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ON AUGMENTED HEGY TESTS FOR SEASONAL UNIT ROOTS

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In this paper we extend the large-sample results provided for the augmented Dickey–Fuller test by Said and Dickey (1984, *Biometrika* 71, 599–607) and Chang and Park (2002, *Econometric Reviews* 21, 431–447) to the case of the augmented seasonal unit root tests of Hylleberg, Engle, Granger, and Yoo (1990, *Journal of Econometrics* 44, 215–238), inter alia. Our analysis is performed under the same conditions on the innovations as in Chang and Park (2002), thereby allowing for general linear processes driven by (possibly conditionally heteroskedastic) martingale difference innovations. We show that the limiting null distributions of the t-statistics for unit roots at the zero and Nyquist frequencies and joint F-type statistics are pivotal, whereas those of the t-statistics at the harmonic seasonal frequencies depend on nuisance parameters that derive from the lag parameters characterizing the linear process. Moreover, the rates on the lag truncation required for these results to hold are shown to coincide with the corresponding rates given in Chang and Park (2002); in particular, an $o(T^{1/2})$ rate is shown to be sufficient.

1. INTRODUCTION

This paper considers testing for *seasonal* unit roots in a univariate time-series process. In the seminal paper in the literature, Hylleberg, Engle, Granger, and Yoo (1990) (HEGY) develop separate regression-based t- and F-tests for unit roots at the zero, Nyquist, and annual (harmonic) frequencies in the context of quarterly data. Recently, Smith, Taylor, and del Bario Castro (2009) have generalized the HEGY approach to allow for an arbitrary seasonal aspect.

In the nonseasonal context, Said and Dickey (1984) establish that the augmented Dickey–Fuller (ADF) statistics have (pivotal) Dickey–Fuller limiting null distributions in the presence of autoregressive moving average (ARMA) processes

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of unknown finite order with independently and identically distributed (i.i.d.) zero mean and constant variance innovations. This result requires that the lag length employed should increase in proportion with the sample size T at rate $o(T^{1/3})$. More recently, Chang and Park (2002) extend these results to allow the shocks to follow a general linear process driven by potentially heteroskedastic martingale difference innovations and show, moreover, that a weaker $o(T^{1/2})$ rate condition is sufficient for the result to hold. To the best of our knowledge, however, analogous results have not been established for the HEGY-type tests. Previous contributions have either assumed that the shocks are serially uncorrelated (e.g., Smith and Taylor, 1998) or follow a finite-order autoregressive (AR) process (e.g., Burridge and Taylor, 2001; Rodrigues and Taylor, 2004a, 2004b; Smith et al., 2009) or a finite-order moving average (MA) process (del Barrio Castro and Osborn, 2011). Therefore, none allow for either the finite-order ARMA assumption of Said and Dickey (1984) or the more general linear process assumptions of Chang and Park (2002). As Taylor (2005, p. 34) notes, "It has been widely conjectured, but never formally proved, that an approach along the lines of that developed in regard of the ADF test by Said and Dickey (1984) will purge the effects of ARMA behavior in the shocks from the limiting null distribution of the augmented HEGY-type statistics."

It has been known since the seminal work of Box and Jenkins (1976) that seasonally observed time series can display MA behavior, in addition to seasonal (AR) unit roots. Indeed, Box and Jenkins (1976) developed the well-known seasonal autoregressive integrated moving average (ARIMA) factorizations (a prominent example of which is the so-called airline model) as a parsimonious device for modeling dependence in seasonal data. Allowing for both AR and MA behavior is therefore very important when testing for unit roots in a seasonal context. ARMA behavior in the shocks can also be a manifestation of neglected periodic autoregressive (PAR) behavior, something that has been largely overlooked in the context of the HEGY tests. A detailed discussion of the PAR class of processes is given in Ghysels and Osborn (2001, Ch. 6). As an example, the first-order stationary PAR process for a series observed with period S, denoted $PAR(1)_S$, admits a stationary and invertible ARMA representation, which combines a seasonal AR term with an MA$(S-1)$ component.

Motivated by these considerations, in this paper we establish that the results given by Said and Dickey (1984) and Chang and Park (2002) for the ADF statistic do extend, at least in part, to the case of the augmented HEGY-type tests. Specifically, and complementing the findings of previous authors for finite-order shocks of either AR or MA form, we show that, provided the order of the lag augmentation polynomial increases in proportion with the sample size at a suitable rate, the t-statistics for unit roots at the zero and Nyquist frequencies, together with all the F-type tests employed in this context, remain pivotal in the presence of general linear processes driven by martingale difference innovations. However, this is not the case for the t-statistics at the harmonic seasonal frequencies, where the asymptotic null distributions depend on nuisance parameters that derive from

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the coefficients characterizing the linear process. Moreover, we show that the rate restriction required on the order of the lag augmentation polynomial coincides with the rate given in Chang and Park (2002).

The remainder of the paper is organized as follows. In Section 2 we outline the seasonal model, define the hypotheses of interest within that model, and briefly review the augmented HEGY-type seasonal unit root tests. The limiting null distributions of the HEGY statistics when the shocks follow a general linear process driven by martingale difference innovations are established in Section 3, along with the necessary rate conditions. Section 4 concludes. Proofs of our main results are contained in a mathematical Appendix.

In what follows $\xrightarrow{d}$ denotes weak convergence and $\xrightarrow{p}$ convergence in probability, in each case as the sample size diverges to positive infinity; $a := b$ ($a =: b$) indicates that a is defined by b (b is defined by a); $\lfloor \cdot \rfloor$ denotes the integer part of its argument; and I_p denotes the $p \times p$ identity matrix. The euclidean norm of the $k \times 1$ vector, x, is defined as $\|x\| := \left(x'x\right)^{1/2}$, and for the $k \times k$ matrix, A, we also define $\|A\| := \max_x \|Ax\| / \|x\|$. Finally we define $\mathrm{i} := \sqrt{(-1)}$.

2. THE SEASONAL UNIT ROOT FRAMEWORK

2.1. The Seasonal Model and Assumptions

Consider the univariate seasonal time-series process $\{x_{St+s}\}$ that satisfies the following data generating process:

$$\alpha(L)x_{St+s} = u_{St+s}, \qquad s = 1 - S, \ldots, 0, \qquad t = 1, 2, \ldots, N, \tag{2.1a}$$

$$u_{St+s} = \psi(L)\varepsilon_{St+s}, \tag{2.1b}$$

where the positive integer S denotes the number of seasons[1] and $\alpha(z) := 1 - \sum_{j=1}^{S} \alpha_j^* z^j$ is an AR(S) polynomial in the conventional lag operator, L. The error process u_{St+s} in (2.1b) is taken to be a linear process with $\psi(z) := 1 + \sum_{j=1}^{\infty} \psi_j z^j$. Precise conditions on this process are given subsequently. The *initial conditions*, $x_{1-S}, \ldots, x_0$, are taken to be of $o_p(T^{1/2})$. In what follows we define $T := SN$.

Following Chang and Park (2002), we make the following assumptions on the innovation sequence $\{\varepsilon_{St+s}\}$ and on the coefficients of the polynomial $\psi(L)$.

Assumption A.1. Let $(\varepsilon_{St+s}, \mathcal{F}_{St+s})$ be a martingale difference sequence, with filtration $(\mathcal{F}_{St+s})$, where $\mathcal{F}_{St+s} \subset \mathcal{F}_{St+s+1}$, for all s, t, and such that

(a) $\mathbb{E}\left[\varepsilon_{St+s}^2\right] = \sigma^2$,
(b) $1/N \sum_{t=1}^{N} \varepsilon_{St+s}^2 \xrightarrow{p} \sigma^2$ for each $s = 1 - S, \ldots, 0$, and
(c) $\mathbb{E}\,|\varepsilon_{St+s}|^r < K$ with $r \geq 4$, where K is some constant depending only upon r.

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Assumption A.2. Let the polynomial $\psi(z)$ be such that

(a) $\psi(z) \neq 0$ for all $|z| \leq 1$ and
(b) $\sum_{j=1}^{\infty} |j|^{\tau} |\psi_j| < \infty$ for some $\tau \geq 1$.

For some of the results in this paper, it will be necessary to substitute Assumption A.1 by the following somewhat stronger assumption.

Assumption A.1′. Let $(\varepsilon_{St+s}, \mathcal{F}_{St+s})$ be a martingale difference sequence, with filtration $(\mathcal{F}_{St+s})$, where $\mathcal{F}_{St+s} \subset \mathcal{F}_{St+s+1}$, for all s, t, such that

(a) $\mathbb{E}\left(\varepsilon_{St+s}^2 | \mathcal{F}_{St+s-1}\right) = \sigma^2$ and
(b) $\mathbb{E}|\varepsilon_{St+s}|^r < K$ with $r \geq 4$, where K is some constant depending only upon r.

