3.3 (splitting triangular cell). Let π_1 and π_2 be two connected proof-structures verifying $\mathcal{N}$ and having, respectively, A and B among their conclusions. Let π be the proof-structure obtained after A and B have been linked by a binary connective $*$. Then $\mathcal{N}(\pi)$ is true.

Proof. Let $\{W_1\}$ (respectively, $\{W_2\}$) be a presentation of $K(\pi_1)$ (respectively, $K(\pi_2)$) and (p_1,q_1) (respectively, (p_2,q_2)) be its genus.

$$\begin{aligned} W_1 &= w_{p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma A c_{q_1}^{-1} \\ W_2 &= w_{p_2} d_1 S_1 d_1^{-1} \dots d_{q_2} B \Delta d_{q_2}^{-1}. \end{aligned}$$

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A presentation of $K(\pi)$ is given by $\{W_1, W_2, (A * B)B^{-1}A^{-1}\}$.

$$
\begin{aligned}
&W_1, W_2, (A * B)B^{-1}A^{-1}\\
&\rightsquigarrow w_{p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma (A * B) B^{-1} c_{q_1}^{-1}, W_2 && \text{by gluing}\\
&\rightsquigarrow w_{p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma (A * B) \Delta d_{q_2}^{-1} w_{p_2} d_1 S_1 d_1^{-1} \dots d_{q_2} c_{q_1}^{-1} && \text{by gluing}\\
&\rightsquigarrow d_{q_2}^{-1} w_{p_2} d_1 S_1 d_1^{-1} \dots d_{q_2-1} S_{q_2-1} d_{q_2-1}^{-1} d_{q_2} w_{p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma (A * B) \Delta c_{q_1}^{-1} && \text{by shifting}\\
&\rightsquigarrow d_{q_2}^{-1} w_{p_2} d_{q_2} w_{p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma (A * B) \Delta c_{q_1}^{-1} d_1 S_1 d_1^{-1} \dots d_{q_2-1} S_{q_2-1} d_{q_2-1}^{-1} && \text{by } q_2 - 1 \text{ shiftings.}
\end{aligned}
$$

It remains to show that $d_{q_2}^{-1} w_{p_2} d_{q_2} w_{p_1}$ can be transformed in $w_{p_1+p_2}$. Let

$$
\begin{aligned}
W &= c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma (A * B) \Delta c_{q_1}^{-1} d_1 S_1 d_1^{-1} \dots d_{q_2-1} S_{q_2-1} d_{q_2-1}^{-1}\\
d &= d_{q_2}^{-1}.
\end{aligned}
$$

We proceed by induction on p_2. If $p_2 = 0$, then $w_{p_2} = aa^{-1}$ and $d w_{p_2} d^{-1} w_{p_1}$ is reduced to w_{p_1} by gluing. For $p_2 > 0$, we write

$$w_{p_2} = aba^{-1}b^{-1}w_{p_2-1}.$$

$$
\begin{aligned}
daba^{-1}b^{-1}w_{p_2-1}d^{-1}w_{p_1}W &\rightsquigarrow da\dot{a}^{-1}, \dot{a}ba^{-1}b^{-1}w_{p_2-1}d^{-1}w_{p_1}W && \text{by cutting out}\\
&\rightsquigarrow \dot{a}b\dot{a}^{-1}db^{-1}w_{p_2-1}d^{-1}w_{p_1}W && \text{by gluing the } a\text{'s}\\
&\rightsquigarrow \dot{a}b\dot{b}^{-1}, \dot{b}a^{-1}db^{-1}w_{p_2-1}d^{-1}w_{p_1}W && \text{by cutting out}\\
&\rightsquigarrow \dot{b}a^{-1}d\dot{b}^{-1}\dot{a}w_{p_2-1}d^{-1}w_{p_1}W && \text{by gluing the } b\text{'s}\\
&\rightsquigarrow \dot{b}a^{-1}\ddot{a}^{-1}, \ddot{a}d\dot{b}^{-1}\dot{a}w_{p_2-1}d^{-1}w_{p_1}W && \text{by cutting out}\\
&\rightsquigarrow \ddot{a}d\dot{b}^{-1}\ddot{a}^{-1}\dot{b}w_{p_2-1}d^{-1}w_{p_1}W && \text{by gluing the } \dot{a}\text{'s}\\
&\rightsquigarrow d\dot{b}^{-1}\ddot{b}^{-1}, \ddot{b}\ddot{a}^{-1}\dot{b}w_{p_2-1}d^{-1}w_{p_1}W\ddot{a} && \text{by cutting out}\\
&\rightsquigarrow \ddot{a}\ddot{b}\ddot{a}^{-1}\ddot{b}^{-1}dw_{p_2-1}d^{-1}w_{p_1}W && \text{by gluing the } \dot{b}\text{'s}\\
&\rightsquigarrow \ddot{a}\ddot{b}\ddot{a}^{-1}\ddot{b}^{-1}w_{p_2+p_1-1}W && \text{by induction.}
\end{aligned}
$$

Thus, we have

$$w_{p_2+p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma (A * B) \Delta c_{q_1}^{-1} d_1 S_1 d_1^{-1} \dots d_{q_2-1} S_{q_2-1} d_{q_2-1}^{-1}$$

as a presentation for $K(\pi)$. $\mathcal{N}(\pi)$ is true and the genus of π is $(p_1 + p_2, q_1 + q_2 - 1)$. □

Lemma 3.4 (non-splitting cut). Let π be a connected proof-structure verifying $\mathcal{N}$ and having formulas A and $A^\perp$ among its conclusions. Let π' be the same proof-structure where A and $A^\perp$ have been linked by a cut. Then $\mathcal{N}(\pi')$ is true.

Proof. The proof is similar to the one used for the non-splitting triangular cell case. There are two possibilities.

1 $K(\pi)$ has (p, q) for genus and is presented by

$$w_p c_1 T_1 c_1^{-1} \dots c_q \Gamma A \Delta A^\perp c_q^{-1}.$$

Then a presentation of $K(\pi')$ is given by

$$w_p c_1 T_1 c_1^{-1} \dots c_{q-1} T_{q-1} c_{q-1}^{-1} c_q \Gamma c_q^{-1} c_{q+1} \Delta c_{q+1}^{-1}.$$

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Its genus is thus $(p, q-1+P(\Gamma)+P(\Delta))$ where $P(\Theta)$ equals 0 if Θ is empty, and 1 otherwise.

