Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 161 SEPTEMBER 2016 PART 2

Math. Proc. Camb. Phil. Soc. (2016), **161**, 193–197 © *Cambridge Philosophical Society* 2016
doi:10.1017/S0305004116000013X
First published online 11 March 2016

An improved lower bound for arithmetic regularity

BY KAAVE HOSSEINI† AND SHACHAR LOVETT†
Department of Computer Science and Engineering, University of California, San Diego, La Jolla, CA 92093*, USA.*
e-mails: skhossei@cse.ucsd.edu; slovett@cse.ucsd.edu

GUY MOSHKOVITZ‡ AND ASAF SHAPIRA§
School of Mathematics, Tel Aviv University, Tel Aviv 69978*, Israel.*
e-mails: guymosko@tau.ac.il; asafico@tau.ac.il

(*Received* 20 *May* 2014; *revised* 25 *January* 2016)

Abstract

The arithmetic regularity lemma due to Green [GAFA 2005] is an analogue of the famous Szemerédi regularity lemma in graph theory. It shows that for any abelian group G and any bounded function $f : G \to [0, 1]$, there exists a subgroup $H \leqslant G$ of bounded index such that, when restricted to most cosets of H, the function f is pseudorandom in the sense that all its nontrivial Fourier coefficients are small. Quantitatively, if one wishes to obtain that for $1 - \epsilon$ fraction of the cosets, the nontrivial Fourier coefficients are bounded by ϵ, then Green shows that $|G/H|$ is bounded by a tower of twos of height $1/\epsilon^3$. He also gives an example showing that a tower of height $\Omega(\log 1/\epsilon)$ is necessary. Here, we give an improved example, showing that a tower of height $\Omega(1/\epsilon)$ is necessary.

1. *Introduction*

As an analogue of Szemerédi's regularity lemma in graph theory [**4**], Green [**2**] proposed an arithmetic regularity lemma for abelian groups. Given an abelian group G and a bounded

† Supported in part by NSF award 1350481.
‡ Supported in part by ISF grant 224/11.
§ Supported in part by ISF Grant 224/11, Marie–Curie CIG Grant 303320 and ERC-Starting Grant 633509

https://doi.org/10.1017/S030500411600013X Published online by Cambridge University Press

function $f : G \to [0, 1]$, Green showed that one can find a subgroup $H \leqslant G$ of bounded index, such that when restricted to most cosets of H, the function f is pseudorandom in the sense that all of its nontrivial Fourier coefficients are small. Quantitatively, the index of H in G is bounded by a tower of twos of height polynomial in the error parameter. The aim of this paper is to provide an example showing that these bounds are essentially tight. This strengthens a previous example due to Green [**2**] which shows that a tower of height logarithmic in the error parameter is necessary; and makes the lower bounds in the arithmetic case analogous to these obtained in the graph case [**1**].

We restrict our attention in this paper to the group $G = \mathbb{Z}_2^n$, and note that our construction can be generalised to groups of bounded torsion in an obvious way. We first make some basic definitions. Let A be an affine subspace (that is, a translation of a vector subspace) of $\mathbb{Z}_2^n$ and let $f : A \to [0, 1]$ be a function. The Fourier coefficient of f associated with $\eta \in \mathbb{Z}_2^n$ is

$$\widehat{f}(\eta) = \frac{1}{|A|} \sum_{x \in A} f(x)(-1)^{\langle x, \eta \rangle} = \mathbb{E}_{x \in A}[f(x)(-1)^{\langle x, \eta \rangle}] \, .$$

Any subspace $H \leqslant \mathbb{Z}_2^n$ naturally determines a partition of $\mathbb{Z}_2^n$ into affine subspaces

$$\mathbb{Z}_2^n / H = \{H + g \, : \, g \in \mathbb{Z}_2^n\} \, .$$

The number $\left|\mathbb{Z}_2^n / H\right| = 2^{n - \dim H}$ of translations is called the *index* of H.