Remark 1. Assumptions A.1, A.2, and A.1′ correspond to Assumptions 1, 2, and 1′, respectively, in Chang and Park (2002)—albeit condition (b) of Assumption A.1 is slightly stronger than condition (b) of Assumption 1 in Chang and Park (2002), reflecting the seasonal aspect of the data—and we therefore refer the reader to Chang and Park (2002, pp. 433–434) for a detailed discussion concerning these. However, notice, in particular, that a special case of Assumption A.2 is where u_{St+s} in (2.1b) admits the causal and invertible ARMA(p, q) representation, $\phi(L)u_{St+s} = \theta(L)\varepsilon_{St+s}$, such that all the roots of $\phi(z) := 1 - \sum_{i=1}^{p} \phi_p z^i$ and $\theta(z) := 1 - \sum_{i=1}^{q} \theta_i z^i$ lie strictly outside the unit circle. Assumption A.1 is also weaker than those previously made about the innovation process, ε_{St+s}, in the regression-based seasonal unit root literature, where either Assumption A.1′ or the even stronger assumption that ε_{St+s} is $iid(0, \sigma^2)$ with finite fourth moment has been adopted.

2.2. The Seasonal Unit Root Hypotheses

Our focus is on tests for seasonal unit roots in $\alpha(L)$ of (2.1a); that is, the null hypothesis of interest is

$$H_0 : \alpha(L) = 1 - L^S =: \Delta_S. \tag{2.2}$$

Under H_0 of (2.2), the DGP (2.1) of $\{x_{St+s}\}$ is a seasonally integrated process. We may factorize the AR(S) polynomial $\alpha(L)$ as $\alpha(L) = \prod_{j=0}^{\lfloor S/2 \rfloor} \omega_j(L)$, where $\omega_0(L) := (1 - \alpha_0 L)$ associates the parameter α_0 with the zero frequency $\omega_0 := 0$, $\omega_j(L) := [1 - 2(\alpha_j \cos \omega_j - \beta_j \sin \omega_j)L + (\alpha_j^2 + \beta_j^2)L^2]$ corresponds to the conjugate (harmonic) seasonal frequencies $(\omega_j, 2\pi - \omega_j)$, $\omega_j := 2\pi j/S$, with associated parameters α_j and β_j, $j = 1, \ldots, S^*$, where $S^* := \lfloor (S-1)/2 \rfloor$, and, for S even, $\omega_{S/2}(L) := (1 + \alpha_{S/2} L)$, associates the parameter $\alpha_{S/2}$ with the Nyquist frequency[2] $\omega_{S/2} := \pi$. Consequently H_0 of (2.2) may be commensurately partitioned as $H_0 = \cap_{j=0}^{\lfloor S/2 \rfloor} H_{0,j}$, where

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$$H_{0,i}:\alpha_i=1,\quad i=0,\ S/2,\qquad \text{and}\qquad H_{0,j}:\alpha_j=1,\quad \beta_j=0,\quad j=1,\ldots,S^*. \tag{2.3}$$

The hypothesis $H_{0,0}$ corresponds to a unit root at the zero frequency, whereas $H_{0,S/2}$ yields a unit root at the Nyquist frequency, $\omega_{S/2}=\pi$. A pair of complex conjugate unit roots at the jth harmonic seasonal frequencies is obtained under $H_{0,j}$, $j=1,...,S^*$.

The alternative hypothesis H_1 is of stationarity at one or more of the zero or seasonal frequencies; that is, $H_1=\cup_{j=0}^{\lfloor S/2\rfloor}H_{1,j}$, where

$$H_{1,i}:|\alpha_i|<1,\qquad i=0,S/2,\quad \text{and}\quad H_{1,j}:\alpha_j^2+\beta_j^2<1,\qquad j=1,\ldots,S^*. \tag{2.4}$$

Consequently, the maintained hypothesis $H_0\cup H_1$ excludes all unit roots, except for a possible single unit root at each of the zero and Nyquist frequencies and a single pair of complex conjugate unit roots at each of the harmonic seasonal frequencies. Explosive roots in $\alpha(L)$ are also excluded.

2.3. The Augmented HEGY Tests

To develop regression-based seasonal unit root tests, note first that under Assumption A.2 $\psi(z)$ is invertible, and let the (unique) inverse of $\psi(z)$ be denoted $d(z):=1-\sum_{j=1}^{\infty}d_jz^j$. Then it follows, using the proposition in HEGY (1990, pp. 221–222) that expanding $\alpha(z)d(z)$ around the zero and seasonal frequency unit roots, $\exp(\pm i2\pi j/S)$, $j=0,\ldots,\lfloor S/2\rfloor$, that the hypotheses in (2.3) may be restated as $H_{0,0}:\pi_0=0$, $H_{0,S/2}:\pi_{S/2}=0$, and $H_{0,j}:\pi_j=\pi_j^*=0$, $j=1,\ldots,S^*$, in the model

$$d^*(L)\Delta_S x_{St+s}=\pi_0x_{0,St+s}+\pi_{S/2}x_{S/2,St+s}+\sum_{j=1}^{S^*}\left(\pi_jx_{j,St+s}+\pi_j^*x_{j,St+s}^*\right)+\varepsilon_{St+s}, \tag{2.5}$$

omitting the term $\pi_{S/2}x_{S/2,St+s}$ where S is odd, where $d^*(z):=1-\sum_{j=1}^{\infty}d_j^*z^j$ is a causal AR polynomial and where

$$x_{0,St+s}:=\sum_{j=0}^{S-1}x_{St+s-j-1},\qquad x_{S/2,St+s}:=\sum_{j=0}^{S-1}\cos[(j+1)\pi]x_{St+s-j-1}, \tag{2.6a}$$

$$x_{i,St+s}:=\sum_{j=0}^{S-1}\cos[(j+1)\omega_i]x_{St+s-j-1},$$

$$x_{i,St+s}^*:=-\sum_{j=0}^{S-1}\sin[(j+1)\omega_i]x_{St+s-j-1},\qquad i=1,\ldots,S^*. \tag{2.6b}$$

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As in Chang and Park (2002, p. 434), we can approximate u_{St+s} from (2.1b) in rth mean by the finite-order AR process

$$u_{St+s} = d_1 u_{St+s-1} + \cdots + d_k u_{St+s-k} + e^k_{St+s} \tag{2.7}$$

with

$$e^k_{St+s} = \varepsilon_{St+s} + \sum_{j=k+1}^{\infty} d_j u_{St+s-j}. \tag{2.8}$$

Using the fact that under H_0 of (2.2), $\Delta_S x_{St+s} = u_{St+s}$ and $d^*(L) = d(L)$, substituting (2.7) into (2.5) yields the auxiliary regression equation

$$\Delta_S x_{St+s} = \pi_0 x_{0,St+s} + \pi_{S/2} x_{S/2,St+s} + \sum_{j=1}^{S^*} \left(\pi_j x_{j,St+s} + \pi_j^* x^*_{j,St+s} \right) + \sum_{j=1}^{k} d_j \Delta_S x_{St+s-j} + e^k_{St+s}, \tag{2.9}$$

again omitting the term $\pi_{S/2} x_{S/2,St+s}$ where S is odd, so that (2.9) may be estimated by ordinary least squares (OLS) over observations $St + s = k + 1, \ldots, T$. Consequently, as discussed for $S = 4$ in HEGY and for general S in Smith et al. (2009), tests for the presence (or otherwise) of a unit root at the zero and Nyquist frequencies may be obtained using conventional lower tailed regression t-tests, denoted t_0 and $t_{S/2}$, for the exclusion of $x_{0,St+s}$ and $x_{S/2,St+s}$, respectively, from (2.9). Similarly, the hypothesis of a pair of complex unit roots at the jth harmonic seasonal frequency may be tested by the lower tailed t_j and two-tailed t^*_j regression t-tests for the exclusion of $x_{j,St+s}$ and $x^*_{j,St+s}$, respectively, or by the (upper tailed) regression F-test, denoted F_j, for the exclusion of both $x_{j,St+s}$ and $x^*_{j,St+s}$ from (2.9), $j = 1, \ldots, S^*$. Ghysels, Lee, and Noh (1994) for $S = 4$ and Smith et al. (2009), again for general S, also consider the joint frequency (upper tail) regression F-tests from (2.9), namely, $F_{1\ldots\lfloor S/2 \rfloor}$ for the exclusion of $x_{S/2,St+s}$, together with $x_{j,St+s}$ and $x^*_{j,St+s}$, $j = 1, \ldots, S^*$, and $F_{0\ldots\lfloor S/2 \rfloor}$ for the exclusion of $x_{0,St+s}$, $x_{S/2,St+s}$, and $x_{j,St+s}$ and $x^*_{j,St+s}$, $j = 1, \ldots, S^*$. The former tests the null hypothesis of unit roots at all seasonal frequencies, whereas the latter tests the overall null, H_0.

As discussed in Chang and Park (2002, p. 434), under Assumption A.2 it holds that $\sum_{j=1}^{\infty} |j|^\tau \left| d_j \right| < \infty$, and consequently $\sum_{j=k+1}^{\infty} \left| d_j \right| = o\left(k^{-\tau}\right)$. Hence, as in Chang and Park (2002), the existence of the rth moment of u_{St+s}, which is implied by Assumptions A.1 and A.2, yields that

$$\mathbb{E} \left| e^k_{St+s} - \varepsilon_{St+s} \right|^r \leq \mathbb{E} \left| u_{St+s} \right|^r \left(\sum_{j=k+1}^{\infty} \left| d_j \right| \right)^r = o\left(k^{-r\tau}\right).$$

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Consequently, the approximation error in (2.7), and hence in (2.9), becomes small as k gets large. However, as in Chang and Park (2002), an assumption is still required concerning the rate permitted on the lag truncation parameter, k, as the sample size increases. Depending on the context, three possible assumptions can be made, as follows.

Assumption A.3. Let $k \to \infty$ and $k = o\left(T^{1/2}\right)$ as $T \to \infty$.