2 A presentation of $K(\pi)$ is

$$w_p c_1 T_1 c_1^{-1} \dots c_{q-1} \Gamma A c_{q-1}^{-1} c_q A^{\perp} \Delta c_q^{-1}.$$

Then $K(\pi')$ is presented by

$$w_{p+1} c_1 T_1 c_1^{-1} \dots c_{q-2} T_{q-2} c_{q-2}^{-1} c_{q-1} \Gamma \Delta c_{q-1}^{-1}.$$

The genus of $K(\pi')$ is $(p+1, q-2+P(\Gamma\Delta))$. □

Lemma 3.5 (splitting cut). Let π_1 and π_2 be two connected proof-structures verifying $\mathcal{N}$ and having A and $A^{\perp}$, respectively, among their conclusions. Let π be the proof-structure obtained after A and $A^{\perp}$ have been linked by a cut-link. Then $\mathcal{N}(\pi)$ is true.

Proof. The proof is similar to the one used for the splitting triangular cell case. $K(\pi_1)$ and $K(\pi_2)$ are presented by

$$W_1 = w_{p_1} c_1 T_1 c_1^{-1} \dots c_{q_1} \Gamma A c_{q_1}^{-1}$$
$$W_2 = w_{p_2} d_1 S_1 d_1^{-1} \dots d_{q_2} A^{\perp} \Delta d_{q_2}^{-1}.$$

Then a presentation of $K(\pi)$ is given by

$$w_{p_1+p_2} c_1 T_1 c_1^{-1} \dots c_{q_1-1} T_{q_1-1} c_{q_1-1}^{-1} d_1 S_1 d_1^{-1} \dots d_{q_2-1} S_{q_2-1} d_{q_2-1}^{-1} e \Gamma \Delta e^{-1}.$$

Its genus is $(p_1+p_2, q_1+q_2-2+P(\Gamma\Delta))$ where P is the predicate of the previous lemma. □

Corollary 3.1 (Métayer 2001, Lemma 1). Every connected proof-structure π verifies $\mathcal{N}(\pi)$. As a consequence, $K(\pi)$ is either a sphere with holes or a connected sum of tori with holes.

Proof. The proof is by induction on the number of links of π. If there is only one link, π is necessarily an axiom link:



$K(\pi)$ has the presentation $\{aa^{-1}cAA^{\perp}c^{-1}\}$. $\mathcal{N}$ is verified. Other cases of the induction are managed by the previous lemmas. □

4. Calculus of surfaces

In this section, we particularise the technical lemmas from Subsection 3.2 to make explicit the effect of each logical rule on Métayer complexes. A decorated sequent calculus formalises this in such a way that, in parallel with the elaboration of a sequential proof with implicit exchange, we compute everything that characterises the associated complex.

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Definition 4.1 (K(MLL) sequent). Basic objects are cycles of formulas

$$C = (A_1, \ldots, A_n).$$

That is, the finite sequence of formulas $A_1, \ldots, A_n$ is ordered up to a circular permutation.

A *K(MLL) sequent* is the data of an integer $p \geqslant 0$ and a finite set of cycles of formulas

$$\vdash_p C_1, \ldots, C_q.$$

The empty cycle () is a neutral element of sequents, that is to say

$$\vdash_p (), C_1, \ldots, C_q = \vdash_p C_1, \ldots, C_q.$$

Remark. A K(MLL) sequent contains everything that characterises a Métayer complex (see also Lemma 4.1): the pair (p, q), where q is the number of non-empty cycles of the sequent, is the genus of the complex; each C_i describes the cyclic order of the conclusions composing a hole T_i; these C_i's are not ordered since holes on a surface are not.

Definition 4.2 (K(MLL) calculus). A K(MLL) proof is built inductively according to the following rules:

$$\text{axiom}\ \frac{}{\vdash_0 (A, A^\perp)}$$

$$\text{cut}\ \frac{\vdash_{p_1} \Gamma, (\Sigma, A) \qquad \vdash_{p_2} (A^\perp, \Theta), \Delta}{\vdash_{p_1+p_2} \Gamma, (\Sigma, \Theta), \Delta}$$

$$\otimes\ \frac{\vdash_{p_1} \Gamma, (\Sigma, A) \qquad \vdash_{p_2} (B, \Theta), \Delta}{\vdash_{p_1+p_2} \Gamma, (\Sigma, A \otimes B, \Theta), \Delta}$$

$$\parr_1\ \frac{\vdash_p (A, \Delta, B, \Sigma), \Gamma}{\vdash_p (\Delta), (A \parr B, \Sigma), \Gamma}$$

$$\parr_2\ \frac{\vdash_p (\Sigma, A), (B, \Delta), \Gamma}{\vdash_{p+1} (\Sigma, A \parr B, \Delta), \Gamma}$$

Obviously, an McyLL (respectively, MXLL, M$\mathbb{X}$LL) proof P can be translated in a K(MLL) proof $\widehat{P}$ in such a way that the conclusion sequents of P and $\widehat{P}$ have the same underlying set of formulas. Notably, if P is a proof P', followed by an exchange rule, then $\widehat{P} = \widehat{P'}$.

Lemma 4.1 (adequacy). Let P be an McyLL (respectively, MXLL, M$\mathbb{X}$LL) proof such that

$$\vdash_p C_1, \ldots, C_q$$

is the conclusion sequent of $\widehat{P}$ ($C_1, \ldots, C_q$ not empty). Then $K(P^*)$ is the surface of genus (p, q) whose holes are described by $C_1, \ldots, C_q$.

Proof. The proof is by induction on P. We use Lemma 3.3 for the $\otimes$, Lemma 3.2 for the $\parr$, and Lemma 3.5 for the cut, in which we can observe the evolution of conclusions and the formation of tori. □

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5. Exchange

The main idea of Métayer's paper was the following: if π is a proof-net, the topological complexity of $K(\pi)$ should be linked to the exchange complexity of its sequentialisations. It would be a kind of measure of the distance to cyclic logic. Actually, the correspondence between cyclicity and zero topological complexity works quite well. Unfortunately, it is another story for strictly positive ranks..