1·1. *Arithmetic regularity and the main result*

For an affine subspace $A = H + g$ of $\mathbb{Z}_2^n$, where $H \leqslant \mathbb{Z}_2^n$ and $g \in \mathbb{Z}_2^n$, we say that a function $f : A \to [0, 1]$ is ϵ-*regular* if all its nontrivial Fourier coefficients are bounded by ϵ, that is,

$$\max_{\eta \notin H^\perp} \left|\widehat{f}(\eta)\right| \leqslant \epsilon \, .$$

Note that a trivial Fourier coefficient (i.e., $\widehat{f}(\eta)$ with $\eta \in H^\perp$) satisfies $|\widehat{f}(\eta)| = |\mathbb{E}_{x \in A} f(x)|$. Henceforth, for any $f : \mathbb{Z}_2^n \to [0, 1]$ we write $f|_A : A \to [0, 1]$ for the restriction of f to A.

Definition 1·1 (ϵ-regular subspace). Let $f : \mathbb{Z}_2^n \to [0, 1]$. A subspace $H \leqslant \mathbb{Z}_2^n$ is ϵ-*regular* for f if $f|_A$ is ϵ-regular for at least $(1 - \epsilon) \cdot \left|\mathbb{Z}_2^n / H\right|$ translations A of H.

Green [**2**] proved that any bounded function has an ϵ-regular subspace H of bounded index, that is, whose index depends only on ϵ (equivalently, H has bounded codimension). In the following, $\mathrm{twr}(h)$ is a tower of twos of height h; formally, $\mathrm{twr}(h) := 2^{\mathrm{twr}(h-1)}$ for a positive integer h, and $\mathrm{twr}(0) = 1$.

THEOREM 1 (Arithmetic regularity lemma in $\mathbb{Z}_2^n$, [**2** theorem 2·1]). *For every* $0 < \epsilon < 1/2$ *there is* $M(\epsilon)$ *such that every function* $f : \mathbb{Z}_2^n \to [0, 1]$ *has an* ϵ*-regular subspace of index at most* $M(\epsilon)$*. Moreover,* $M(\epsilon) \leqslant \mathrm{twr}(\lceil 1/\epsilon^3 \rceil)$.

A lower bound on $M(\epsilon)$ of about $\mathrm{twr}(\log_2(1/\epsilon))$ was given in the same paper [**2**], following the lines of Gowers' lower bound on the order of ϵ-regular partitions of graphs [**1**]. While Green's lower bound implies that $M(\epsilon)$ indeed has a tower-type growth, it is still quite far from the upper bound in Theorem 1.

Our main result here nearly closes the gap between the lower and upper bounds on $M(\epsilon)$, showing that $M(\epsilon)$ is a tower of twos of height at least linear in $1/\epsilon$. Our construction follows

https://doi.org/10.1017/S030500411600013X Published online by Cambridge University Press

the same initial setup as in [**2**], but will diverge from that point on. Our proof is inspired by the recent simplified lower bound proof for the graph regularity lemma in [**3**] by a subset of the authors.

THEOREM 2. *For every* $\epsilon > 0$ *it holds that* $M(\epsilon) \geqslant \mathrm{twr}(\lfloor 1/16\epsilon \rfloor)$.

1·2. *A variant of Theorem* 2 *for binary functions*

One can also deduce from Theorem 2 a similar bound for ϵ-regular *sets*, that is, for binary functions $f : \mathbb{Z}_2^n \to \{0, 1\}$. For this, all we need is the following easy probabilistic argument.

CLAIM 1·2. *Let* $\tau > 0$ *and* $f : \mathbb{Z}_2^n \to [0, 1]$. *There exists a binary function* $S : \mathbb{Z}_2^n \to \{0, 1\}$ *satisfying, for every affine subspace* A *of* $\mathbb{Z}_2^n$ *of size* $|A| \geqslant 4n^2/\tau^2$ *and any vector* $\eta \in \mathbb{Z}_2^n$, *that*

$$\left|\widehat{S|_A}(\eta) - \widehat{f|_A}(\eta)\right| \leqslant \tau.$$

Proof. Choose $S : \mathbb{Z}_2^n \to \{0, 1\}$ randomly by setting $S(x) = 1$ with probability $f(x)$, independently for each $x \in \mathbb{Z}_2^n$. Let A, η be as in the statement. The random variable

$$\widehat{S|_A}(\eta) = \frac{1}{|A|} \sum_{x \in A} S(x)(-1)^{\langle x, \eta \rangle}$$