Assumption A.3′. Let $k \to \infty$ and $k = o\left(\left[T/\log T\right]^{1/2}\right)$ as $T \to \infty$.

Assumption A.3″. Let $k \to \infty$ and $k = o\left(T^{1/3}\right)$ as $T \to \infty$.

Remark 2. As noted in Chang and Park (2002), Assumptions A.2 and A.3 are considerably weaker than the corresponding assumptions used in Said and Dickey (1984), which, in particular, rule out the possibility of a logarithmic rate on k. As for the nonseasonal case considered in Chang and Park (2002), we will show in the next section that the rate conditions given in either Assumption A.3′ or Assumption A.3″ are sufficient to guarantee the consistency of the estimators of the coefficients, d_j, $j = 1, \ldots, k$, on the lagged dependent variables in (2.9). The rate imposed by the former is sufficient for consistency under homogeneous martingale difference innovations, as in Assumption A.1′, whereas the rate imposed by the latter is sufficient for possibly heterogeneous martingale difference innovations, as allowed under Assumption A.1.

3. ASYMPTOTIC RESULTS

Under H_0 of (2.2), $\{x_{St+s}\}$ of (2.1a) admits the so-called vector of seasons representation

$$X_t = X_{t-1} + U_t, \qquad t = 1, 2, \ldots, N, \tag{3.1}$$

where $X_t := [x_{St-(S-1)}, x_{St-(S-2)}, \ldots, x_{St}]'$, $t = 0, \ldots, N$, and $U_t := [u_{St-(S-1)}, u_{St-(S-2)}, \ldots, u_{St}]'$, $t = 1, \ldots, N$. As shown in Burridge and Taylor (2001), the error process U_t satisfies the vector $MA(\infty)$ representation

$$U_t = \sum_{j=0}^{\infty} \Psi_j E_{t-j}, \tag{3.2}$$

where $E_t := [\varepsilon_{St-(S-1)}, \varepsilon_{St-(S-2)}, \ldots, \varepsilon_{St}]'$, and the $S \times S$ matrices

$$\Psi_0 := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \psi_1 & 1 & 0 & 0 & \cdots & 0 \\ \psi_2 & \psi_1 & 1 & 0 & \cdots & 0 \\ \psi_3 & \psi_2 & \psi_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{S-1} & \psi_{S-2} & \psi_{S-3} & \psi_{S-4} & \cdots & 1 \end{bmatrix}$$

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and

$$\Psi_j := \begin{bmatrix} \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \psi_{jS-3} & \cdots & \psi_{jS-(S-1)} \\ \psi_{jS+1} & \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \cdots & \psi_{jS-(S-2)} \\ \psi_{jS+2} & \psi_{jS+1} & \psi_{jS} & \psi_{jS-1} & \cdots & \psi_{jS-(S-3)} \\ \psi_{jS+3} & \psi_{jS+2} & \psi_{jS+1} & \psi_{jS} & \cdots & \psi_{jS-(S-4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{jS+S-1} & \psi_{jS+S-2} & \psi_{jS+S-3} & \psi_{jS+S-4} & \cdots & \psi_{jS} \end{bmatrix}, \qquad j = 1, 2, \ldots.$$

In Lemma 1 we now provide a multivariate invariance principle for X_t of (3.1)–(3.2). This provides the basic building block for the asymptotic results given in this paper. In Lemma 2 we then establish the large-sample properties of the focal variables $x_{j,St+s}$, $j = 0, \ldots, \lfloor S/2 \rfloor$, and $x^*_{i,St+s}$, $i = 1, \ldots, S^*$, from (2.9).

LEMMA 1. *Let X_t be generated by (3.1)–(3.2). Then under Assumptions A.1 and A.2,*

$$N^{-1/2} X_{\lfloor rN \rfloor} \xrightarrow{d} \sigma\, \Psi(1)\, \mathbf{W}(r) := \mathbf{B}(r), \qquad r \in [0, 1], \tag{3.3}$$

where $\mathbf{W}(r)$ is a $S \times 1$ standard Brownian motion process and $\Psi(1) := \sum_{j=0}^{\infty} \Psi_j$. Notice, therefore, that $\mathbf{B}(r)$ is a $S \times 1$ vector Brownian motion process with variance matrix $\mathbf{\Omega} := \sigma^2 \Psi(1) \Psi(1)'$. Moreover, the right member of the convergence result in (3.3) can also be written as $\sigma / S \left[\psi(1) C_0 + \psi(-1) C_{S/2} + 2 \sum_{i=1}^{S^*} (b_i C_i + a_i C_i^*) \right] \mathbf{W}(r)$, where $C_0 := \mathsf{Circ}[1, 1, 1, \ldots, 1]$, $C_{S/2} = \mathsf{Circ}[1, -1, 1, \ldots, -1]$, and, for $\omega_i = 2\pi i / S$, $C_i = \mathsf{Circ}[\cos(0), \cos(\omega_i), \cos(2\omega_i), \ldots, \cos((S-1)\omega_i)]$ and $C_i^* = \mathsf{Circ}[\sin(0), \sin((S-1)\omega_i), \sin((S-2)\omega_i), \ldots, \sin(\omega_i)]$, $i = 1, \ldots, S^*$, are $S \times S$ circulant matrices, and where $a_i := \mathcal{I}m(\psi[\exp(\mathrm{i}\omega_i)])$ and $b_i := \mathcal{R}e(\psi[\exp(\mathrm{i}\omega_i)])$, $i = 1, \ldots, S^*$, $\mathcal{R}e(\cdot)$ and $\mathcal{I}m(\cdot)$ denoting the real and imaginary parts of their arguments, respectively.

Remark 3. Of the circulant matrices appearing in Lemma 1, both C_0 and $C_{S/2}$ have rank one, whereas C_j and C_j^*, $j = 1, \ldots, S^*$, are all of rank two. For further details on circulant matrices see, for example, Davis (1979), Osborn and Rodrigues (2002), and Smith et al. (2009).

LEMMA 2. *Let the conditions of Lemma 1 hold. Then for $X_{j,t} := [x_{j,St-(S-1)}, x_{j,St-(S-2)}, \ldots, x_{j,St}]'$, $j = 0, \ldots, \lfloor S/2 \rfloor$, and $X^*_{i,t} := [x^*_{i,St-(S-1)}, x^*_{i,St-(S-2)}, \ldots, x^*_{i,St}]'$, $i = 1, \ldots, S^*$, we have that*

$$N^{-1/2} X_{0,\lfloor rN \rfloor} \xrightarrow{d} \sigma\, \psi(1)\, C_0 \mathbf{W}(r), \tag{3.4}$$

$$N^{-1/2} X_{S/2,\lfloor rN \rfloor} \xrightarrow{d} \sigma\, \psi(-1)\, C_{S/2} \mathbf{W}(r), \tag{3.5}$$

$$N^{-1/2} X_{i,\lfloor rN \rfloor} \xrightarrow{d} \sigma \left(b_i C_i + a_i C_i^* \right) \mathbf{W}(r), \qquad i = 1, \ldots, S^*, \tag{3.6}$$

$$N^{-1/2} X^*_{i,\lfloor rN \rfloor} \xrightarrow{d} \sigma \left(b_i C_i^* - a_i C_i \right) \mathbf{W}(r), \qquad i = 1, \ldots, S^*, \tag{3.7}$$

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where the vector standard Brownian motion, $\mathbf{W}(r)$*, the constants* a_i *and* b_i*,* $i = 1, \ldots, S^*$*, and the circulant matrices,* C_j*,* $j = 0, \ldots, \lfloor S/2 \rfloor$*, and* C_i^**,* $i = 1, \ldots, S^*$*, are as defined in Lemma 1.*

Remark 4. It can be seen from the results in Lemma 2 that the right members of (3.4)–(3.7) are formed from linear combinations of the S independent standard Brownian motions that comprise $\mathbf{W}(r)$. Recalling that C_0 and $C_{S/2}$ both have rank one, whereas C_j and C_j^* for $j = 1, \ldots, S^*$ all have rank two, it is seen that each element of $C_i\mathbf{W}(r)$, $i = 0, S/2$, and of $C_j\mathbf{W}(r)$, $C_j^*\mathbf{W}(r)$, $j = 1, \ldots, S^*$, is, after rescaling, a function of a scalar standard Brownian motion and of two standard Brownian motions, respectively. Moreover, because the products $C_0C_{S/2}$ and C_iC_j and $C_iC_j^*$, $i = 0, S/2$, $j = 1, \ldots, S^*$, are all zero matrices, it is seen that these Brownian motions arising from the linear combinations that feature in the right members of (3.4) and (3.5) are independent of one another and of those that arise in the right members of (3.6) and (3.7). Moreover, by virtue of the fact that the products $C_iC_j^*$, C_iC_j, and $C_i^*C_j^*$, $i, j = 1, \ldots, S^*$, $i \neq j$, are also all zero matrices, the pairs of Brownian motions featuring in (3.6) and (3.7) are also seen to be independent across $i = 1, \ldots, S^*$.

Drawing on the results in Lemmas 1 and 2 we may now state our main result, which details the large-sample behavior of the unit root statistics from (2.9) under the general linear process assumptions adopted in this paper.