As we have seen earlier (Section 2.1), several exchange rules can be considered to sequentialise a proof-net in commutative logic. In fact, only the $\mathbb{X}$-rule, allowing transpositions by blocks, is relevant, to some extent, from the geometrical point of view.

After the reminder of Métayer's results, we shall give answers to some open questions in his paper.

5.1. *Cyclic case*

Lemma 5.1 (Métayer 2001, Proposition 1). For each McyLL proof P, we have $K(P^*)$ is a disk whose boundary is the cycle of conclusions of P.

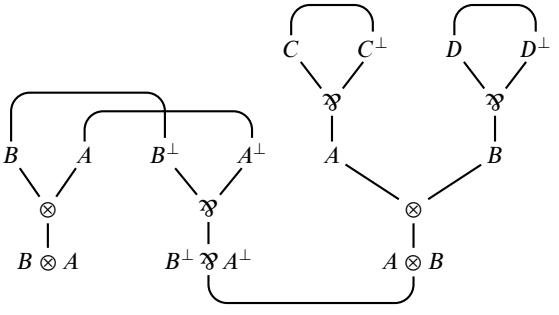
Proof. The proof is trivial by translating McyLL into K(MLL). □

The converse is only partial. We reformulate Métayer's Proposition 2 (Métayer 2001) and its proof in terms of proof-nets and with the help of the K(MLL) calculus.

Proposition 5.1. If π is a cut-free proof-net (for MXLL) such that $\mathrm{rk}(\pi)$ is zero, then π has an McyLL sequentialisation.

Proof. Let P be an MXLL sequentialisation of π and $\widehat{P}$ be its translation in K(MLL). The nullity of the rank of π means that $K(\pi)$ is a disk. Thus $\widehat{P}$ cannot make use of the $\parr_2$ rule, which increases the number of tori, since no rule decreases it. As a consequence, each application of the $\parr_1$ rule is done on consecutive formulas, otherwise new holes would appear and only cut and $\parr_2$ rules can merge two holes. Hence, we see that, apart from the decoration, $\widehat{P}$ is the required McyLL sequentialisation. □

The restriction to normal proof-nets cannot be removed precisely because the cut-rule can hide part of the complexity. We can see this on the following proof-net π:



where $A = C \parr C^\perp$ and $B = D \parr D^\perp$. In a sequentialisation of π, the only possible final rule is the cut-rule. Note that π_1 and π_2 the two sub-proof-nets of the respective

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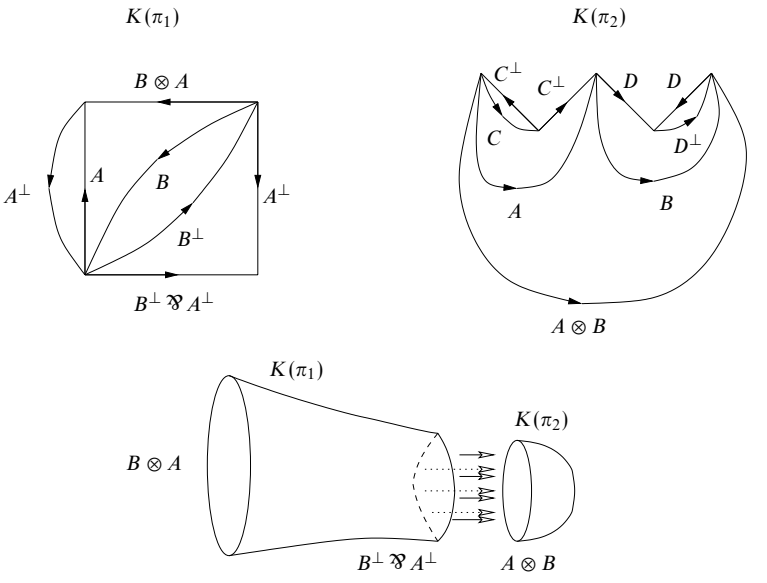


Fig. 4. Weakening of the complexity by a cut

conclusions $B \otimes A$, $B^\perp \parr A^\perp$ and $A \otimes B$. $K(\pi)$ is a disk while $K(\pi_1)$ is a cylinder. Figure 4 shows the effect of the cut. We have $\mathrm{rk}(\pi_1) = 1$, and thus, by Lemma 5.1, it is clear that an exchange rule is necessary to sequentialise π_1.

5.2. $\mathbb{X}$ *case*

Notation. For an MIXLL proof P, the number of applications of $\mathbb{X}$ is denoted $\varepsilon(P)$.

Theorem 5.1 (Métayer 2001, Theorem 2). For each MIXLL proof P, $\mathrm{rk}(P^*) \leqslant 2\varepsilon(P)$.

The important point of the proof is the possibility of defining a complex $J(P)$ that is no longer a surface but whose topology effectively translates the presence of exchange rules. $J(P)$, defined by induction on P, is built in such a way that, if P has $\vdash A_1, \ldots, A_n$ for conclusion, then $(A_1, \ldots, A_n)$ is an oriented path in $J(P)$. Using homology, we deduce that $\mathrm{rk}(P^*) \leqslant \mathrm{rk}(J(P))$. Then, the proof consists of showing that $\mathrm{rk}(J(P)) \leqslant 2\varepsilon(P)$, again by homological methods. The path property is essential for the induction case of $\parr$.

5.3. *Impossible extension to the X rule*

The previous method works for MIXLL but not for MXLL because an $\mathbb{X}$ application has a bounded effect on the rank of $J(P)$. If we wanted to keep the same path property passing through an X application, we could not keep the control on $\mathrm{rk}(J(P))$, because the number of neighbourhood changes on the sequent is not bounded. Moreover, this potential increase of complexity, made by an X rule, can be realised on $\mathrm{rk}(P^*)$ by $\parr$-rules. The next proposition exploits this idea.

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Proposition 5.2. For each pair $(p,q) \in \mathbb{N} \times \mathbb{N}^* \setminus \{(0,1)\}$, there exists a proof-net $\pi_{p,q}$ of genus (p,q), which is sequentialisable in MXLL with only one application of X.