is an average of $|A|$ mutually independent random variables taking values in $[-1, 1]$, and its expectation is $\widehat{f|_A}(\eta)$. By Hoeffding's bound, the probability that $\left|\widehat{S|_A}(\eta) - \widehat{f|_A}(\eta)\right| > \tau$ is smaller than

$$2\exp(-\tau^2 |A|/2) \leqslant 2^{-2n^2+1}.$$

The number of affine subspaces over $\mathbb{Z}_2^n$ can be trivially bounded by 2^{n^2}, the number of sequences of n vectors in $\mathbb{Z}_2^n$. Hence, the number of pairs (A, η) is bounded by 2^{n^2+n}. The claim follows by the union bound.

Applying Claim 1·2 with $\tau = \epsilon/2$ (say) implies that if $f : \mathbb{Z}_2^n \to [0, 1]$ has no ϵ-regular subspace of index smaller than $\mathrm{twr}(\lfloor 1/16\epsilon \rfloor)$ then, provided n is sufficiently large in terms of ϵ, there is $S : \mathbb{Z}_2^n \to \{0, 1\}$ that has no $\epsilon/2$-regular subspace of index smaller than $\mathrm{twr}(\lfloor 1/16\epsilon \rfloor)$.

2. *Proof of Theorem* 2

2·1. *The Construction*

To construct a function witnessing the lower bound in Theorem 2 we will use pseudo-random spanning sets.

CLAIM 2·1. *Let* V *be a vector space over* $\mathbb{Z}_2$ *of dimension* d. *Then there is a set of* $8d$ *nonzero vectors in* V *such that any* $6d$ *of them span* V.

Proof. Choose random vectors $v_1, \ldots, v_{8d} \in V \setminus \{0\}$ independently and uniformly. Let U be a subspace of V of dimension $d - 1$. The probability that a given v_i lies in U is at most $1/2$. By Chernoff's bound, the probability that more than $6d$ of our vectors v_i lie in U is smaller than $\exp(-2(2d)^2/8d) = \exp(-d)$. By the union bound, the probability that there exists a subspace U of dimension $d-1$ for which the above holds is at most $2^d \exp(-d) < 1$. This completes the proof.

https://doi.org/10.1017/S030500411600013X Published online by Cambridge University Press

We now describe a function $f : \mathbb{Z}_2^n \to [0,1]$ which, as we will later prove, has no ϵ-regular subspace of small index. Henceforth set $s = \lfloor 1/16\epsilon \rfloor$. Furthermore, let d_i be the following sequence of integers of tower-type growth:

$$d_{i+1} = \begin{cases} 2^{D_i} & \text{if } i = 1,2,3 \\ 2^{D_i - 3} & \text{if } i > 3 \end{cases} \qquad \text{where } D_i = \sum_{j=1}^{i} d_j \text{ and } D_0 = 0 .$$

Note that the first values of d_i for $i \geqslant 1$ are $1, 2, 8, 2^8, 2^{264}$, etc., and it is not hard to see that $d_i \geqslant \operatorname{twr}(i-1)$ for every $i \geqslant 1$. Set $n = D_s$ ($\geqslant \operatorname{twr}(s-1)$). For $x \in \mathbb{Z}_2^n$, partition its coordinates into s blocks of sizes $d_1, \ldots, d_s$, and identify $x = (x^1, \ldots, x^s) \in \mathbb{Z}_2^{d_1 + \cdots + d_s} = \mathbb{Z}_2^n$.

Let $1 \leqslant i \leqslant s$. Bijectively associate with each $v \in \mathbb{Z}_2^{D_{i-1}} = \mathbb{Z}_2^{d_1 + \cdots + d_{i-1}}$ a nonzero vector $\xi_i(v) \in \mathbb{Z}_2^{d_i}$ such that the set of vectors $\{\xi_i(v) : v \in \mathbb{Z}_2^{D_{i-1}}\}$ has the property that any subset of $3/4$ fraction of its elements spans $\mathbb{Z}_2^{d_i}$. The existence of such a set, which is a subset of size $2^{D_{i-1}}$ in a vector space of dimension d_i, follows from Claim 2·1 when $i > 3$, since then $2^{D_{i-1}} = 8d_i$. When $i \leqslant 3$ the existence of such a set is trivial since $\lceil (3/4)i \rceil = i$, hence any basis would do (and we take $2^{D_{i-1}} = d_i$). With a slight abuse of notation, if $x \in \mathbb{Z}_2^n$ we write $\xi_i(x)$ for $\xi_i((x^1, \ldots, x^{i-1}))$.