PROPOSITION 1. *Let the conditions of Lemma 1 hold. Moreover, let Assumption A.3 hold in the auxiliary HEGY regression (2.9). Then the* t_0*,* $t_{S/2}$ *(*S *even),* t_j*, and* t_j^**,* $j = 1, \ldots, S^*$*, statistics from (2.9) are such that*

$$t_i \xrightarrow{d} \frac{\int_0^1 W_i dW_i}{\sqrt{\int_0^1 W_i^2 dr}} =: \eta_i, \qquad i = 0, S/2, \tag{3.8}$$

$$t_j \xrightarrow{d} \frac{a_j\left[\int_0^1 W_j^* dW_j - \int_0^1 W_j dW_j^*\right] + b_j\left[\int_0^1 W_j dW_j + \int_0^1 W_j^* dW_j^*\right]}{\sqrt{(a_j^2 + b_j^2)\left[\int_0^1 W_j^2 dr + \int_0^1 W_j^{*2} dr\right]}},$$
$$j = 1, \ldots, S^*, \tag{3.9}$$

$$t_j^* \xrightarrow{d} \frac{a_j\left[\int_0^1 W_j^* dW_j^* + \int_0^1 W_j dW_j\right] + b_j\left[\int_0^1 W_j dW_j^* - \int_0^1 W_j^* dW_j\right]}{\sqrt{(a_j^2 + b_j^2)\left[\int_0^1 W_j^2 dr + \int_0^1 W_j^{*2} dr\right]}},$$
$$j = 1, \ldots, S^*, \tag{3.10}$$

where W_0*,* $W_{S/2}$*,* W_j^**, and* W_j*,* $j = 1, \ldots, S^*$*, are independent standard (scalar) Brownian motions and the constants* a_j *and* b_j*,* $j = 1, \ldots, S^*$*, are as defined in Lemma 1.*

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Remark 5. For the quarterly case, $S = 4$, the representations given in (3.8)–(3.10) coincide with the corresponding representations given in, for example, Theorem 2.1 of Burridge and Taylor (2001). Burridge and Taylor (2001) derive their results under conditions on the innovation process ε_{St+s} that are analogous to our Assumption A.1′ coupled with the much stronger assumption than our Assumption A.2 that the inverse of $\psi(z)$ is finite-ordered; that is, they assume that u_{St+s} in (2.1b) follows an AR(p) process with p finite. Correspondingly, their results require that the lag truncation in (2.9) is a fixed number (i.e., not a function of the sample size) no smaller than p. Other authors have made the same or stronger assumptions than those made in Burridge and Taylor (2001). We have therefore demonstrated that the limiting distributions obtained by previous authors can be maintained under much weaker assumptions on both the serial dependence in the shocks, u_{St+s}, and on the moments of the innovation process, ε_{St+s}, than in this previous literature. Moreover, our results highlight the fact that the conjecture made by a number of these authors that the lag length would need to increase at rate $o(T^{1/3})$ when MA behavior is permitted in u_{St+s} to obtain this result is in fact more stringent than is necessary.

Remark 6. In light of Remark 5 it follows immediately, as in Burridge and Taylor (2001), that the F-type statistics, F_j, $j = 1, \ldots, S^*$, $F_{1\ldots\lfloor S/2\rfloor}$, and $F_{0\ldots\lfloor S/2\rfloor}$ from (2.9) have the following limiting null distributions:

$$F_j \xrightarrow{d} \frac{\left[\int_0^1 W_j dW_j + \int_0^1 W_j^* dW_j^*\right]^2 + \left[\int_0^1 W_j dW_j^* - \int_0^1 W_j^* dW_j\right]^2}{2\left(\int_0^1 W_j^2 dr + \int_0^1 W_j^{*2} dr\right)} := \eta_j,$$
$$j = 1, \ldots, S^*, \tag{3.11}$$

$$F_{1\ldots\lfloor S/2\rfloor} \xrightarrow{d} \frac{1}{S-1}\left(\eta_2^2 + 2\sum_{j=1}^{S^*}\eta_j\right), \qquad F_{0\ldots\lfloor S/2\rfloor} \xrightarrow{d} \frac{1}{S}\left(\eta_0^2 + \eta_2^2 + 2\sum_{j=1}^{S^*}\eta_j\right), \tag{3.12}$$

omitting η_2^2 from both expressions in (3.12) when S is odd. For $S = 4$, the representations given in (3.8), (3.11), and (3.12) coincide with those given for the case where u_{St+s} is serially uncorrelated in Smith and Taylor (1998, pp. 279–280). The limiting null distributions of the t_0, $t_{S/2}$, F_k, $k = 1, \ldots, S^*$, $F_{1\ldots\lfloor S/2\rfloor}$, and $F_{0\ldots\lfloor S/2\rfloor}$ statistics from (2.9) are therefore invariant to the serial correlation nuisance parameters $\{\psi_j\}_{j=1}^{\infty}$ that characterize the serial dependence in u_{St+s}. Previously tabulated asymptotic critical values for the tests based on these statistics may therefore still be used; for example, (3.8) is the standard Dickey–Fuller distribution tabulated in Fuller (1996, Table 10.A.2, p. 642).

Remark 7. Regardless of the serial dependence in u_{St+s}, it is seen from the results in Proposition 1 and Remark 6 that the harmonic frequency statistics t_j, t_j^*, and F_j are asymptotically independent across $j = 1, \ldots, S^*$ and are

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asymptotically independent of the zero and Nyquist frequency statistics, t_0 and $t_{S/2}$, respectively, under H_0 of (2.2) by virtue of the mutual independence of W_0, W_j, W_j^*, $j = 1, \ldots, S^*$, and $W_{S/2}$. Moreover, t_0 is asymptotically independent of $t_{S/2}$ and $F_{1\ldots\lfloor S/2\rfloor}$.

Remark 8. The proof of Proposition 1 includes the result that under Assumptions A.1–A.3 the usual OLS residual variance estimator from (2.9), $\hat{\sigma}^2$ say, has the property that $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$. This result is, of course, required for using information criteria–based methods, such as, for example, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), to determine the lag length k in (2.9).

Remark 9. Under Assumptions A.1, A.2, and A.3″ it can be shown, following the same lines as in the proof of Lemma 3.4 in Chang and Park (2002), that the OLS estimator of $\Phi := [d_1, \ldots, d_k]'$ from (2.9) satisfies $\|\hat{\Phi} - \Phi\| = o_p\left(k^{-1/2}\right)$, for N large. Moreover, under Assumptions A.1′, A.2, and A.3′ it can also be shown that Lemma 3.5 in Chang and Park (2002) carries over to the seasonal case and, hence, that

$$\hat{\Phi} = \Phi + O_p\left(k\left[\frac{\log T}{T}\right]^{1/2}\right) + o_p\left(k^{-\tau}\right),$$

where τ is defined in Assumption A.2. These results show which assumptions are sufficient to establish the consistency of the estimators of $d_1, \ldots, d_k$, as required to make use of sequential algorithms, such as those of Ng and Perron (1995) and Beaulieu and Miron (1993), to determine the lag length in (2.9).

Remark 10. Thus far we have considered the case where the process $\{x_{St+s}\}$ admits no deterministic component. It is straightforward to extend the foregoing results to the case where the series contains deterministic elements. To that end, consider the following generalization of (2.1a) and (2.1b):

$$x_{St+s} = \mu_{St+s} + y_{St+s}, \qquad s = 1 - S, \ldots, 0, \qquad t = 0, 1, \ldots, N, \tag{3.13a}$$

$$\alpha(L) y_{St+s} = u_{St+s}, \qquad s = 1 - S, \ldots, 0, \qquad t = 1, 2, \ldots, N, \tag{3.13b}$$

$$u_{St+s} = \psi(L)\varepsilon_{St+s}. \tag{3.13c}$$

In (3.13a), $\mu_{St+s} := \gamma' Z_{St+s}$ where Z_{St+s} are purely deterministic. The right member of the auxiliary regression in (2.9) must now be correspondingly augmented by the addition of the deterministic component $\mu^*_{St+s} := \gamma^* Z_{St+s}$, where μ_{St+s} and μ^*_{St+s} are linear in the mapping $\gamma \mapsto \gamma^*$. Smith et al. (2009) present a typology of six cases of interest for μ_{St+s}, namely: no deterministic component (as considered earlier); nonseasonal intercept; nonseasonal intercept and nonseasonal trend; seasonal intercepts; seasonal intercepts and nonseasonal

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trend; and seasonal intercepts and seasonal trends. It is important to notice, as shown in Smith et al. (2009), that the inclusion of seasonal intercepts in (2.9) renders the resulting unit root tests similar with respect to the initial conditions, $y_{1-S}, \ldots, y_0$. Where such deterministic components are included in (2.9), the results given in this section still hold provided the standard Brownian motions, $W_0, W_1, W_1^*, \ldots, W_{S^*}, W_{S^*}^*$, and (where S is even) $W_{S/2}$, are redefined as appropriate to the deterministic scenario of interest; cf. Sections 4.1–4.5 of Smith and Taylor (1998) and Smith and Taylor (1999). As an example, if seasonal intercepts are included in (2.9) then the preceding standard Brownian motions are all replaced by their demeaned analogues, so that (e.g.) W_0 is replaced by the process $W_0 - \int_0^1 W_0(s)ds$. The same can also be shown to hold for the corresponding HEGY-type tests based on local generalized least squares (GLS) detrending as outlined in Rodrigues and Taylor (2007), again provided the standard Brownian motions are replaced by their relevant local GLS detrended analogues; see Theorem 5.1 of Rodrigues and Taylor (2007, pp. 559–560).