Proof. For $n \geqslant 1$, we define P_n, an McyLL proof of conclusion

$$\vdash a_{2n}, a_{2n-1}, \ldots, a_1, A_n$$

where

$$A_n = \left(\ldots\left(\left(a_1^\perp \otimes a_2^\perp\right) \otimes a_3^\perp\right)\ldots\right) \otimes a_{2n}^\perp$$

by induction on n.

$$P_1 = \dfrac{\dfrac{}{\vdash a_1, a_1^\perp} \quad \dfrac{}{\vdash a_2, a_2^\perp}}{\vdash a_2, a_1, a_1^\perp \otimes a_2^\perp}$$

Then P_{n+1} is

$$\dfrac{\dfrac{\dfrac{P_n}{\vdash a_{2n}, \ldots, a_1, A_n} \quad \dfrac{}{\vdash a_{2n+1}^\perp, a_{2n+1}}}{\vdash a_{2n+1}, \ldots, a_1, A_n \otimes a_{2n+1}^\perp} \quad \dfrac{}{\vdash a_{2n+2}^\perp, a_{2n+2}}}{\vdash a_{2n+2}, \ldots, a_1, \left(A_n \otimes a_{2n+1}^\perp\right) \otimes a_{2n+2}^\perp}$$

P_n is cyclic, thus $K(P_n^*)$ is a disk whose oriented boundary is $(a_{2n}, a_{2n-1}, \ldots, a_1, A_n)$. In other words, the translation of P_n in K(MLL) has

$$\vdash_0 (a_{2n}, a_{2n-1}, \ldots, a_1, A_n)$$

as conclusion sequent.

Given $(p,q) \in \mathbb{N} \times \mathbb{N}^* \setminus \{(0,1)\}$, we take $n = p + q - 1$. Now, the MXLL proof $Q_{p,q}$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{P_n}{\vdash a_{2n}, a_{2n-1}, a_{2n-2}, a_{2n-3} \ldots, a_1, A_n}}{\vdash a_{2n-1}, a_{2n}, a_{2n-3}, a_{2n-2}, \ldots, a_1, a_2, A_n}\, X}{\vdash B_1, \ldots, B_n, B_{n+1}}\, \parr \ (n \text{ times})}{\vdash (\ldots(B_1 \parr B_2) \parr \ldots) \parr B_{p+1}, B_{p+2}, \ldots, B_n, B_{n+1}}\, \parr \ (p \text{ times})$$

where $B_i := a_{2n-2i+1} \parr a_{2n-2i+2}$ for $1 \leqslant i \leqslant n$ and $B_{n+1} := A_n$.

Let us translate this into K(MLL).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash_{p-1} (B_1 \parr B_2 \parr \ldots \parr B_p), (B_{p+1}), \ldots (B_n), (B_{n+1})}\, \parr_2}{\vdash_p (B_1 \parr B_2 \parr \ldots \parr B_{p+1}), (B_{p+2}), \ldots, (B_n), (B_{n+1})}\, \parr_2}}{\vdots}}{\vdash_1 (B_1 \parr B_2), (B_3), \ldots, (B_n), (B_{n+1})}\, \parr_2}{\vdash_0 (B_1), \ldots, (B_{n-1}), (B_n), (B_{n+1})}\, \parr_2}{\vdash_0 (B_1), \ldots, (B_{n-1}), (a_2, a_1, A_n)}\, \parr_1 \text{ with } B_n = a_1 \parr a_2 \text{ and } B_{n+1} = A_n}{\vdots}}{\vdash_0 (B_1), (a_{2n-2}, \ldots, a_1, A_n)}\, \parr_1}{\vdash_0 (a_{2n}, a_{2n-1}, \ldots, a_1, A_n)}\, \parr_1 \text{ with } B_1 = a_{2n-1} \parr a_{2n}$$

Of course, we take $\pi_{p,q} = Q_{p,q}^*$. The K(MLL) sequentialization shows that its genus is indeed $(p, n+1-p) = (p,q)$. □

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As an immediate corollary, we have that no (interesting) function f of the genus can be defined to be a lower bound of the number of X applications, because we must have

$$f(p,q) \leqslant 1$$

for each pair $(p,q) \in \mathbb{N} \times \mathbb{N}^*$.

If we use $\varepsilon(\pi)$ to denote the least number of X applications in sequentialisations of a proof-net π, an extension of Theorem 5.1 would be to find a function f of $\varepsilon(\pi)$ as an upper bound of the rank $\mathrm{rk}(\pi)$. In fact, it is easy to see that for each n, there exist proof-nets π with $\varepsilon(\pi) = n$ and a rank arbitrary large: for any p, it suffices to join n copies of $\pi_{p,1}$ by tensor links to obtain a proof-net π such that $\mathrm{rk}(\pi) = 2np$ and $\varepsilon(\pi) = n$.

Naturally, one can wonder about the converse, that is to say, can we find an upper bound of the exchange complexity as a function of the genus? The next subsection dashes any hope.

5.4. *Absence of converse*

The observation is the following: Métayer's complexes and sequents do not treat conclusions in the same way. On a complex, conclusions are split into several cycles, whereas a sequentialisation deals with only one. This 'linearity' of sequents imposes exchanges to bring together formulas that are, on the complex, side-by-side. Moreover, by an appropriate series of changes $\otimes / \parr$, exchange applications can be fixed at precise moments of the proof, in such a way that no rule commutations allow one to group them together.

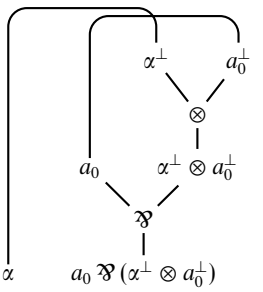
Proposition 5.3. For all $n \in \mathbb{N}^*$, there exists a cut-free proof-net π_n of rank 1, such that each of its sequentialisations requires at least n exchanges (all rules taken into account).

It follows, of course, that neither the genus nor the rank can be used to find an upper bound of the necessary number of exchanges. Here too, the proposition is easily generalised to any rank $r \geqslant 1$.

