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Linear forms in algebraic points of Abelian functions. II

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5. *Extrapolation techniques.* In this paper we continue to develop the apparatus needed for the proof of the theorem announced in (11). We retain the notation of (11) together with the assumptions made there about the field $\mathfrak{F}$ of Abelian functions. This section deals with properties of more general functions holomorphic on $\mathbf{C}^n$. When $n = 1$ the extrapolation procedure in problems of transcendence is essentially the maximum modulus principle together with the act of dividing out zeros of an analytic function. For $n > 1$, however, this approach is not possible, and some mild theory of several complex variables is required. This was first used in the context of transcendence by Bombieri and Lang in (2) and (12), and we now give a brief account of the basic constructions of their papers.

Let $f(\mathbf{z})$ be a function holomorphic on $\mathbf{C}^n$ and let $\mathscr{W}$ be the set of zeros of $f(\mathbf{z})$. For a point $\mathbf{a}$ and $r > 0$ we define $\Theta(\mathbf{a}, r)$ by

$$\pi^{n-1} r^{2n-2} \Theta(\mathbf{a}, r) = (n-1)!\, \mu(\mathbf{a} + r\mathscr{B}),$$

where μ is the measure on $\mathbf{C}^n$ induced by Hausdorff measure h on $\mathscr{W}$ of $2n-2$ real dimensions, i.e.

$$\mu(\mathscr{R}) = h(\mathscr{W} \cap \mathscr{R}).$$

This function indicates the extent to which f vanishes in $\mathbf{a} + r\mathscr{B}$. It is monotone non-decreasing in r for fixed $\mathbf{a}$, so that the limit

$$\Theta(\mathbf{a}) = \lim_{r \to 0} \Theta(\mathbf{a}, r)$$

exists. In fact this is a non-negative integer equal to the order of the zero (if any) of $f(\mathbf{z})$ at $\mathbf{a}$; for example, $\Theta(\mathbf{a}) = 0$ if $f(\mathbf{a}) \neq 0$. The limit of $\Theta(\mathbf{a}, r)$ as $r \to \infty$ also exists if f is a polynomial; we shall not use this remark but it is the basis of the proof of Lemma 13.

The next lemma serves as a Schwarz inequality in $\mathbf{C}^n$.

Lemma 18. *For $r > 0$ we have the inequality*

$$\mathfrak{M}(f, r\mathscr{B}) \leqslant \exp\left(-\tfrac{1}{4}\Theta(\mathbf{0}, r)\right) \mathfrak{M}(f, 24nr\mathscr{B}).$$

Proof. This is an immediate consequence of Proposition 4 of (2) if we make the choices

$$\tau = \tfrac{1}{2}, \quad |\mathbf{w}| \leqslant r, \quad R = 24nr.$$

We shall apply this lemma to deduce that f is small in some ball if it and its derivatives are small on a set of the type considered in the previous section. But first we need to make the following calculation, which implies the independence of distinct derivatives of a polynomial at distinct points.

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LEMMA 19. *Suppose k is a positive integer and r, s are real numbers with $r \geqslant 2, 0 < s \leqslant 1$ and let $\mathscr{S} \subseteq r\mathscr{B}$ be a set of m distinct points with separation s. For each $\boldsymbol{\zeta}$ in $\mathscr{S}$ and each differential operator ∂ of order strictly less than k let $a(\boldsymbol{\zeta}, \partial)$ be a complex number with absolute value at most A. Then there exists a polynomial P in $\mathbf{C}[\mathbf{z}]$ of degree at most mk with*

$$|P| \leqslant (r/s)^{c_1 mk} A$$

such that

$$\partial P(\boldsymbol{\zeta}) = a(\boldsymbol{\zeta}, \partial)$$

for all $\boldsymbol{\zeta}, \partial$. Here c_1 depends only on n.

Proof. Fix $\boldsymbol{\zeta}$ in $\mathscr{S}$, and let $Q = Q(\mathbf{z})$ be the polynomial of degree at most $k-1$ satisfying

$$\partial Q(\boldsymbol{\zeta}) = a(\boldsymbol{\zeta}, \partial)$$

for all ∂. For $\boldsymbol{\zeta}'$ in $\mathscr{S}$ distinct from $\boldsymbol{\zeta}$, there is a number i such that $|\zeta_i - \zeta_i'| \geqslant s/n$; put

$$R(\mathbf{z}, \boldsymbol{\zeta}') = (z_i - \zeta_i')/(\zeta_i - \zeta_i')$$

so that

$$R(\boldsymbol{\zeta}', \boldsymbol{\zeta}') = 0, \quad R(\boldsymbol{\zeta}, \boldsymbol{\zeta}') = 1.$$

Hence if

$$R = R(\mathbf{z}) = \{\prod_{\boldsymbol{\zeta}' \neq \boldsymbol{\zeta}} R(\mathbf{z}, \boldsymbol{\zeta}')\}^k$$

we have

$$R(\boldsymbol{\zeta}) = 1, \quad \partial R(\boldsymbol{\zeta}') = 0$$

whenever $\boldsymbol{\zeta}' \neq \boldsymbol{\zeta}$ and $|\partial| < k$. Now let $S = S(\mathbf{z})$ be the truncated power series of degree at most $k-1$ of $R^{-1}Q$ about $\mathbf{z} = \boldsymbol{\zeta}$; thus

$$S = R^{-1}Q + w$$

for some remainder $w = w(\mathbf{z})$, and

$$Q = T - Rw$$

where $T = RS$. It follows that

$$\partial T(\boldsymbol{\zeta}') = 0 \quad (\boldsymbol{\zeta}' \neq \boldsymbol{\zeta}),$$

$$\partial T(\boldsymbol{\zeta}) = \partial Q(\boldsymbol{\zeta}) = a(\boldsymbol{\zeta}, \partial) \quad (|\partial| < k).$$

The required polynomial P is now constructed by adding together the m polynomials T found in this way for each $\boldsymbol{\zeta}$ in $\mathscr{S}$. The asserted bound for its degree is clear, and its coefficients are not difficult to estimate using the dominance techniques of (6), p. 23.