Remark 11. The foregoing results also extend straightforwardly to the near seasonally integrated case considered in Rodrigues and Taylor (2007, p. 551). Here the polynomial $\alpha(L)$ in (2.1a) is written in local-to-unity form using $\alpha_0 = (1 + (\nu_0/T))$, $\alpha_{S/2} = \left(1 + (\nu_{S/2}/T)\right)$ (S even), and $\alpha_k = (1 + (\nu_k/T))$ and $\beta_k = 0$, $k = 1, \ldots, S^*$, where the noncentrality parameters, $\nu_0, \nu_1, \ldots, \nu_{\lfloor S/2 \rfloor}$, are finite constants. Under this framework the results given in this section continue to hold but with the standard Brownian motions, $W_0, W_1, W_1^*, \ldots, W_{S^*}, W_{S^*}^*$, and (where S is even) $W_{S/2}$, replaced by the corresponding Ornstein–Uhlenbeck processes, as detailed in Rodrigues and Taylor (2007), so that for example $W_0(r)$ would be replaced by $J_{c,0}(r) := \int_0^r \exp(\nu_0(r - \lambda))dW_0(\lambda)$.

4. CONCLUSIONS

In this paper we have extended the results relating to the asymptotic null distribution for the ADF unit root tests given in Chang and Park (2002) to the case of augmented HEGY seasonal unit root tests. Specifically, we have shown that regression t-statistics for unit roots at the zero and Nyquist frequencies and all F-type statistics have pivotal limiting null distributions in the case where the shocks follow a general linear process driven by martingale difference innovations, but that this is not the case for the t-statistics at the harmonic seasonal frequencies whose asymptotic null distributions depend on serial correlation nuisance parameters. The (deterministic) rate at which the length of the lag augmentation polynomial used in the test regression is required to increase for these results to hold was also explored and shown to coincide with the rate derived for the nonseasonal ADF statistic by Chang and Park (2002).

The focus of this paper has been to establish the theoretical validity of the use of lag augmentation (of deterministic order) in the HEGY seasonal unit root tests, paralleling the contribution of Chang and Park (2002) for ADF tests. Empirical

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practice, however, takes matters a stage further and employs data-dependent methods to select the lag augmentation polynomial. In the accompanying working paper, del Barrio Castro, Osborn, and Taylor (2011), we use Monte Carlo methods to explore and compare the performance of a variety of such procedures: the sequential test procedures employed by Hall (1994) and Ng and Perron (1995), seasonal variants of these as suggested by Rodrigues and Taylor (2004a) and Beaulieu and Miron (1993), and methods based on information criteria including AIC, BIC, and a seasonal generalization of the modified information criteria (MAIC, MBIC) of Ng and Perron (2001). Our findings can be summarized as follows. In general, the procedure of Beaulieu and Miron (1993) applied using a 10% significance level on the lag specification tests was shown to perform well when taking both size and power considerations into account. This method avoided the severe size distortions seen in the presence of ARMA shocks by many of the procedures and at the same time was generally competitive on power with the other procedures. Deleting intermediate lags, in the spirit of the seasonal factorizations of Box and Jenkins (1976), was shown to be advantageous in terms of power, but it was also shown that such deletion could lead to increased size distortions, especially in the presence of MA disturbances. The results showed that information criteria approaches could have potentially very poor size (e.g., AIC/BIC with MAs) or poor power (MAIC/MBIC for disturbance processes with negative seasonal AR coefficients). However, of all the procedures considered, the most reliable size properties were demonstrated by the MAIC method of Ng and Perron (2001) even in cases where a near-cancellation occurs across AR unit roots with corresponding roots in the MA disturbance process.

NOTES

1. So that, for example, $S = 4$ yields the case of quarterly data, $S = 12$ monthly data, and $S = 1$ nonseasonal data.

2. As a point of notation, throughout the paper where reference is made to the Nyquist frequency this is understood only to apply where S is even. Where S is odd, elements and discussion pertaining to the Nyquist frequency should simply be deleted.

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APPENDIX

For the purposes of the Appendix, and without loss of generality, we simplify our exposition by setting the initial vector $X_0 := (x_{1-S}, \ldots, x_0)' = \mathbf{0}$ throughout.

Proof of Lemma 1. The conditions placed on $\{E_t\}$ through Assumption A.1 imply that it satisfies a multivariate invariance principle (see, e.g., Phillips and Durlauf, 1996; Davidson, 1994, pp. 454–455), such that

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$$N^{-1/2}\sum_{j=1}^{\lfloor rN\rfloor} E_j \xrightarrow{d} \sigma \mathbf{W}(r), \tag{A.1}$$

where $\mathbf{W}(r)$ is a $S \times 1$ vector standard Brownian motion. Next observe from (3.1) and (3.2) that

$$\begin{aligned} N^{-1/2}X_{\lfloor rN\rfloor} &= N^{-1/2}\sum_{j=1}^{\lfloor rN\rfloor} U_j \\ &= \Psi(1)\, N^{-1/2}\sum_{j=1}^{\lfloor rN\rfloor} E_j + o_p(1), \end{aligned}$$

where the approximation in the second equality follows from the same argument as in Boswijk and Franses (1996, p. 238). Using (A.1) and the continuous mapping theorem (CMT) the result in (3.3) then follows immediately.

To establish the remainder of the lemma, observe first that $I_S = (1/S)C_0 + (1/S)C_{S/2} + (2/S)\sum_{j=1}^{S^*} C_j$. Moreover, noting that $\Psi(1)$ is also a circulant matrix, then by the properties of products of circulant matrices it can be shown that $C_0\Psi(1) = \psi(1)C_0$, $C_{S/2}\Psi(1) = \psi(-1)C_{S/2}$, $C_j\Psi(1) = b_jC_j + a_jC_j^*$, and $C_j^*\Psi(1) = -a_jC_j + b_jC_j^*$ for $j = 1, \ldots S^*$; see, inter alia, Theorem 3.2.4 of Davis (1979), Theorem 3.1 of Gray (2006), and Smith et al. (2009) for further details. The stated result then follows immediately. ■

Proof of Lemma 2. Noting that $X_{j,t} = C_jX_t$, $j = 0, \ldots, \lfloor S/2\rfloor$, and that $X_{i,t}^* = C_i^*X_t$, $i = 1, \ldots, S^*$, the stated results follow immediately from Lemma 1, using the following identities: $C_0C_0 = SC_0$, $C_{S/2}C_{S/2} = SC_{S/2}$, $C_jC_j = (S/2)C_j$, $C_jC_j^* = (S/2)C_j^*$, and $C_j^*C_j^* = (S/2)C_j$, $j = 1, \ldots, S^*$, also recalling from Remark 4 that the remaining matrix products between C_0, $C_{S/2}$, C_j, and C_j^*, $j = 1, \ldots, S^*$, are all zero matrices and noting that multiplication between circulant matrices is commutative.

For later reference, noting that $C_0 = \mathbf{v}_0\mathbf{v}_0'$, where $\mathbf{v}_0' = [1, 1, 1, \ldots, 1]$, $C_{S/2} = \mathbf{v}_{S/2}\mathbf{v}_{S/2}'$, where $\mathbf{v}_{S/2}' = [-1, 1, -1, \ldots, 1]$, and that $C_j = \mathbf{v}_j\mathbf{v}_j'$ and $C_j^* = \mathbf{v}_j\mathbf{v}_j^{*\prime}$, where

$$\mathbf{v}_j' = \begin{bmatrix} \cos(\omega_j[1-S])\cos(\omega_j[2-S])\cdots\cos(0) \\ \sin(\omega_j[1-S])\ \sin(\omega_j[2-S])\cdots\ \sin(0) \end{bmatrix}$$

and

$$\mathbf{v}_j^{*\prime} = \begin{bmatrix} -\sin(\omega_j[1-S]) & -\sin(\omega_j[2-S]) & \cdots & -\sin(0) \\ \cos(\omega_j[1-S]) & \cos(\omega_j[2-S]) & \cdots & \cos(0) \end{bmatrix},$$

it is straightforwardly seen that

$$N^{-1/2}x_{0,S\lfloor rN\rfloor+s} = \frac{\psi(1)}{\sqrt{N}}\left(\sum_{h=1}^{S}\left[\sum_{j=1}^{\lfloor rN\rfloor}\varepsilon_{Sj-(S-h)}\right]\right) + o_p(1), \tag{A.2}$$

$$N^{-1/2}x_{S/2,S\lfloor rN\rfloor+s} = \frac{\psi(-1)(-1)^s}{\sqrt{N}}\left(\sum_{h=1}^{S}\left[(-1)^h\sum_{j=1}^{\lfloor rN\rfloor}\varepsilon_{Sj-(S-h)}\right]\right) + o_p(1) \tag{A.3}$$