Proof. A family of proof-nets $\{\pi_n\}_n$ is defined by recurrence; for each n, π_n has α and A_n as conclusions where

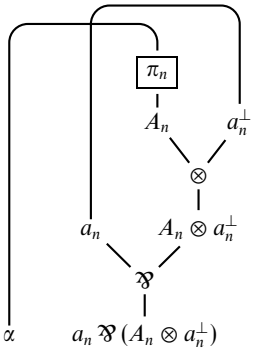
$$A_0 = \alpha^\perp$$
$$A_{n+1} = a_n \parr (A_n \otimes a_n^\perp).$$

π_1 is given by



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Then, by induction, π_{n+1} is



We prove by induction that $\mathrm{rk}(\pi_n) = 1$ exhibiting a sequentialisation $\widetilde{\pi}_n$ in K(MLL). We have, $\widetilde{\pi}_1$ is

$$\otimes \frac{\dfrac{}{\vdash_0 (\alpha, \alpha^\perp)} \quad \dfrac{}{\vdash_0 (a_0^\perp, a_0)}}{\parr_1 \dfrac{\vdash_0 (\alpha, \alpha^\perp \otimes a_0^\perp, a_0)}{\vdash_0 (\alpha), (A_1)}}$$

Thus $K(\pi_1)$ is a cylinder and $\mathrm{rk}(\pi_1) = 1$. Then, by induction, π_{n+1} is sequentialised by

$$\frac{\dfrac{\widetilde{\pi}_n}{\vdash_0 (\alpha), (A_n)} \quad \dfrac{}{\vdash_0 (a_n^\perp, a_n)}}{\dfrac{\vdash_0 (\alpha), (A_n \otimes a_n^\perp, a_n)}{\vdash_0 (\alpha), (A_{n+1})}}$$

So $\mathrm{rk}(\pi_n) = 1$ for each n.

It is easy to see that, for each n, π_n requires exactly n transpositions to be sequentialised in MXLL or M$\mathbb{X}$LL. The only possible final rule is the $\parr$. Its premise is necessarily the sequent

$$\vdash \alpha, a_{n-1}, A_{n-1} \otimes a_{n-1}^\perp.$$

But then the only possible *logical* rule is the $\otimes$ whose conclusion is

$$\vdash \alpha, A_{n-1} \otimes a_{n-1}^\perp, a_{n-1}.$$

Thus, an exchange is required (and sufficient) here between the $\otimes$ and the $\parr$. □

6. Drawing proof-structures

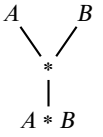
The subject of this final section is to show that Métayer's complex, associated to a proof-structure, is the surface of minimal complexity to draw it without crossing and respecting local orientation.

Notation. If π is a proof-structure, we use $G(\pi)$ to denote a graph-like object containing graphical information about π, notably the local orientation. $G(\pi) = (F, C, E, \{O_*\}_{*\in C})$

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where

— F is the set of occurrences of formulas of π;
— C is the set of occurrences of connectives of π;
— E is the set of edges of π. An edge is a pair belonging to F^2 or $F \times C$. Each axiom or cut link is represented by a pair of the form $(A, A^{\perp})$. Each logical link



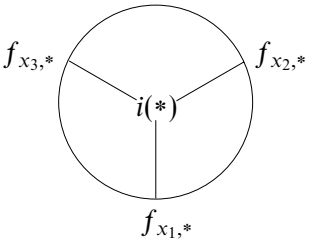
is represented by edges $(A, *)$, $(B, *)$ et $(A * B, *)$. Finally, O_* is the triple $(A * B, B, A)$.

Definition 6.1. Let π be a connected proof-structure and $G(\pi) = (F, C, E, \{O_*\}_{*\in C})$ be its associated graph. Let S be a surface presented by a polygon P and $j : P \to S$ be the canonical surjection. The set of non-paired edges of P is $j^{-1}(\partial(S))$.

A *drawing* $\mathscr{D}_S(\pi)$ *of* π *on* S is the following data:

— A one-to-one map $i : F \cup C \longrightarrow P$ such that:
  – $i(x) \in j^{-1}(\partial(S))$ if x is a conclusion of π;
  – $i(x) \in P \setminus \partial(P)$ otherwise.
— For each edge $a = (x, y)$ belonging to E, a piecewise-linear map $f_a : [0, 1] \longrightarrow P$ such that:
  – $f_a(0) = i(x)$ and $f_a(1) = i(y)$;
  – $j \circ f_a$ is continuous and one-to-one.

The drawing must preserve the local orientation around each connective $* \in C$, that is, segments of ends $i(*)$ in images of $f_{x_1,*}$, $f_{x_2,*}$ and $f_{x_3,*}$ respect the order $O_* = (x_1, x_2, x_3)$.



The drawing is said to be *without crossing* if

$$\forall (a, b) \in E^2 (a \neq b \implies f_a([0, 1]) \cap f_b(]0, 1[) = \varnothing).$$

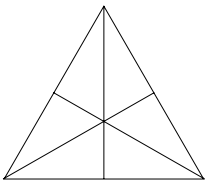
Theorem 6.1 (minimality of Métayer complexes). Let π be a connected proof-structure drawn without crossing on a surface S with genus (p, q). Let (p_0, q_0) be the genus of $K(\pi)$.

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Then

$$p_0 \leqslant p$$
$$\mathrm{rk}(\pi) \leqslant \mathrm{rk}(S).$$

Proof. First step: A tubular neighbourhood
Let P be the polygon representing S on which π is drawn. There exists a triangulation of P (Ahlfors and Sario 1960), that is, a paving of P by triangles such that the intersection of two triangles is either empty, or a single point, or a side, modulo identifications of $\partial(P)$. Let T be the subset of P formed by triangles that intersect the drawing (T is relative to the triangulation). The barycentric subdivision of a triangulation is constructed by dividing a triangle into six triangles:



After a sufficient number of barycentric subdivisions (two in fact), T can be arbitrarily close to the drawing of π and be a tubular neighbourhood of it, that is, a surface retracting on the drawing (Hudson 1969).

Second step: Identification of a ribbon
T is a surface characterised by its genus (p_T, q_T). The q_T holes of T are of two kinds:

— $\mathscr{B}_1$ denotes the set of holes containing conclusions of π and q_1 its cardinal;
— $\mathscr{B}_2$ denotes its complementary with cardinal q_2.