It would be interesting to know the best possible dependence of the degree of P on m even in the case $k = 1$.

LEMMA 20. *Let f be an entire function on $\mathbf{C}^n$, and suppose $\mathscr{S} \subseteq r\mathscr{B}$ is a set of m distinct points with separation s for $r \geqslant 2$, $s \leqslant 1$. Then for a positive integer k and a real number $r' \geqslant 2r$ we have the inequality*

$$\mathfrak{M}(f, r'\mathscr{B}) \leqslant (r'/s)^{c_2 mk} \max_{|\partial| < k} \mathfrak{M}(\partial f, \mathscr{S}) + \exp(-c_2^{-1} mk(s/r)^{2n-2}) \mathfrak{M}(f, c_2 r'\mathscr{B})$$

where c_2 depends only on n.

Proof. We use the preceding lemma to find a polynomial $P(\mathbf{z})$ such that

$$\partial P(\boldsymbol{\zeta}) = \partial f(\boldsymbol{\zeta})$$

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for all $\boldsymbol{\zeta}$ in $\mathscr{S}$ and ∂ with $|\partial| < k$. We proceed to apply Lemma 18 to the function $g(\mathbf{z}) = f(\mathbf{z}) - P(\mathbf{z})$, and to do this we must estimate $\Theta(\mathbf{0}, r')$ for this function. Since the balls $\boldsymbol{\zeta} + \frac{1}{4}s\mathscr{B}$ are disjoint and contained in $2r\mathscr{B}$, we have

$$\Theta(\mathbf{0}, r') \geqslant \Theta(\mathbf{0}, 2r) \geqslant c_3^{-1} r^{-2n+2} \mu(2r\mathscr{B})$$

and

$$\mu(2r\mathscr{B}) \geqslant \sum_{\boldsymbol{\zeta}} \mu(\boldsymbol{\zeta} + \tfrac{1}{4}s\mathscr{B}) \geqslant c_4^{-1} s^{2n-2} \sum_{\boldsymbol{\zeta}} \Theta(\boldsymbol{\zeta}, \tfrac{1}{4}s).$$

But by the basic properties of Θ

$$\Theta(\boldsymbol{\zeta}, \tfrac{1}{4}s) \geqslant \Theta(\boldsymbol{\zeta}) \geqslant k$$

because $g(\mathbf{z})$ has a zero of order at least k at $\boldsymbol{\zeta}$. It follows that

$$\Theta(\mathbf{0}, r') > c_5^{-1} mk(s/r)^{2n-2}$$

whence Lemma 18 gives

$$\mathfrak{M}(g, r'\mathscr{B}) \leqslant \exp(-c_6^{-1} mk(s/r)^{2n-2})\, \mathfrak{M}(g, c_6 r'\mathscr{B}).$$

Finally the estimates of Lemma 19 take us back to f by means of the inequality

$$\mathfrak{M}(f-g, r''\mathscr{B}) = \mathfrak{M}(P, r''\mathscr{B}) \leqslant (r''/s)^{c_7 mk} \max_{|\partial|<k} \mathfrak{M}(\partial f, \mathscr{S})$$

valid for any $r'' \geqslant r$.

This result is the basic extrapolation formula that replaces the well-known Hermite identity in the case $n = 1$. In applications the term $\mathfrak{M}(\partial f, \mathscr{S})$ will be very small indeed, and this means that the precision of Lemma 19 was not very critical. More delicately balanced is the second term on the right-hand side; it is just enough to permit the inductive extrapolation associated with Baker's method.

We shall also use this lemma with $k = 1$ for quite a different purpose. This is to obtain an analogue of Lemma 14 for the function $\boldsymbol{f} = \mathbf{A}$, which is not entire. The first step in this direction is the following, which asserts, roughly speaking, that the zero divisor of an entire function is not dense in balls however small.

LEMMA 21. *Suppose $f(\mathbf{z})$ is a non-zero entire function and $\mathscr{R}$ is a non-empty closed ball of $\mathbf{C}^n$. Then there is a constant c_8 depending only on f and $\mathscr{R}$ such that any subball of $\mathscr{R}$ of radius $s < c_8^{-1}$ contains a subball of radius s^2 on which*

$$|f(\mathbf{z})| > \exp(-s^{-3n}).$$

Proof. Note that c_8 may depend on f; this possibility was forbidden throughout section 4. If the lemma is false, there is a ball $\mathbf{a} + s\mathscr{B}$ with $|\mathbf{a}| < c_9$, $s \leqslant 1$ such that any subball of radius s^2 contains a point $\boldsymbol{\zeta}$ where

$$|f(\boldsymbol{\zeta})| \leqslant \exp(-s^{-3n}).$$

Let $g(\mathbf{z}) = f(\mathbf{a} + s\mathbf{z})$; then there clearly exists a set $\mathscr{S}$ containing at least $c_{10}^{-1} s^{-2n}$ points of $\mathscr{B}$ with separation at least $c_{10}^{-1} s$ at which

$$|g(\mathbf{z})| \leqslant \exp(-s^{-3n}).$$

Suppose the ball $\mathbf{a} + p\mathscr{B}$ contains $\mathscr{R}$ for some p with $1 \leqslant p \leqslant c_{11}$; if we apply Lemma 20 with $r = 2$, $k = 1$, $r' = 4ps^{-1}$ we find that

$$\mathfrak{M}(g, r'\mathscr{B}) \leqslant \exp(c_{12}^{-1} s^{-2n} \log s^{-1}) \exp(-s^{-3n}) + \exp(-c_{12}^{-1} s^{-2n} s^{2n-2})\, \mathfrak{M}(g, c_{12} r'\mathscr{B}).$$