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and that for $i = 1, \ldots, S^*$,

$$N^{-1/2} x_{i,S\lfloor rN\rfloor+s} = \frac{b_i}{\sqrt{N}} \left[\cos(\omega_i [s]) \left(\sum_{h=1}^{S} \cos(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) + \sin(\omega_i [s]) \left(\sum_{h=1}^{S} \sin(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) \right] + \frac{a_i}{\sqrt{N}} \left[\sin(\omega_i [s]) \left(\sum_{h=1}^{S} \cos(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) - \cos(\omega_i [s]) \left(\sum_{h=1}^{S} \sin(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) \right] + o_p(1) \tag{A.4}$$

and

$$N^{-1/2} x^*_{i,S\lfloor rN\rfloor+s} = \frac{b_i}{\sqrt{N}} \left[\sin(\omega_i [s]) \left(\sum_{h=1}^{S} \cos(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) - \cos(\omega_i [s]) \left(\sum_{h=1}^{S} \sin(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) \right] - \frac{a_i}{\sqrt{N}} \left[\cos(\omega_i [s]) \left(\sum_{h=1}^{S} \cos(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) + \sin(\omega_i [s]) \left(\sum_{h=1}^{S} \sin(\omega_i [h-S]) \left[\sum_{j=1}^{\lfloor rN\rfloor} \varepsilon_{Sj-(s-h)} \right] \right) \right] + o_p(1). \tag{A.5}$$

■

Proof of Proposition 1. First rewrite (2.9) in vector form, viz.,

$$\mathbf{y} = (\mathbf{Y}, \mathbf{Z}_k)\beta_0 + \mathbf{u},$$

where $\mathbf{y}$ is a $T \times 1$ vector with generic element $\Delta_S x_{St+s}$; $\mathbf{Y} := \left[\mathbf{y}_0 | \mathbf{y}_1 | \mathbf{y}^*_1 | \mathbf{y}_2 | \mathbf{y}^*_2 | \ldots \mathbf{y}_{S^*} | \mathbf{y}^*_{S^*} | \mathbf{y}_{S/2}\right]$ is a $T \times S$ matrix where $\mathbf{y}_i$, $i = 0, \ldots, \lfloor S/2 \rfloor$, are $T \times 1$ vectors with generic element $x_{i,St+s}$, and $\mathbf{y}^*_i$, $i = 1, \ldots, S^*$, are $T \times 1$ vectors with generic element $x^*_{i,St+s}$; $\mathbf{Z}_k := \left[\mathbf{z}_1 | \mathbf{z}_2 | \ldots | \mathbf{z}_k\right]$ is a $T \times k$ matrix with $\mathbf{z}_j$ being $T \times 1$ vectors with generic element $\Delta_S x_{St+s-j}$ for $j = 1, \ldots, k$; $\beta_0 = \left[\Pi' | \Phi'\right]' := \left[\pi_0, \pi_1, \pi^*_1, \pi_2, \pi^*_2, \ldots, \pi_{S^*}, \pi^*_{S^*}, \pi_{S/2,}, d_1, \ldots, d_k\right]'$; finally $\mathbf{u}$ is a $T \times 1$ vector with generic element e^k_{St+s}. Commensurate with the partitions of β_0, define the $(S+k) \times (S+k)$ scaling matrix, $M := \text{diag}\left[T, \ldots, T, T^{1/2}, \ldots, T^{1/2}\right]$. It is then straightforwardly seen that the OLS estimator, $\hat{\beta}_0$ say, from (2.9) is such that

$$M\left(\hat{\beta}_0 - \beta_0\right) = \begin{bmatrix} T^{-2}\mathbf{Y}'\mathbf{Y} & T^{-3/2}\mathbf{Y}'\mathbf{Z}_k \\ T^{-3/2}\mathbf{Z}'_k\mathbf{Y} & T^{-1}\mathbf{Z}'_k\mathbf{Z}_k \end{bmatrix}^{-1} \times \begin{bmatrix} T^{-1}\mathbf{Y}'\mathbf{u} \\ T^{-1/2}\mathbf{Z}'_k\mathbf{u} \end{bmatrix}. \tag{A.6}$$

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As a result of the unit root nonstationary of the elements of $\mathbf{Y}$ and the stationarity of the elements of $\mathbf{Z}_k$, it follows immediately that $T^{-3/2}\mathbf{Y}'\mathbf{Z}_k \xrightarrow{p} \mathbf{0}$, so that the inverse matrix in (A.6) is asymptotically block diagonal. As a consequence, the scaled estimators $T\left(\hat{\Pi} - \Pi\right)$ and $T^{1/2}\left(\hat{\Phi} - \Phi\right)$ are asymptotically orthogonal. Moreover, it is straightforward to show that $T^{-2}\mathbf{Y}'\mathbf{Y}$ weakly converges to a $S \times S$ diagonal matrix. We may therefore consider the large-sample behavior of the OLS estimators of π_j, $j = 0, \ldots, \lfloor S/2 \rfloor$, and π_i^*, $i = 1, \ldots, S^*$, separately.

Consequently, defining $\mathbf{Q} := I_T - \mathbf{Z}_k \left(\mathbf{Z}_k'\mathbf{Z}_k\right)^{-1} \mathbf{Z}_k'$, we can write the so-called normalized bias statistics as follows:

$$T\hat{\pi}_j = \frac{T^{-1}\mathbf{y}_j'\mathbf{Q}\mathbf{u}}{T^{-2}\mathbf{y}_j'\mathbf{Q}\mathbf{y}_j} + o_p(1)$$
$$= \frac{T^{-1}\left(\mathbf{y}_j'\mathbf{u} - \mathbf{y}_j'\mathbf{Z}_k\left(\mathbf{Z}_k'\mathbf{Z}_k\right)^{-1}\mathbf{Z}_k'\mathbf{u}\right)}{T^{-2}\left(\mathbf{y}_j'\mathbf{y}_j - \mathbf{y}_j'\mathbf{Z}_k\left(\mathbf{Z}_k'\mathbf{Z}_k\right)^{-1}\mathbf{Z}_k'\mathbf{y}_j\right)} + o_p(1), \qquad j = 0, \ldots, \lfloor S/2 \rfloor, \tag{A.7}$$

and

$$T\hat{\pi}_i^* = \frac{T^{-1}\mathbf{y}_i^{*\prime}\mathbf{Q}\mathbf{u}}{T^{-2}\mathbf{y}_i^{*\prime}\mathbf{Q}\mathbf{y}_i^*} + o_p(1)$$
$$= \frac{T^{-1}\left(\mathbf{y}_i^{*\prime}\mathbf{u} - \mathbf{y}_i^{*\prime}\mathbf{Z}_k\left(\mathbf{Z}_k'\mathbf{Z}_k\right)^{-1}\mathbf{Z}_k'\mathbf{u}\right)}{T^{-2}\left(\mathbf{y}_i^{*\prime}\mathbf{y}_i^* - \mathbf{y}_i^{*\prime}\mathbf{Z}_k\left(\mathbf{Z}_k'\mathbf{Z}_k\right)^{-1}\mathbf{Z}_k'\mathbf{y}_j^*\right)} + o_p(1), \qquad j = 1, \ldots, S^*. \tag{A.8}$$

The following lemma will allow us to simplify the expressions given previously.

LEMMA A.1. *Under the conditions of Proposition 1 the following results hold:*

(i) $\left\|\mathbf{Z}_k'\mathbf{Z}_k^{-1}\right\| = O_p\left(T^{-1}\right)$;

(ii) $\left\|\mathbf{Z}_k'\mathbf{u}\right\| = o_p\left(Tk^{-1/2}\right)$; *and*

(iii) $\left\|\mathbf{Z}_k'\mathbf{y}_j\right\| = O_p\left(Tk^{1/2}\right)$, $j = 0, \ldots, \lfloor S/2 \rfloor$, *and* $\left\|\mathbf{Z}_k'\mathbf{y}_h^*\right\| = O_p\left(Tk^{1/2}\right)$, $h = 1, \ldots, S^*$.

Proof. Parts (i) and (ii) follow immediately from parts (a) and (c), respectively, of Lemma 3.2 of Chang and Park (2002) simply by replacing the standard first differences that appear there by the seasonal differences, $\Delta_S x_{St+s}$. To establish the validity of part (iii), first use the fact that for $j = 0, \ldots, \lfloor S/2 \rfloor$, $\left\|\mathbf{Z}_k'\mathbf{y}_j\right\|^2 = \sum_{i=1}^k \left(\mathbf{z}_i'\mathbf{y}_j\right)^2$, where $\mathbf{z}_i'\mathbf{y}_j = \sum_{t=1}^N \sum_{s=1-S}^0 \Delta_S x_{St+s-i} x_{j,St+s}$, $i = 1, \ldots, k$, and, for $h = 1, \ldots, S^*$, $\|\mathbf{Z}_k'\mathbf{y}_h^*\|^2 = \sum_{i=1}^k \left(\mathbf{z}_i'\mathbf{y}_h^*\right)^2$, where $\mathbf{z}_i'\mathbf{y}_h^* = \sum_{t=1}^N \sum_{s=1-S}^0 \Delta_S x_{St+s-i} x_{h,St+s}^*$ $i = 1, \ldots, k$. Next observe that the following equalities hold for each $i = 1, \ldots, k$:

$$\sum_{t=1}^N \sum_{s=1-S}^0 x_{0,St+s} \Delta_S x_{St+s-i} = \sum_{t=1}^N \sum_{s=1-S}^0 \left(\sum_{j=1}^{(St+s)-1} \Delta_S x_{St+s-j} \right) \Delta_S x_{St+s-i}, \tag{A.9}$$