For example, Figure 5 shows the tubular neighbourhood of a drawing with $q_1 = 1$ and $q_2 = 3$.

Each hole is connected, compact and of dimension 1, and so is homeomorphic to a circle. Therefore, we can form a new surface $\widehat{T}$ by pasting a disk on each hole of $\mathscr{B}_2$ ($\widehat{T}$ is the connected sum of T with q_2 discs). The genus of $\widehat{T}$ is (p_T, q_1). Our goal is to show that $\widehat{T}$ and $K(\pi)$ are homeomorphic and thus that $p_0 = p_T$ and $q_0 = q_1$. To this end, we shall compare a presentation of T with one of $K(\pi)$. We may suppose that π is not an axiom since, in this case, everything is trivial.

The orientation of T induces an orientation of its holes. Since T is nothing more than a thickening of the drawing of π, we can talk about *the links of* T. We cut out axiom and cut links of T to obtain a set $\mathscr{W}$ of pieces that is indeed a presentation of T. In Figure 5, the places where the ribbon is cut are marked by arrows; the result of the cutting out can be seen in Figure 6.

Let $W \in \mathscr{W}$. We have W retracts on the tree of a formula (either a conclusion, or a premise of a cut link of π). Thus W is planar since its homology, equal to the homology of the tree, is zero in every dimension. Its border ∂W is a cycle of oriented edges. There are three kinds of edges in $\mathscr{W}$:

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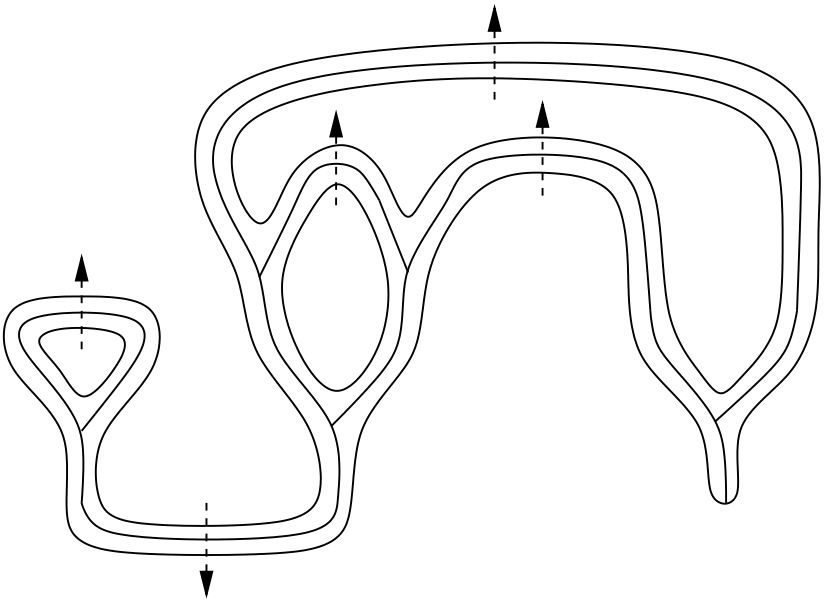


Fig. 5. The tubular neighbourhood of a drawing with the places where it is cut

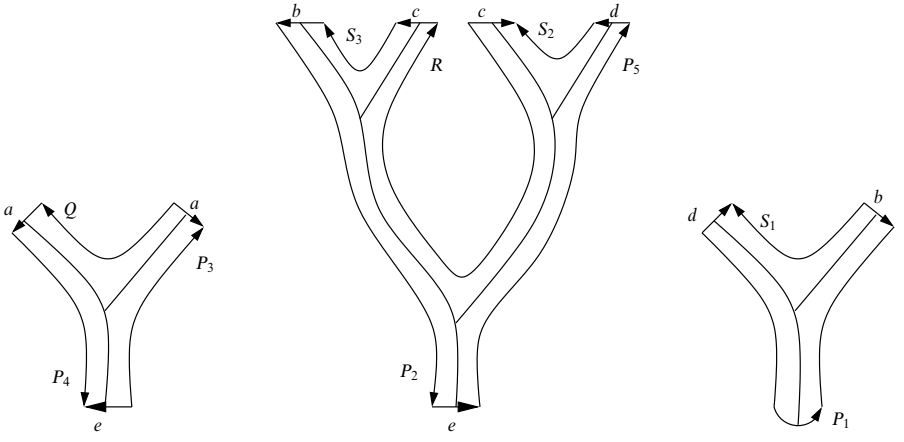


Fig. 6. Labelling of pieces resulting from the cutting out

— $\mathscr{E}_1$ the set of edges resulting from the cutting out of T. They are paired and can be labelled by the corresponding formula of π (we use the same label for the dual formulas of axiom or cut link);
— $\mathscr{E}_2$ the set of edges that do not contain a conclusion. They are not paired and can be labelled by any names (that are not used in π);
— $\mathscr{E}_3$ the set of edges containing a conclusion. They are not paired and are labelled by the conclusion formula.

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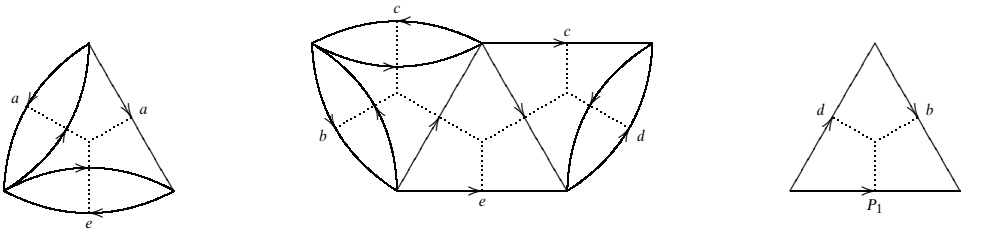


Fig. 7. Métayer's construction corresponding to Figure 6

In Figure 6, we do not follow these labelling instructions, since the proof-structure that is drawn is not typed. Edges of $\mathscr{E}_1$ are labelled by the small letters $a, b, \ldots, e$. The set $\mathscr{E}_2$ contains $P_2, P_3, P_4, S_1, S_2, S_3, Q, R$, and P_1 is the only edge of $\mathscr{E}_3$. The only hole of $\mathscr{B}_1$ is scattered on the three pieces and is presented by the cycle $P_1P_2P_3P_4$. Two holes of $\mathscr{B}_2$ are presented by only one edge each, Q and R, while the third is $S_1S_2S_3$.