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But by our choice of p we have

$$\mathfrak{M}(g, c_{12}r'\mathscr{B}) \leqslant \mathfrak{M}(f, c_{13}\mathscr{B}) \leqslant c_{14}$$
$$\mathfrak{M}(g, r'\mathscr{B}) \geqslant \mathfrak{M}(f, \mathscr{R}) \geqslant c_{15}^{-1}$$

and so if s is sufficiently small we have a contradiction.

LEMMA 22. *Let $\phi = P(\mathbf{A})$ where P is a polynomial of degree at most l sufficiently large with respect to $\mathfrak{F}$. Suppose $\mathscr{R}$ is a closed ball of positive radius $s \leqslant l^{-1}$ on which $\mathbf{A}$ is regular. Then if $\nu = 2^6 n$ we have*

$$|P| \leqslant \exp(s^{-\nu})\, \mathfrak{M}(\phi, \mathscr{R}).$$

Proof. Suppose $\theta(\mathbf{z})$, $\theta(\mathbf{z})\,\mathbf{A}(\mathbf{z})$ are entire functions and assume l sufficiently large. Using the periodicity of $\mathbf{A}$ we may take $\mathscr{R}$ to lie in a bounded region. By the preceding lemma there is a subball $\mathscr{R}' \subseteq \mathscr{R}$ with radius s^2 on which

$$|\theta(\mathbf{z})| \geqslant \exp(-s^{-3n})$$

and consequently

$$|\mathbf{A}(\mathbf{z})| \leqslant \exp(s^{-4n}).$$

Similarly since $\theta^{2n}J$ is entire there is a subball $\mathscr{R}'' \subseteq \mathscr{R}'$ of radius s^4 on which

$$|J(\mathbf{z})| \geqslant \exp(-s^{-7n}).$$

If $\boldsymbol{\zeta}$ is the centre of $\mathscr{R}''$, we put

$$\mathbf{g}(\mathbf{z}) = \exp(-s^{-4n})\, \mathbf{A}(\boldsymbol{\zeta} + s^4\mathbf{z})$$

so that for $\mathbf{z}$ in $\mathscr{B}$ we have

$$|\mathbf{g}(\mathbf{z})| \leqslant 1, \quad |J_g(\mathbf{z})| \geqslant \exp(-s^{-8n}).$$

Also $\psi(\mathbf{z}) = \phi(\boldsymbol{\zeta} + s^4\mathbf{z})$ is a polynomial Q in the function $\mathbf{g}(\mathbf{z})$ of degree at most

$$l < s^{-16n} = m,$$

and we have

$$\mathfrak{M}(\psi, \mathscr{B}_m) \leqslant \mathfrak{M}(\psi, \mathscr{B}) \leqslant \mathfrak{M}(\phi, \mathscr{R}'') \leqslant \mathfrak{M}(\phi, \mathscr{R}).$$

Hence we may apply Lemma 14 to $\psi(\mathbf{z}) = Q(\mathbf{g})$, and we conclude that

$$|Q| \leqslant e^{m^2}\mathfrak{M}(\phi, \mathscr{R})$$

whence the estimate for the coefficients of P follows immediately.

6. *Preliminary modifications.* Before introducing the auxiliary function we shall simplify the basic linear form in (3) in a number of ways. Among other things we wish to exclude the mythical algebraic points mentioned in connexion with Lemma 11. In fact we assert that it suffices to prove the theorem when the linear form in (3) satisfies the following conditions:

(i) $C = 1$,
(ii) H sufficiently large with respect to $\epsilon, \mathbf{u}_1, \ldots, \mathbf{u}_m$ and $\mathfrak{F}$,
(iii) $\mathbf{u}_1, \ldots, \mathbf{u}_m$ non-mythical,
(iv) $\mathbf{u}_1, \ldots, \mathbf{u}_m$ congruent to points of $\frac{1}{2}\mathscr{B}$ modulo the periods of $\mathfrak{F}$,
(v) $\sigma_1, \ldots, \sigma_{m-1}$ non-zero,
(vi) $\sigma_m = -1$,
(vii) σ_1 bottom-heavy of height at least $H^{1/\nu}$, where $\nu > 0$ depends only on n.

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The first two of these are usual in this type of work and obviously involve no loss of generality. Next, any non-zero period $\boldsymbol{\omega}$ is non-mythical; thus for any algebraic points $\mathbf{u}_1, \ldots, \mathbf{u}_m$ we can find an integer k with $0 \leqslant k \leqslant mn$ such that $\mathbf{u}_i' = \mathbf{u}_i + k\boldsymbol{\omega}$ are non-mythical for $1 \leqslant i \leqslant m$. The theorem for $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is a consequence of the extended version of the theorem for $\boldsymbol{\omega}, \mathbf{u}_1', \ldots, \mathbf{u}_m'$ mentioned in section 1. So we may assume (iii).