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$$\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{S/2,St+s}\,\Delta_S x_{St+s-i} = \sum_{t=1}^{N}\sum_{s=1-S}^{0}\left(\sum_{j=1}^{(St+s)-1}(-1)^j\,\Delta_S x_{St+s-j}\right)\Delta_S x_{St+s-i}, \tag{A.10}$$

and, moreover, that for each $h = 1, \ldots, S^*$, the following equalities also hold for each $i = 1, \ldots, k$:

$$\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{h,St+s}\,\Delta_S x_{St+s-i} = \sum_{t=1}^{N}\sum_{s=1-S}^{0}\left(\sum_{j=1}^{(St+s)-1}\cos(j\omega_h)\,\Delta_S x_{St+s-j}\right)\Delta_S x_{St+s-i}, \tag{A.11}$$

$$\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}\,\Delta_S x_{St+s-i} = \sum_{t=1}^{N}\sum_{s=1-S}^{0}\left(\sum_{j=1}^{(St+s)-1}-\sin(j\omega_h)\,\Delta_S x_{St+s-j}\right)\Delta_S x_{St+s-i}. \tag{A.12}$$

The stated result is then established using (A.9)–(A.12) and following along the lines of the proof of part (b) of Lemma 3.2 in Chang and Park (2002, pp. 443–444). ■

Applying the results in Lemma A.1 to (A.7) and (A.8), we then have that

$$\begin{aligned} T\hat{\pi}_j &= \frac{T^{-1}\mathbf{y}_j'\mathbf{u}}{T^{-2}\mathbf{y}_j'\mathbf{y}_j} + o_p(1) \\ &= \frac{T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}e^k_{St+s}}{T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^2_{j,St+s}} + o_p(1), \qquad j = 0, \ldots, \lfloor S/2\rfloor \end{aligned} \tag{A.13}$$

and

$$\begin{aligned} T\hat{\pi}^*_h &= \frac{T^{-1}\mathbf{y}_h^{*\prime}\mathbf{u}}{T^{-2}\mathbf{y}_h^{*\prime}\mathbf{y}^*_h} + o_p(1) \\ &= \frac{T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}e^k_{St+s}}{T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^{*2}_{h,St+s}} + o_p(1), \qquad h = 1, \ldots, S^*. \end{aligned} \tag{A.14}$$

For each of $j = 0, \ldots, \lfloor S/2\rfloor$ in (A.13) and $h = 1, \ldots, S^*$ in (A.14), rewrite the numerators of (A.13) and (A.14), respectively, as

$$\begin{aligned} &T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}e^k_{St+s} \\ &\quad = T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}\varepsilon_{St+s} + T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}\left(e^k_{St+s} - \varepsilon_{St+s}\right), \\ &T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}e^k_{St+s} \\ &\quad = T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}\varepsilon_{St+s} + T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}\left(e^k_{St+s} - \varepsilon_{St+s}\right). \end{aligned}$$

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Substituting into (A.13) and (A.14), respectively, we therefore obtain that for, $j = 0, \ldots, \lfloor S/2 \rfloor$,

$$T\hat{\pi}_j = \frac{T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}\varepsilon_{St+s} + T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}\left(e^k_{St+s} - \varepsilon_{St+s}\right)}{T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^2_{j,St+s}} + o_p(1) \tag{A.15}$$

and that, for $h = 1, \ldots, S^*$,

$$T\hat{\pi}^*_h = \frac{T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}\varepsilon_{St+s} + T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}\left(e^k_{St+s} - \varepsilon_{St+s}\right)}{T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^{*2}_{h,St+s}} + o_p(1). \tag{A.16}$$

Consider first the denominators of (A.15) and (A.16). Using the results $\Psi(1)' C_0 \Psi(1) = \psi(1)^2 C_0$, $\Psi(1)' C_{S/2} \Psi(1) = \psi(-1)^2 C_{S/2}$, and $\Psi(1)' C_j \Psi(1) = \left(a_j^2 + b_j^2\right) C_j$, for $j = 1, \ldots, S^*$, from the multivariate invariance principle in (3.3) and the CMT we obtain that

$$\begin{aligned} T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^2_{j,St+s} &= T^{-2}\sum_{t=1}^{N} S\left(X'_{t-1} C_j X_{t-1}\right) + o_p(1) \qquad j = 0, S/2 \\ &\xrightarrow{d} \frac{\sigma^2}{S}\int_0^1 \mathbf{W}(r)' \Psi(1)' C_j \Psi(1) \mathbf{W}(r)\, dr \\ &= \begin{cases} \sigma^2 \psi(1)^2 \int_0^1 \mathbf{W}^*(r)' C_0 \mathbf{W}^*(r)\, dr & j = 0 \\ \sigma^2 \psi(-1)^2 \int_0^1 \mathbf{W}^*(r)' C_{S/2} \mathbf{W}^*(r)\, dr & j = S/2, \end{cases} \end{aligned} \tag{A.17}$$

where $\mathbf{W}^*(r) := (1/\sqrt{S})\mathbf{W}(r)$ and

$$\begin{aligned} T^{-2}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^2_{j,St+s} &= T^{-2}\sum_{t=1}^{N}\left(\frac{S}{2}\right)\left(X'_{t-1} C_j X_{t-1}\right) + o_p(1) \qquad j = 1, \ldots, S^* \\ &\xrightarrow{d} \frac{\sigma^2}{S^2}\left(\frac{S}{2}\right)\int_0^1 \mathbf{W}(r)' \Psi(1)' C_j \Psi(1) \mathbf{W}(r)\, dr \\ &= \frac{\sigma^2\left(a_j^2 + b_j^2\right)}{4}\int_0^1 \mathbf{W}^\dagger(r)' C_j \mathbf{W}^\dagger(r)\, dr, \end{aligned} \tag{A.18}$$

where $\mathbf{W}^\dagger(r) := (1/\sqrt{S/2})\mathbf{W}(r)$.

Consider next the numerators of (A.15) and (A.16). In each case, for the first term it is straightforward to show that

$$T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}\varepsilon_{St+s} = T^{-1}\sum_{t=1}^{N} X'_{t-1} C_j E_t + o_p(1), \qquad j = 0, \ldots, \lfloor S/2 \rfloor,$$

$$T^{-1}\sum_{t=1}^{N}\sum_{s=1-S}^{0} x^*_{h,St+s}\varepsilon_{St+s} = T^{-1}\sum_{t=1}^{N} X'_{t-1} C^*_h E_t + o_p(1), \qquad h = 1, \ldots, S^*.$$

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Again using (3.3), applications of the CMT, and the identities $\Psi(1)' C_0 = \psi(1) C_0$, $\Psi(1)' C_{S/2} = \psi(-1) C_{S/2}$, and, for $j = 1, \ldots, S^*$, $\Psi(1)' C_j = b_j C_j - a_j C_j^*$ and $\Psi(1)' C_j^* = a_j C_j + b_j C_j^*$, the following results then obtain:

$$T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{0,St+s} \varepsilon_{St+s} \xrightarrow{d} \frac{\sigma^2}{S} \int_0^1 \mathbf{W}(r)' \Psi(1)' C_0 d\mathbf{W}(r)$$

$$= \sigma^2 \psi(1) \int_0^1 \mathbf{W}^*(r)' C_0 d\mathbf{W}^*(r), \tag{A.19}$$

$$T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{S/2,St+s} \varepsilon_{St+s} \xrightarrow{d} \frac{\sigma^2}{S} \int_0^1 \mathbf{W}(r)' \Psi(1)' C_{S/2} d\mathbf{W}(r)$$

$$= \sigma^2 \psi(-1) \int_0^1 \mathbf{W}^*(r)' C_{S/2} d\mathbf{W}^*(r), \tag{A.20}$$

and for $j = 1, \ldots, S^*$,

$$T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s} \varepsilon_{St+s} \xrightarrow{d} \frac{\sigma^2}{S} \int_0^1 \mathbf{W}(r)' \Psi(1)' C_j d\mathbf{W}(r)$$

$$= \frac{\sigma^2 b_j}{2} \int_0^1 \mathbf{W}^\dagger(r)' C_j d\mathbf{W}^\dagger(r)$$

$$- \frac{\sigma^2 a_j}{2} \int_0^1 \mathbf{W}^\dagger(r)' C_j^* d\mathbf{W}^\dagger(r), \tag{A.21}$$

$$T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s}^* \varepsilon_{St+s} \xrightarrow{d} \frac{\sigma^2}{S} \int_0^1 \mathbf{W}(r)' \Psi(1)' C_j^* d\mathbf{W}(r)$$

$$= \frac{\sigma^2 a_j}{2} \int_0^1 \mathbf{W}^\dagger(r)' C_j d\mathbf{W}^\dagger(r)$$

$$+ \frac{\sigma^2 b_j}{2} \int_0^1 \mathbf{W}^\dagger(r)' C_j^* d\mathbf{W}^\dagger(r). \tag{A.22}$$

Turning to the second term in the numerators of (A.15) and (A.16), using the results established in (A.2)–(A.5) we have that

$$\sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{0,St+s} \left(e_{St+s}^k - \varepsilon_{St+s} \right)$$

$$= \psi(1) \sum_{t=1}^{N} \sum_{s=1-S}^{0} \left(\sum_{j=1}^{(St+s)-1} \varepsilon_{St+s-j} \right) \sum_{i=k+1}^{\infty} \pi_{k,i} \varepsilon_{St+s-i}$$