We define a particular presentation $\mathscr{W}_0$ of $K(\pi)$ that can be compared to $\mathscr{W}$. To each $W \in \mathscr{W}$, we associate a piece $W_0 \in \mathscr{W}_0$: W retracts on the tree of a formula, we take the basic cells corresponding to the links of this tree and we paste them to obtain W_0. Axiom cells are glued, for example, along left conclusion and cut cells along left premise. Apart from renaming (for questions of orientation of cuttings out), if $W \in \mathscr{W}$ and $W_0 \in \mathscr{W}_0$ correspond to the same formula tree, then the *word* W_0 is a subword of W: if W is $Ca_1h_1a_2h_2 \ldots h_{n-1}a_n$, where the a_i's are edges of $\mathscr{E}_1$, h_i's are in $\mathscr{E}_2$ and either C is in $\mathscr{E}_3$ or results from the cutting out of a cut link, then $W_0 = Ca_1 \ldots a_n$.

Figure 7 shows this particular presentation of the Métayer's complex for the proof-structure and its ribbon of Figures 5 and 6.

Lemma 6.1. If $\mathscr{W}$ is a presentation of an orientable surface, there exists $f : \mathscr{W} \to \{+1, -1\}$ such that in the presentation $\mathscr{W}' = \{W^{f(W)}; W \in \mathscr{W}\}$, each label, a, carried by two edges occurs as a and a^{-1}.

Proof. The proof is by induction on the cardinal of $\mathscr{W}$ and uses the orientability of the surface. If $\mathscr{W}$ is a singleton, a paired label a appears necessarily as a and a^{-1}, otherwise it would mean there is a Möbius strip inside the surface, which would contradict the orientability. If $\mathrm{card}(\mathscr{W}) \geqslant 1$, we glue two pieces to use the induction hypothesis. □

T is an orientable surface, so we can suppose that $\mathscr{W}$ has been modified as in Lemma 6.1. A vertex of ∂T has at most two antecedents in $\mathscr{W}$ by the canonical surjection because the cutting out of T has been made such that there is an alternation of paired and non-paired edges in $\mathscr{W}$. The following lemma will be useful.

Lemma 6.2. Let $\mathscr{W}$ be a presentation of an orientable surface T such that each point of ∂T has at most two antecedents by the canonical surjection. Let H be a hole of T presented in $\mathscr{W}$ by the cycle ΓA_1A_2 (in general, scattered on different pieces of $\mathscr{W}$). We suppose that $\mathscr{W}$ has been changed as in the previous lemma. If A_2 appears in $\mathscr{W}$ in a word $W = waA_2$ where a is a paired label, then A_1 appears in a word $W' = w'A_1a^{-1}$ (W' is possibly equal to W).

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Proof. In T, the end of A_1 is the same point as the beginning of A_2. The word W tells us that this is also the end of a. Since this point has only two antecedents in $\mathscr{W}$, in the word W' where A_1 appears, it must be followed by a^{-1}. □

We define two procedures that have to be applied to the holes of $\mathscr{B}_1$ and $\mathscr{B}_2$.

— *Processing of the holes of* $\mathscr{B}_2$: Let $H \in \mathscr{B}_2$ scattered on $\mathscr{W}$. Then $\#H$ denotes the number of edges of $\mathscr{W}$ whose image in T belongs to H. We show that we can reduce this number without modifying the context of paired edges. H is presented in $\mathscr{W}$ by a cycle of edges $h_1 \dots h_m$ with $h_i \in \mathscr{E}_2$ and $m = \#H > 1$. The edge h_m appears on a word $W = wah_m$. By the lemma, we have a word $W' = w'h_{m-1}a^{-1}$.

$$\begin{aligned} W, W' &\rightsquigarrow w\dot{a}, \dot{a}^{-1}ah_m, w'h_{m-1}a^{-1} && \text{by cutting out} \\ &\rightsquigarrow w\dot{a}, w'h_{m-1}h_m\dot{a}^{-1} && \text{by gluing} \end{aligned}$$

Renaming $\dot{a}$ by a and $h_{m-1}h_m$ by h_{m-1} (which is possible because they are non-paired edges), we have a new presentation of T:

$$\mathscr{W}' = \mathscr{W} \setminus \{W\} \cup \{wa\}.$$

H is now presented by $h_1 \dots h_{m-1}$, $\#H = m-1$. Lemma 6.2 still holds because the other points of the boundary have not been modified. Thus, iterating this process a finite number of times, we reach to a presentation of T where, for each $H \in \mathscr{B}_2$, $\#H = 1$, that is, each hole of the second kind is constituted of only one edge.

— *Processing of the holes of* $\mathscr{B}_1$: Let $H \in \mathscr{B}_1$ scattered on $\mathscr{W}$. This hole H is presented by $C_1w_1C_2w_2 \dots C_mw_m$ where, for each $i \in \{1, \dots, m\}$, w_i is only formed by edges from $\mathscr{E}_2$ (possibly none), and C_i is an edge of $\mathscr{E}_3$. For each i such that w_i is not empty, we follow the same process as for the holes of $\mathscr{B}_2$: we progressively bring back edges of w_i to C_i, reducing $\#w_i$. The goal is to keep the edges of $\mathscr{E}_3$ fixed. After a finite number of steps, each $H \in \mathscr{B}_1$ is uniquely constituted of edges of $\mathscr{E}_3$.

After applying these two algorithms, the only remaining difference between $\mathscr{W}$ and $\mathscr{W}_0$ is that edges-holes of $\mathscr{B}_1$ appear on $\mathscr{W}$. But a presentation $\widehat{\mathscr{W}}$ of $\widehat{T}$ is precisely obtained by removal of these edges. Thus $\widehat{T}$ and $K(\pi)$ have a common presentation and are homeomorphic.