As for (iv), given any $s > 0$ we can find from Lemma 9 a non-zero σ in $\mathsf{E}(c_1)$ such that

$$\|\sigma\mathbf{u}_i\boldsymbol{\omega}^{-1}\| < s \quad (1 \leqslant i \leqslant m)$$

where c_1 depends on s. We shall actually choose s so small that the resulting $\sigma\mathbf{u}_i$ are congruent to points of $\mathscr{B}/8n$. They are algebraic points by Lemma 1 and obviously the theorem for $\sigma\mathbf{u}_1, \ldots, \sigma\mathbf{u}_m$ implies the theorem for $\mathbf{u}_1, \ldots, \mathbf{u}_m$. This sharper form of (iv) will be useful in ensuring (vii).

Condition (v) is trivial; (vi) comes from routine height calculations. Finally for (vii) we can assume σ_1 has height exactly H and is not bottom-heavy, otherwise (vii) holds for $\nu = 1$. We have for any integer r

$$-\sigma_1'(\sigma_1\mathbf{u}_1 + \ldots + \sigma_{m-1}\mathbf{u}_{m-1} - \mathbf{u}_m) = \sigma_1'\mathbf{u}_1' + \ldots + \sigma_{m-1}'\mathbf{u}_{m-1}' - \mathbf{u}_m'$$

where
$$\mathbf{u}_1' = r\mathbf{u}_1 + \mathbf{u}_m, \quad \mathbf{u}_m' = \mathbf{u}_1, \quad \mathbf{u}_i' = \mathbf{u}_i \quad (1 < i < m)$$

and
$$\sigma_1' = (\sigma_1 + r)^{-1}, \quad \sigma_i' = -\sigma_i(\sigma_1 + r)^{-1} \quad (1 < i < m).$$

We choose r with the aid of the following simple lemma.

LEMMA 23. *If σ is a number of F with sufficiently large height that is not bottom-heavy, then at least $n+1$ of the numbers $(\sigma + r)^{-1}$ for $0 \leqslant r \leqslant 3n$ are bottom-heavy.*

Proof. If σ has height H it has a denominator $a \leqslant H^{(8n)^{-1}} \leqslant H^{(7n)^{-1}}$. Assume for the moment that the numbers $(\sigma + r_i)^{-1}$ also have denominators $a_i \leqslant H^{(7n)^{-1}}$ for $2n+1$ distinct integers r_i between 0 and $3n$. There follow the estimates

$$|\mathrm{norm}\,(\sigma + r_i)| \leqslant a_i^{2n} \leqslant H^{\frac{2}{7}}.$$

Let $f(X)$ be the characteristic polynomial of σ multiplied by a^{2n}; it is then defined over Z and we have

$$|f(-r_i)| = a^{2n}|\mathrm{norm}\,(\sigma + r_i)| \leqslant H^{\frac{4}{7}}.$$

From the Lagrange interpolation formula the absolute values of the coefficients of $f(X)$ cannot exceed $H^{\frac{5}{7}}$, and from the lemma on p. 14 of (4) this implies that the height of σ is at most $H^{\frac{5}{7}}$, a contradiction. Thus at least $n+1$ of the numbers of the present lemma have no denominator less than $H^{(7n)^{-1}}$, and since their heights are at most $c_2 H < H^{\frac{5}{7}}$ these numbers are all bottom-heavy.

If $0 \leqslant r \leqslant 3n$ the above points $\mathbf{u}_1', \ldots, \mathbf{u}_m'$ are congruent to points of $\frac{1}{2}\mathscr{B}$ from our stronger version of (iv), and hence they are algebraic points by Lemma 1. All except possibly $\mathbf{u}_1'$ are non-mythical from (iii); but the exception can be genuine for at most n values of r. Hence for some r the new linear form satisfies (iii)–(vi) and σ_1' is bottom-heavy. But now the maximum H' of the heights of $\sigma_1', \ldots, \sigma_{m-1}'$ does not exceed H^{c_3} while since $\sigma_1 = (\sigma_1')^{-1} - r$, the height of σ_1' exceeds $c_4^{-1}H > H^{\frac{1}{2}}$, and since c_3, c_4 here depend only on n this completes the verification of (vii).

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7. *The auxiliary function.* We have just seen that if the theorem is false for $m+1$ algebraic points then for some $\epsilon > 0$ there exist non-mythical algebraic points $\mathbf{u}_0, \dots, \mathbf{u}_m$ congruent to points of $\frac{1}{2}\mathscr{B}$ such that

$$|\boldsymbol{\sigma}_1\mathbf{u}_1 + \dots + \boldsymbol{\sigma}_m\mathbf{u}_m - \mathbf{u}_0| < e^{-H^\epsilon}, \tag{21}$$

where $\sigma_1, \dots, \sigma_m$ are nonzero numbers of F with heights at most H and σ_1 is bottom-heavy with height at least $H^{1/\nu}$. Furthermore we can assume that H is sufficiently large with respect to ϵ, $\mathbf{u}_0, \dots, \mathbf{u}_m$ and $\mathfrak{F}$.

We introduce m independent complex vector variables $\mathbf{z}_1, \dots, \mathbf{z}_m$ and an additional dependent variable $\mathbf{z}_0$ given by

$$\mathbf{u}_0\mathbf{z}_0 = \boldsymbol{\sigma}_1\mathbf{u}_1\mathbf{z}_1 + \dots + \boldsymbol{\sigma}_m\mathbf{u}_m\mathbf{z}_m.$$

We write

$$k = (\log H)^{8n}$$

and assume $k > c$, where $c, c_1, \dots$ will be positive constants depending only on ϵ, $\mathbf{u}_0, \dots, \mathbf{u}_m$ and $\mathfrak{F}$. No matter how large c is, an eventual contradiction will establish the theorem for $m+1$ points.