We define our function $f : \mathbb{Z}_2^n \to [0,1]$ as

$$f(x) = \frac{\left|\{1 \leqslant i \leqslant s \,:\, \langle x^i, \xi_i(x) \rangle = 0\}\right|}{s} .$$

The following is our main technical lemma, from which Theorem 2 immediately follows.

LEMMA 2·2. *The only ϵ-regular subspace for f is the zero subspace* $\{0\}$.

Proof of Theorem 2. The index of $\{0\}$ is $\left|\mathbb{Z}_2^n / \{0\}\right| = 2^n \geqslant \operatorname{twr}(s) = \operatorname{twr}(\lfloor 1/16\epsilon \rfloor)$.

2·2. *Proof of Lemma* 2·2

Let $H \neq \{0\}$ be a subspace of $\mathbb{Z}_2^n$. Let $1 \leqslant i \leqslant s$ be minimal such that there is $v \in H$ for which $v^i \neq 0$. For any $g \in \mathbb{Z}_2^n$ let

$$\gamma_g = (0, \ldots, 0, \xi_i(g), 0, \ldots, 0) \in \mathbb{Z}_2^n$$

where only the ith component is nonzero. We will show that for more than an ϵ fraction of the translations $H + g$ of H it holds that $\gamma_g \notin H^\perp$ yet

$$\widehat{f|_{H+g}}(\gamma_g) > \epsilon .$$

This will imply that H is not ϵ-regular for f, thus completing the proof.

First, we argue that $\gamma_g \notin H^\perp$ for a noticeable fraction of $g \in \mathbb{Z}_2^n$. We henceforth let $B = \{g \in \mathbb{Z}_2^n : \gamma_g \in H^\perp\}$ be the set of "bad" elements.

CLAIM 2·3. $|B| \leqslant \frac{3}{4}\left|\mathbb{Z}_2^n\right|$.

Proof. If $g \in B$ then $\langle \xi_i(g), v^i \rangle = 0$. Hence, $\{\xi_i(g) : g \in B\}$ does not span $\mathbb{Z}_2^{d_i}$. By the construction of ξ_i, this means that $\{(g^1, \ldots, g^{i-1}) : g \in B\}$ accounts to at most $\frac{3}{4}$ fraction of the elements in $\mathbb{Z}_2^{D_{i-1}}$, and hence $|B| \leqslant \frac{3}{4}\left|\mathbb{Z}_2^n\right|$.

Next, we argue that typically $\widehat{f|_{H+g}}(\gamma_g)$ is large. Let $W \leqslant \mathbb{Z}_2^n$ be the subspace spanned by the last $s - i$ blocks, that is, $W = \{w \in \mathbb{Z}_2^n : w^1 = \cdots = w^i = 0\}$. Note that for any $g \in \mathbb{Z}_2^n$, $w \in W$ we have $\gamma_{g+w} = \gamma_g$. In particular, $g + w \in B$ if and only if $g \in B$.

https://doi.org/10.1017/S030500411600013X Published online by Cambridge University Press

CLAIM 2·4. *Fix $g \in \mathbb{Z}_2^n$ such that $\gamma_g \notin H^\perp$. Then*

$$\mathbb{E}_{w\in W}\left[\widehat{f|_{H+g+w}}(\gamma_g)\right] = \frac{1}{2s}\,.$$

Proof. Write $f(x) = \frac{1}{s}\sum_{j=1}^{s} B_j(x)$ where $B_j(x) : \mathbb{Z}_2^n \to \{0,1\}$ is the characteristic function for the set of vectors x satisfying $\langle x^j, \xi_j(x)\rangle = 0$. Hence, for any affine subspace A in $\mathbb{Z}_2^n$,

$$\widehat{f|_A}(\gamma_g) = \frac{1}{s}\sum_{j=1}^{s} \widehat{B_j|_A}(\gamma_g)\,. \tag{2·1}$$