Third step: Minimality

It remains to compare the genera of T and of the initial surface S. On each hole of S, we paste a little cylinder. The resulting surface, denoted by $\widetilde{S}$, is homeomorphic to S. The ribbon T can be seen as a subsurface of $\widetilde{S}$ in such a way that $\partial(T) \subset \widetilde{S} \setminus \partial(\widetilde{S})$. Thus $\widetilde{S}$ appears as the gluing of T with other surfaces along one or several holes.

We first prove that $p \geqslant p_0$. We just observe one step of the reconstruction of $\widetilde{S}$ from T. Let R be a surface of genus $(p_R, q_R + q'_R)$ with $1 \leqslant q_R \leqslant q_T$ and $q'_R \geqslant 0$ such that $q'_R + q_T - q_R > 0$. We use $R \amalg T$ to denote the surface resulting from the identification of q_R holes of R with q_R holes of T. Let $(p_{\amalg}, q_{\amalg})$ be its genus. Since $q_{\amalg} = q'_R + q_T - q_R > 0$, $R \amalg T$ is a surface with boundary. This condition is verified all along the reconstruction of $\widetilde{S}$ because $\widetilde{S}$ has a boundary. This point is of importance for homological reason. It is enough to prove that $p_{\amalg} \geqslant p_T$ to obtain, by iteration, that $p \geqslant p_T$, that is, $p \geqslant p_0$.

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We take $\{U, V\}$ to be an open covering of $R \amalg T$ such that U and V retract on R and T, respectively. Then $U \cap V$ retracts on the disjoint union of q_R circles (that is, holes that have been identified).

Here, we have to remember some statements of homology theory:

— If X is the surface of genus $(p, q) \in \mathbb{N} \times \mathbb{N}^*$, its reduced homology is

$$\begin{cases} \mathrm{H}_1(X) = \mathbb{Z}^{2p+q-1} \\ \mathrm{H}_k(X) = 0 \quad \text{if } k \neq 1. \end{cases}$$

— The reduced homology of a disjoint union Y of q circles is

$$\begin{cases} \mathrm{H}_0(Y) = \mathbb{Z}^{q-1} \\ \mathrm{H}_1(Y) = \mathbb{Z}^q \\ \mathrm{H}_k(Y) = 0 \quad \text{if } k \notin \{0, 1\}. \end{cases}$$

The Mayer–Vietoris sequence (in reduced homology) with the covering $\{U, V\}$ amounts to:

$$0 \longrightarrow \mathrm{H}_1(U \cap V) \longrightarrow \mathrm{H}_1(U) \oplus \mathrm{H}_1(V) \longrightarrow \mathrm{H}_1(R \amalg T) \longrightarrow \mathrm{H}_0(U \cap V) \longrightarrow 0$$

where

$$\begin{aligned} \mathrm{H}_1(U \cap V) &= \mathbb{Z}^{q_R} \\ \mathrm{H}_1(U) &= \mathbb{Z}^{2p_R+q_R+q'_R-1} \\ \mathrm{H}_1(V) &= \mathbb{Z}^{2p_T+q_T-1} \\ \mathrm{H}_0(U \cap V) &= \mathbb{Z}^{q_R-1}. \end{aligned}$$

We write that the alternating sum of ranks is zero:

$$q_R - (2p_R + q_R + q'_R - 1) - (2p_T + q_T - 1) + \mathrm{rk}(R \amalg T) - (q_R - 1) = 0.$$

Thus,

$$\mathrm{rk}(R \amalg T) = 2(p_T + p_R) + q_T + q_R + q'_R - 3 = \mathrm{rk}(T) + \mathrm{rk}(R) - 1.$$

We know that $q_\amalg = q_T - q_R + q'_R$ and $\mathrm{rk}(R \amalg T) = 2p_\amalg + q_\amalg - 1$, so

$$p_\amalg = \frac{1}{2}(\mathrm{rk}(R \amalg T) - q_\amalg + 1) = p_T + p_R + q_R - 1.$$

Since $p_R + q_R - 1 \geqslant 0$, we have $p_\amalg \geqslant p_T$.

Then we prove that $\mathrm{rk}(\pi) \leqslant \mathrm{rk}(S)$. We start with $K(\pi)$. We know that $K(\pi)$ is homeomorphic to $\widehat{T}$. Each hole of $\mathscr{B}_2$, cut out of $K(\pi)$ to find T again, increases the rank by one. So $\mathrm{rk}(T) = \mathrm{rk}(\pi) + \mathrm{card}(\mathscr{B}_2)$. Next, surfaces must be glued on T to find $\widetilde{S}$ again. We have seen above that

$$\mathrm{rk}(R \amalg T) = \mathrm{rk}(R) + \mathrm{rk}(T) - 1.$$

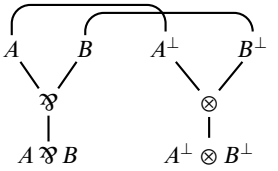
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When R is a disk, the rank decreases by one, and this case must be considered carefully. On the one hand, if a disk is glued on a hole of $\mathscr{B}_2$, the cutting out of this hole has previously increased the rank by one. Thus the operation has no net effect. On the other hand, it never happens that a disk is glued on a hole of $\mathscr{B}_1$. There must be at least one more hole on R corresponding to a hole of S where conclusions could lie. In all other cases of gluing a surface R, $\mathrm{rk}(R) \geqslant 1$ and the operation increases the rank. Finally, we have

$$\mathrm{rk}(\pi) \leqslant \mathrm{rk}(\widetilde{S}) = \mathrm{rk}(S).$$

□

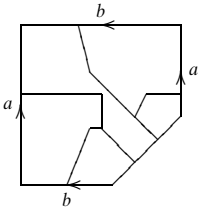
Remark. The inequality $q_0 \leqslant q$ is false. Take the following proof-net:



Its sequentialisation in K(MLL) is

$$\dfrac{\dfrac{\vdash_0 (A, A^\perp) \qquad \vdash_0 (B^\perp, B)}{\vdash_0 (A, A^\perp \otimes B^\perp, B)}}{\vdash_0 (A \parr B), (A^\perp \otimes B^\perp)}$$

Thus $K(\pi)$ is a cylinder. But π can be drawn on a torus with only one hole:



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