We define

$$L = L_0 = L_1 = \dots = L_{m-1} = k^{1-(8mn)^{-1}}$$

and

$$L_m = k^{(2n)^{-1}}$$

and for as yet undetermined coefficients

$$p = p(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_m)$$

we set

$$\Phi(\mathbf{z}_1, \dots, \mathbf{z}_m) = \Sigma p A(\mathbf{u}_0\mathbf{z}_0, \boldsymbol{\lambda}_0) \dots A(\mathbf{u}_m\mathbf{z}_m, \boldsymbol{\lambda}_m)$$

where the summation is over all index vectors $\boldsymbol{\lambda}_i$ $(0 \leqslant i \leqslant m)$ of non-negative integers with $|\boldsymbol{\lambda}_i| \leqslant L_i$. It will be useful to consider an associated function $\Xi_\sigma(\mathbf{x}_1, \dots, \mathbf{x}_m) = \Xi_\sigma$ defined for σ in E by

$$\Xi_\sigma = \Sigma p A(\boldsymbol{\sigma}\mathbf{x}_0, \boldsymbol{\lambda}_0) \dots A(\boldsymbol{\sigma}\mathbf{x}_m, \boldsymbol{\lambda}_m)$$

where $\mathbf{x}_0$ is given in terms of the independent vector variables $\mathbf{x}_1, \dots, \mathbf{x}_m$ by

$$\mathbf{x}_0 = \boldsymbol{\sigma}_1(\mathbf{x}_1 - \mathbf{u}_1) + \dots + \boldsymbol{\sigma}_m(\mathbf{x}_m - \mathbf{u}_m) + \mathbf{u}_0.$$

Now let ∂ be a differential operator in $\mathbf{z}_1, \dots, \mathbf{z}_m$ and denote also by ∂ the corresponding operator in $\mathbf{x}_1, \dots, \mathbf{x}_m$. Write $u(\partial)$, $\sigma(\partial)$ for the multiplicative factors in $\partial\Phi$, $\partial\Xi_\sigma$ resulting from the coefficients $\mathbf{u}_1, \dots, \mathbf{u}_m$ and $\boldsymbol{\sigma}$ in front of the variables. The differentiation of Ξ_σ imitates that of Φ, and when we make the substitutions

$$\mathbf{x}_i = \mathbf{u}_i, \quad \mathbf{z}_i = \boldsymbol{\sigma} \quad (1 \leqslant i \leqslant m)$$

we shall have $\mathbf{x}_0 = \mathbf{u}_0$, while $\mathbf{z}_0$ will differ slightly from $\boldsymbol{\sigma}$ because of (21). Thus we should expect $(u(\partial))^{-1}\partial\Phi(\boldsymbol{\sigma}, \dots, \boldsymbol{\sigma})$ to approximate to

$$(\sigma(\partial))^{-1}\partial\Xi_\sigma(\mathbf{u}_1, \dots, \mathbf{u}_m) = E(\sigma, \partial)$$

provided the parameters involved are not too large. Furthermore, if the p's are reasonably small rational integers, it is clear from the algebraic nature of differentiation on $\mathfrak{F}$ that $E(\sigma, \partial)$ will be an algebraic number (subject to possible restrictions on σ). These

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observations underlie the sequence of lemmas we shall now prove. We denote the set of differential operators ∂ of order $|\partial| < k$ by $\mathfrak{D}(k)$, and we recall the subset $\hat{\mathsf{E}}$ of E defined just before Lemma 12 in terms of the algebraic points $\mathbf{u}_0, \ldots, \mathbf{u}_m$.

LEMMA 24. *Suppose ∂ is in $\mathfrak{D}(k)$ and σ is in $\hat{\mathsf{E}}(\exp(k^{(16n)^{-1}}))$. Then we have*

$$|\partial\Phi(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma}) - u(\partial)\, E(\sigma, \partial)| \leqslant \exp(-\tfrac{3}{4}H^e) \max|p|.$$

Proof. Let

$$F = F(\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_m)$$

be the monomial whose coefficient is $p(\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_m)$ in the auxiliary function, and let G be the corresponding monomial in Ξ_σ. From Lemma 3 it is easy to see that $(u(\partial))^{-1}\,\partial F$ can be written as a polynomial in the functions

$$\mathbf{A}(\mathbf{u}_i\mathbf{z}_i), \quad B(\mathbf{u}_i\mathbf{z}_i), \quad (C(\mathbf{u}_i\mathbf{z}_i))^{-1} \quad (0 \leqslant i \leqslant m). \tag{22}$$

If $|\partial| < k$ this polynomial has degree at most $c_1(k+L) < c_2 k$ and its coefficients are in $\mathfrak{k}$ with sizes at most

$$c_3^{k+L}(c_3 L)^k\, k! < k^{c_4 k}.$$

Since σ is in $\hat{\mathsf{E}}$ the points $\boldsymbol{\sigma}\mathbf{u}_i$ are congruent to points of $\frac{1}{2}\mathscr{B}$, and Lemma 8 shows that the absolute values of the functions (22) at the point $\mathbf{z}_i = \boldsymbol{\sigma}$ $(1 \leqslant i \leqslant m)$ are at most $S^{c_5 S^2}$ where

$$S = \exp(k^{(16n)^{-1}}).$$

Hence at this point $(u(\partial))^{-1}\,\partial F$ may be expressed as a polynomial of degree $c_2 k$ in

$$\mathbf{A}(\mathbf{u}_0\mathbf{z}_0), \quad B(\mathbf{u}_0\mathbf{z}_0), \quad (C(\mathbf{u}_0\mathbf{z}_0))^{-1} \tag{23}$$

with coefficients of absolute values at most $S^{c_6 k S^2}$.