Set $A = H+g+w$ for an arbitrary $w \in W$. We next analyze the Fourier coefficient $\widehat{B_j|_A}(\gamma_g)$ for each $j \leqslant i$, and note that in these cases we have $\xi_j(x) = \xi_j(g)$ for any $x \in A$. First, if $j < i$ then for every $x \in A$ we have $x^j = g^j$, which implies that $B_j|_A$ is constant. Since a nontrivial Fourier coefficient of a constant function equals 0, we have

$$\widehat{B_j|_A}(\gamma_g) = 0, \qquad \forall j < i. \tag{2·2}$$

Next, for $j = i$, write $B_i|_A(x) = \frac{1}{2}((-1)^{\langle x^i,\xi_i(x)\rangle}+1)$. Since $\langle x, \gamma_g\rangle = \langle x^i, \xi_i(x)\rangle$, we have

$$\widehat{B_i|_A}(\gamma_g) = \mathbb{E}_{x\in A}\left[\frac{1}{2}((-1)^{\langle x^i,\xi_i(x)\rangle}+1)\cdot(-1)^{\langle x^i,\xi_i(x)\rangle}\right] = \mathbb{E}_{x\in A}\left[B_i(x)\right] = \frac{1}{2}\,. \tag{2·3}$$

Finally, for $j > i$ we average over all $w \in W$. Let $H + W$ be the subspace spanned by H, W. Writing $B_j(x) = ((-1)^{\langle x^j,\xi_j(x)\rangle}+1)/2$, the average Fourier coefficient is

$$\mathbb{E}_{w\in W}\mathbb{E}_{x\in H+g+w}\left[B_j(x)(-1)^{\langle x^i,\xi_i(g)\rangle}\right] = \frac{1}{2}\mathbb{E}_{x\in H+W+g}\left[(-1)^{\langle x^i,\xi_i(g)\rangle+\langle x^j,\xi_j(x)\rangle}\right].$$

Note that for every fixing of $x^1, \ldots, x^{j-1}$, we have that x^j is uniformly distributed in $\mathbb{Z}_2^{d_j}$ (due to W), and that $(-1)^{\langle x^i,\xi_i(g)\rangle}$ is constant. Since $\xi_j(x) \neq 0$, we conclude that

$$\mathbb{E}_{w\in W}\left[\widehat{B_j|_{H+g+w}}(\gamma_g)\right] = 0, \qquad \forall j > i. \tag{2·4}$$

The proof now follows by substituting (2·2), (2·3) and (2·4) into (2·1).

As $\widehat{f|_{H+g+w}}(\gamma_g) \leqslant 1$, we infer (via a simple averaging argument) the following corollary.

COROLLARY 2·5. *If $\gamma_g \notin H^\perp$ then for more than $1/4s$ fraction of all $w \in W$,*

$$\widehat{f|_{H+g+w}}(\gamma_g) > \frac{1}{4s}\,.$$

We can now conclude the proof of Lemma 2·2. Partition $\mathbb{Z}_2^n$ into translations of W. By Claim 2·3, for at least $1/4$ fraction of the translations $g + W$ we have $\gamma_g \notin H^\perp$. By Corollary 2·5, for each such g, more than $1/4s$ fraction of the elements $g + w \in g + W$ satisfy $\widehat{f|_{H+g+w}}(\gamma_g) > 1/4s$. As $1/16s \geqslant \epsilon$, this means that $f|_{H+x}$ is not ϵ-regular for more than ϵ fraction of all $x \in \mathbb{Z}_2^n$, implying that the subspace H is not ϵ-regular for f.

REFERENCES

[1] T. GOWERS. Lower bounds of tower type for Szemerédi's uniformity lemma. *GAFA* **7** (1997), 322–337.

[2] B. GREEN. A Szemerédi-type regularity lemma in abelian groups. *GAFA* **15** (2005), 340–376.

[3] G. MOSHKOVITZ AND A. SHAPIRA. A short proof of Gowers' lower bound for the regularity lemma. *Combinatorica*, to appear.

[4] E. SZEMERÉDI. Regular partitions of graphs. *Proc. Colloque Inter. CNRS*. (1978), 399–401.

https://doi.org/10.1017/S030500411600013X Published online by Cambridge University Press