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and that

$$\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{S/2,St+s}\left(e_{St+s}^{k}-\varepsilon_{St+s}\right)$$

$$=\psi\,(-1)\sum_{t=1}^{N}\sum_{s=1-S}^{0}\left(\sum_{j=1}^{(St+s)-1}(-1)^{j}\,\varepsilon_{St+s-j}\right)\sum_{i=k+1}^{\infty}\pi_{k,i}\varepsilon_{St+s-i}$$

and that for $j=1,\ldots,S^{*}$,

$$\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}\left(e_{St+s}^{k}-\varepsilon_{St+s}\right)$$

$$=b_j\sum_{t=1}^{N}\sum_{s=1-S}^{0}\left[\cos\left(\omega_j\,[s]\right)\left(\sum_{h=1}^{S}\cos\left(\omega_j\,[h-S]\right)\left[\sum_{g=1}^{t-1}\varepsilon_{Sg-(s-h)}\right]\right)\right.$$

$$\left.+\sin\left(\omega_j\,[s]\right)\left(\sum_{h=1}^{S}\sin\left(\omega_j\,[h-S]\right)\left[\sum_{g=1}^{t-1}\varepsilon_{Sg-(s-h)}\right]\right)\right]$$

$$\times\sum_{i=k+1}^{\infty}\pi_{k,i}\varepsilon_{St+s-i}$$

$$+a_j\sum_{t=1}^{N}\sum_{s=1-S}^{0}\left[\sin\left(\omega_j\,[s]\right)\left(\sum_{h=1}^{S}\cos\left(\omega_j\,[h-S]\right)\left[\sum_{g=1}^{t-1}\varepsilon_{Sg-(s-h)}\right]\right)\right.$$

$$\left.-\cos\left(\omega_j\,[s]\right)\left(\sum_{h=1}^{S}\sin\left(\omega_j\,[h-S]\right)\left[\sum_{g=1}^{t-1}\varepsilon_{Sg-(s-h)}\right]\right)\right]$$

$$\times\sum_{i=k+1}^{\infty}\pi_{k,i}\varepsilon_{St+s-i},$$

$$\sum_{t=1}^{N}\sum_{s=1-S}^{0} x_{j,St+s}^{*}\left(e_{St+s}^{k}-\varepsilon_{St+s}\right)$$

$$=b_j\sum_{t=1}^{N}\sum_{s=1-S}^{0}\left[\sin\left(\omega_j\,[s]\right)\left(\sum_{h=1}^{S}\cos\left(\omega_j\,[h-S]\right)\left[\sum_{g=1}^{t-1}\varepsilon_{Sg-(s-h)}\right]\right)\right.$$

$$\left.-\cos\left(\omega_j\,[s]\right)\left(\sum_{h=1}^{S}\sin\left(\omega_j\,[h-S]\right)\left[\sum_{g=1}^{t-1}\varepsilon_{Sg-(s-h)}\right]\right)\right]$$

$$\times\sum_{i=k+1}^{\infty}\pi_{k,i}\varepsilon_{St+s-i}$$

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$$
\begin{aligned}
&- a_j \sum_{t=1}^{N} \sum_{s=1-S}^{0} \left[\cos\left(\omega_j [s]\right) \left(\sum_{h=1}^{S} \cos\left(\omega_j [h-S]\right) \left[\sum_{g=1}^{t-1} \varepsilon_{Sg-(s-h)} \right] \right) \right. \\
&\left. + \sin\left(\omega_j [s]\right) \left(\sum_{h=1}^{S} \sin\left(\omega_j [h-S]\right) \left[\sum_{g=1}^{t-1} \varepsilon_{Sg-(s-h)} \right] \right) \right] \\
&\times \sum_{i=k+1}^{\infty} \pi_{k,i} \varepsilon_{St+s-i},
\end{aligned}
$$

where, in each case, as in Lemma 2 of Berk (1974), we have made use of the substitution $e^k_{St+s} - \varepsilon_{St+s} = \sum_{i=k+1}^{\infty} d_i u_{St+s-i} = \sum_{i=k+1}^{\infty} \pi_{k,i} \varepsilon_{St+s-i}$, where $\sum_{i=k+1}^{\infty} \pi^2_{k,i} \leq \sum_{i=k+1}^{\infty} d_i^2 = o(k^{-2\tau})$ and where τ is as defined in Assumption A.2. It is then straightforward to establish, paralleling the proof of part (a) of Lemma 3.1 in Chang and Park (2002, p. 441), that $\sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s} \left(e^k_{St+s} - \varepsilon_{St+s} \right) = o_p(T)$ for $j = 0, \ldots, \lfloor S/2 \rfloor$ and that $\sum_{t=1}^{N} \sum_{s=1-S}^{0} x^*_{j,St+s} \left(e^k_{St+s} - \varepsilon_{St+s} \right) = o_p(T)$ for $j = 1, \ldots, S^*$. As an immediate consequence of these results, we then obtain, under Assumption A.2,

$$
T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s} e^k_{St+s} = T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x_{j,St+s} \varepsilon_{St+s} + o_p(1), \quad j = 0, \ldots, \lfloor S/2 \rfloor, \tag{A.23}
$$

$$
T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x^*_{j,St+s} e^k_{St+s} = T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} x^*_{j,St+s} \varepsilon_{St+s} + o_p(1), \quad j = 1, \ldots, S^*. \tag{A.24}
$$

Combining the results in (A.17)–(A.23), and using applications of the CMT, we obtain the following results:

$$
T\hat{\pi}_0 \xrightarrow{d} \frac{\int_0^1 \mathbf{W}^*(r)' C_0 d\mathbf{W}^*(r)}{\psi(1) \int_0^1 \mathbf{W}^*(r)' C_0 \mathbf{W}^*(r) dr}, \tag{A.25}
$$

$$
T\hat{\pi}_{S/2} \xrightarrow{d} \frac{\int_0^1 \mathbf{W}^*(r)' C_{S/2} d\mathbf{W}^*(r)}{\psi(-1) \int_0^1 \mathbf{W}^*(r)' C_{S/2} \mathbf{W}^*(r) dr}, \tag{A.26}
$$

and for $j = 1, \ldots, S^*$,

$$
T\hat{\pi}_j \xrightarrow{d} \frac{b_j \int_0^1 \mathbf{W}^\dagger(r)' C_j d\mathbf{W}^\dagger(r) - a_j \int_0^1 \mathbf{W}^\dagger(r)' C_j^* d\mathbf{W}^\dagger(r)}{\frac{\left(a_j^2 + b_j^2\right)}{2} \int_0^1 \mathbf{W}^\dagger(r)' C_j \mathbf{W}^\dagger(r) dr}, \tag{A.27}
$$

$$
T\hat{\pi}_j^* \xrightarrow{d} \frac{a_j \int_0^1 \mathbf{W}^\dagger(r)' C_j d\mathbf{W}^\dagger(r) + b_j \int_0^1 \mathbf{W}^\dagger(r)' C_j^* d\mathbf{W}^\dagger(r)}{\frac{\left(a_j^2 + b_j^2\right)}{2} \int_0^1 \mathbf{W}^\dagger(r)' C_j \mathbf{W}^\dagger(r) dr}. \tag{A.28}
$$

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Next observe that the corresponding t-statistics from (2.9) can be written as

$$t_j = \hat{\sigma}^{-1} T \hat{\pi}_j \times \sqrt{T^{-2} \sum_{t=1}^{N} \sum_{s=1-S}^{0} \left(x_{j,St+s}\right)^2} + o_p(1), \qquad j = 0, \ldots, \lfloor S/2 \rfloor, \quad \textbf{(A.29)}$$

$$t_h^* = \hat{\sigma}^{-1} T \hat{\pi}_h^* \times \sqrt{T^{-2} \sum_{t=1}^{N} \sum_{s=1-S}^{0} \left(x_{h,St+s}^*\right)^2} + o_p(1), \qquad h = 1, \ldots, S^*, \quad \textbf{(A.30)}$$

where $\hat{\sigma}^2$ is the usual OLS variance estimator from (2.9); that is, $\hat{\sigma}^2 := T^{-1} \sum_{t=1}^{N} \sum_{s=1-S}^{0} (\hat{e}_{St+s}^k)^2 = T^{-1} \left(\mathbf{u}'\mathbf{u} - \mathbf{u}'\mathbf{Z}_k \left(\mathbf{Z}_k' \mathbf{Z}_k \right)^{-1} \mathbf{Z}_k' \mathbf{u} \right)$. It then follows immediately from parts (i) and (ii) of Lemma A.1 that $\hat{\sigma}^2 = T^{-1} \mathbf{u}'\mathbf{u} + o_p(1)$. Then because part (c) of Lemma 3.1 in Chang and Park (2002) also applies here, we obtain the result that $\hat{\sigma}^2 \xrightarrow{p} \sigma^2$. Substituting this result together with those in (A.25)–(A.28) into (A.29)–(A.30) and using applications of the CMT we then, after some simple manipulations, finally obtain the stated results in Proposition 1, where we have defined the independent standard Brownian motions, $W_i(r) := \mathbf{v}_i' \mathbf{W}^*(r)$, $i = 0, S/2$, $W_j(r) := \mathbf{c}_j' \mathbf{W}^\dagger(r)$, and $W_j^*(r) := \mathbf{c}_j^{*\prime} \mathbf{W}^\dagger(r)'$, where $\mathbf{c}_j'$ and $\mathbf{c}_j^{*\prime}$ are the first rows of $\mathbf{v}_j'$ and $\mathbf{v}_j^{*\prime}$, respectively for $j = 1, \ldots, S^*$. ■

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