Now by the remarks above we see that $(\sigma(\partial))^{-1}\,\partial G(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ is the same polynomial in the numbers (when $\mathbf{x}_i = \mathbf{u}_i$)

$$\mathbf{A}(\boldsymbol{\sigma}\mathbf{x}_0), \quad B(\boldsymbol{\sigma}\mathbf{x}_0), \quad (C(\boldsymbol{\sigma}\mathbf{x}_0))^{-1}. \tag{24}$$

We also have

$$\mathbf{u}_0\mathbf{z}_0 = \boldsymbol{\sigma}(\boldsymbol{\sigma}_1\mathbf{u}_1 + \ldots + \boldsymbol{\sigma}_m\mathbf{u}_m)$$

whereas $\boldsymbol{\sigma}\mathbf{x}_0 = \boldsymbol{\sigma}\mathbf{u}_0$, so that from (21)

$$|\mathbf{u}_0\mathbf{z}_0 - \boldsymbol{\sigma}\mathbf{x}_0| < c_7 S e^{-H^e}.$$

Thus both $\mathbf{u}_0\mathbf{z}_0, \boldsymbol{\sigma}\mathbf{x}_0$ lie in $\frac{3}{4}\mathscr{B}$, and standard estimates for holomorphic functions show that

$$|f(\mathbf{u}_0\mathbf{z}_0) - f(\boldsymbol{\sigma}\mathbf{x}_0)| < c_8 S e^{-H^e},$$

where f is A_i, B, or C^{-1}. The identity

$$X_1X_2\ldots X_N - Y_1Y_2\ldots Y_N = \sum_{i=1}^{N} (X_i - Y_i)\, Y_1\ldots Y_{i-1}X_{i+1}\ldots X_N$$

can be used to estimate the difference between a monomial of degree $c_2 k$ in (23) and the same monomial in (24). We find the upper bound $c_9^k S e^{-H^e}$, and by putting such monomials together we conclude that

$$|(u(\partial))^{-1}\,\partial F - (\sigma(\partial))^{-1}\,\partial G| < S^{c_{10} k S^2} e^{-H^e}$$

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and since

$$H = \exp(k^{(8n)^{-1}})$$

the estimate of the lemma is apparent.

In the next lemma we shall determine the coefficients of Φ so that Φ has many approximate zeros of high multiplicity. We write

$$h = \log H = k^{(8n)^{-1}}.$$

LEMMA 25. *There are rational integers $p(\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_m)$, not all zero, with absolute values at most $h^{c_{11}Lh^2}$, such that for all ∂ in $\mathfrak{D}(k)$ and σ in $\hat{\mathsf{E}}(h)$ we have*

$$|\partial\Phi(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma})| \leqslant \exp(-\tfrac{1}{2}H^{\epsilon}).$$

Proof. We shall choose the coefficients so that $E(\sigma, \partial) = 0$ for the desired ranges of σ and ∂. During the proof of the preceding lemma we saw that $(\sigma(\partial))^{-1}\partial G$ is a polynomial in the functions

$$\mathbf{A}(\boldsymbol{\sigma}\mathbf{x}_i), \quad B(\boldsymbol{\sigma}\mathbf{x}_i), \quad (C(\boldsymbol{\sigma}\mathbf{x}_i))^{-1} \quad (0 \leqslant i \leqslant m)$$

of degree at most $c_2 k$ with coefficients in I of sizes at most $k^{c_4 k}$. Thus at $\mathbf{x}_i = \mathbf{u}_i$ the functions $(\sigma(\partial))^{-1}\partial G$ take algebraic values and Lemma 8 shows that for fixed σ in $\hat{\mathsf{E}}(h)$ and ∂ in $\mathfrak{D}(k)$ their sizes and a common denominator do not exceed

$$k^{c_{12}k}(h^{c_{12}h^2})^{c_{12}k} < h^{c_{13}kh^2}.$$

Furthermore they lie in the field L obtained by adjoining $\mathbf{A}(\mathbf{u}_i)$, $B(\mathbf{u}_i)$ to K for $0 \leqslant i \leqslant m$. Thus the equations $E(\sigma, \partial) = 0$ become linear equations in the numbers p, and after multiplying by suitable denominators we may take their coefficients to be algebraic integers of L with sizes at most $h^{c_{14}kh^2}$. The number of equations does not exceed

$$c_{15}k^{mn}h^{2n} < c_{15}k^{mn+\frac{1}{4}}$$

while there are at least

$$c_{16}^{-1}(L_0 L_1 \ldots L_m)^n = c_{16}^{-1}k^{mn+\frac{3}{8}}$$

unknowns. Hence the exponent appearing in Hilfssatz 30 on p. 141 of (9) does not exceed $w = c_{17}k^{-\frac{1}{8}}$, and so there is a non-trivial solution in rational integers p with absolute values at most

$$(k^{c_{17}}h^{c_{14}kh^2})^w < h^{c_{11}Lh^2}$$

since $L \geqslant k^{\frac{7}{8}}$. The inequalities of the present lemma are now immediate consequences of Lemma 24.

With the coefficients of Φ now fixed, we may revise Lemma 24 to yield the following:

LEMMA 26. *For ∂ in $\mathfrak{D}(k)$ and σ in $\hat{\mathsf{E}}(S)$ we have*

$$|\partial\Phi(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma}) - u(\partial)E(\sigma, \partial)| < \exp(-\tfrac{1}{2}H^{\epsilon}).$$

Proof. Clear.

The next two lemmas will contain essential estimates for the inductive extrapolation procedure. The first is an analogue of an inequality used in the treatment (7) of

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elliptic functions. Suppose ∂ is in $\mathfrak{D}(k)$ and σ is in $\hat{\mathbf{E}}(r)$ with $r \geqslant h$; then if $E(\sigma, \partial)$ does not vanish it is not difficult to deduce the lower bound $r^{-c_{18}kr^2}$ for its absolute value. What is needed for the extrapolation is a sharpening of this estimate with $k^{1-\delta}$ for some fixed $\delta > 0$ replacing k in the exponent. The higher dimensional version is rather delicate, and the usual partial ordering of differential operators turns out to be inadequate. We introduce a total ordering as follows. Certainly we take $\partial_1 \leqslant \partial_2$ if $|\partial_1| \leqslant |\partial_2|$; and we refine this by adopting a lexicographic ordering among operators of the same order. Thus if $\partial_1 \leqslant \partial_2$ we have $\partial\partial_1 \leqslant \partial\partial_2$, and the inequalities $\partial_1 \leqslant \partial_2$, $\partial_2 \leqslant \partial_1$ together imply $\partial_1 = \partial_2$.

LEMMA 27. *Suppose $r \geqslant h$, σ in $\hat{\mathbf{E}}(r)$, ∂ in $\mathfrak{D}(k)$ are such that $E(\sigma, \partial') = 0$ for all $\partial' < \partial$. Then either $E(\sigma, \partial) = 0$ or*

$$|E(\sigma, \partial)| > r^{-c_{19}Lr^2}.$$

Proof. We use the formula of Lemma 7 to obtain an alternative expression for Ξ_σ by finding a denominator for Ξ_σ in the following sense. There is a polynomial H in $\mathsf{I}[\mathbf{X}]$ of degree at most $c_{20}Lr^2$ with coefficients of size at most $r^{c_{21}Lr^2}$, such that if

$$D(\mathbf{x}) = H(\mathbf{A}(\mathbf{x}))$$

then the function

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_m) = D(\mathbf{x}_0)\, D(\mathbf{x}_1) \ldots D(\mathbf{x}_m)\, \Xi_\sigma$$

has the following property. It may be written as a polynomial in

$$\mathbf{A}(\mathbf{x}_i), \quad B(\mathbf{x}_i) \quad (0 \leqslant i \leqslant m)$$

of degree at most $c_{22}Lr^2$ and coefficients in I with sizes at most $r^{c_{23}Lr^2}$. In fact H may be taken as an L-th power, but we shall not use this fact. For $0 \leqslant i \leqslant m$ we let ∂_i be the least operator in the total ordering such that $\partial_i D(\mathbf{x}_i) \neq 0$ at $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$. In particular ∂_i is an operator in $\mathbf{x}_i$ for $1 \leqslant i \leqslant m$, and because $J(\mathbf{u}_i) \neq 0$ the order of ∂_i cannot exceed the degree of H. Since, e.g. $\sigma_1 \neq 0$, the same bound holds for the order of ∂_0.

Using the total ordering it now follows that (recall the minimality of ∂)

$$\partial\partial_0 \ldots \partial_m \Psi = \partial_0 D(\mathbf{x}_0) \ldots \partial_m D(\mathbf{x}_m)\, \partial\Xi_\sigma$$

at $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$. From Lemma 3 the numbers $\partial\partial_0 \ldots \partial_m \Psi$, $\partial_i D(\mathbf{x}_i)$ $(0 \leqslant i \leqslant m)$ at this point are algebraic numbers of L with sizes and denominators at most $r^{c_{24}Lr^2}$ (note that $Lr^2 \geqslant Lh^2 > k$), and now the conditional lower bound for $\partial\Xi_\sigma(\mathbf{u}_1, \ldots, \mathbf{u}_m)$ and thus for $E(\sigma, \partial)$ are immediate consequences of the usual norm inequality.

The next lemma provides a denominator for Φ in a rather different sense. We set

$$\Theta = \Theta(\mathbf{z}_1, \ldots, \mathbf{z}_m) = (\theta(\mathbf{u}_0 \mathbf{z}_0))^L \ldots (\theta(\mathbf{u}_m \mathbf{z}_m))^L$$

so that $\Omega = \Theta\Phi$ is entire.

LEMMA 28. *If $r \geqslant h$ and $|\mathbf{z}| \leqslant r$ we have*

$$|\partial\Theta(\mathbf{z}, \ldots, \mathbf{z})| < r^{c_{25}Lr^2},$$
$$|\partial\Omega(\mathbf{z}, \ldots, \mathbf{z})| < r^{c_{25}Lr^2}$$

for any ∂ in $\mathfrak{D}(k)$. Furthermore if $\mathbf{z} = \boldsymbol{\sigma}$ for σ in $\hat{\mathbf{E}}$ we have

$$|\Theta(\mathbf{z}, \ldots, \mathbf{z})| > r^{-c_{26}Lr^2}.$$

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Proof. Suppose $|\mathbf{z}| \leqslant r$ and let $\mathbf{z}_1, \ldots, \mathbf{z}_m$ be points with $|\mathbf{z}_i - \mathbf{z}| \leqslant H^{-2}$. It follows that

$$|\mathbf{z}_0 - \mathbf{z}| \leqslant c_{27} H^{-2} \sum_{i=1}^{m} |\boldsymbol{\sigma}_i| < c_{28} H^{-1},$$

and since θ is of growth order two, we see that

$$|\Theta(\mathbf{z}_1, \ldots, \mathbf{z}_m)| < c_{29}^{Lr^2}.$$

To estimate the derivatives we use (13) in $\mathbf{C}^{mn}$ with a ball of radius H^{-3}; we obtain

$$|\partial\Theta(\mathbf{z}, \ldots, \mathbf{z})| \leqslant k! H^{4k} c_{29}^{Lr^2} < r^{c_{25} Lr^2}.$$

The upper bounds for Ω are found similarly by writing it as a polynomial in the entire functions $\theta(\mathbf{u}_i \mathbf{z}_i)$, $\theta(\mathbf{u}_i \mathbf{z}_i)\, \mathbf{A}(\mathbf{u}_i \mathbf{z}_i)$ for $0 \leqslant i \leqslant m$. Finally if $\mathbf{z} = \boldsymbol{\sigma}$ for σ in $\hat{\mathbf{E}}$ the points $\mathbf{u}_i \mathbf{z}$ are congruent modulo the periods of $\mathfrak{F}$ to points of $\frac{1}{2}\mathscr{B}$. On translating them to this ball the period factors reduce $|\Theta|$ by a multiplicative factor of at most $c_{30}^{Lr^2}$, and since θ does not vanish on $\mathscr{B}$ we obtain the desired lower bound.

This completes the preparations for the extrapolation lemma. We introduce the abbreviations

$$\eta = (2^7 mn)^{-1}, \quad g = k^{\eta}$$

so that we can conveniently state and prove this lemma.

LEMMA 29. *Let j be an integer with $0 \leqslant j \leqslant g^2$. Then for all ∂ in $\mathfrak{D}(k - jkg^{-4})$ and all σ in $\hat{\mathbf{E}}(g^{4j}h)$ we have*

$$|\partial\Phi(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma})| \leqslant \exp(-\tfrac{1}{2}H^{\epsilon}).$$

Proof. We write $l = kg^{-4}$ and for an integer j we put

$$r_j = g^{4j}h, \quad t_j = k - jl.$$

Now we may regard the result of Lemma 25 as the initial step $j = 0$ of an induction proof. We assume that i is an integer with $0 \leqslant i < g^2$ such that the present lemma is true for all j with $0 \leqslant j \leqslant i$. We proceed to deduce its validity for $j = i + 1$.

If the lemma is false for $i + 1$, there exists σ' in $\hat{\mathbf{E}}(r_{i+1})$ such that

$$|\partial'\Phi(\boldsymbol{\sigma}', \ldots, \boldsymbol{\sigma}')| > \exp(-\tfrac{1}{2}H^{\epsilon}) \tag{25}$$

for some ∂' in $\mathfrak{D}(t_{i+1})$, and we may take ∂' to be minimal in the total ordering. We put

$$f(\mathbf{z}) = \partial'\Omega(\mathbf{z}, \ldots, \mathbf{z}).$$

Let Δ be a differential operator of order at most l in $\mathbf{z}$; to evaluate Δf we have to replace Δ by a sum of at most m^l operators in $\mathfrak{D}(l)$. When composed with ∂' these become operators of $\mathfrak{D}(l + t_{i+1}) = \mathfrak{D}(t_i)$, and hence

$$\Delta f(\mathbf{z}) = \sum_{\partial} \partial\Omega(\mathbf{z}, \ldots, \mathbf{z})$$

taken over a collection of at most m^l operators of $\mathfrak{D}(t_i)$. Suppose now that σ is in $\hat{\mathbf{E}}(r_i)$. Then

$$|\Delta f(\boldsymbol{\sigma})| \leqslant m^l \sum_{\partial_1} |\partial_1\Theta(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma})| \sum_{\partial_2} |\partial_2\Phi(\boldsymbol{\sigma}, \ldots, \boldsymbol{\sigma})|$$

where each sum is over all operators of $\mathfrak{D}(t_i)$. The right-hand side of this can be estimated by Lemma 28 and the induction hypothesis; since

$$r_i = g^{4i}h < e^{g^3}$$

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we have

$$|\Delta f(\boldsymbol{\sigma})| \leqslant k^{c_{31}k} \exp(c_{31} L r_i^2 \log r_i) \exp(-\tfrac{1}{2}H^\epsilon) < \exp(-\tfrac{1}{4}H^\epsilon).$$

We now use Lemma 20 with $\mathscr{S}$ as the set of points $\boldsymbol{\sigma}$ with σ in $\hat{\mathsf{E}}(r_i)$; this has separation c_{32}^{-1} and is contained in $r\mathscr{B}$ with $r = c_{33} r_i$. We take $\frac{1}{2}r'$ as the maximum of the absolute values of the vectors corresponding to numbers of $\mathsf{E}(r_{i+1})$, so that r' lies between constant multiples of r_{i+1}. From Lemma 12 $\mathscr{S}$ contains at least $c_{34}^{-1} r_i^{2n}$ points, and so we get

$$\mathfrak{M}(f, r'\mathscr{B}) \leqslant \exp(c_{35} l r_i^{2n} \log r_{i+1}) \exp(-\tfrac{1}{4}H^\epsilon) + \exp(-c_{35}^{-1} l r_i^2)\, \mathfrak{M}(f, c_{35} r'\mathscr{B}).$$

Now

$$\mathfrak{M}(f, c_{35} r'\mathscr{B}) \leqslant \exp(c_{36} L r_{i+1}^2 \log r_{i+1})$$

by Lemma 28; but this is dominated by the exponential factor because of the crucial observations

$$L r_{i+1}^2 \log r_{i+1} \leqslant L g^{11} r_i^2, \quad l r_i^2 = k g^{-4} r_i^2 \tag{26}$$

and our choice of g so that $k = Lg^{16}$. Thus we see that

$$\mathfrak{M}(f, r'\mathscr{B}) \leqslant \exp(-c_{37}^{-1} l r_i^2)$$

and in particular this upper bound holds for the absolute value of $f(\boldsymbol{\sigma}')$. By expanding $\partial'(\Theta\Phi)$ and using the minimality of ∂' it is easily checked that

$$|f(\boldsymbol{\sigma}') - \Theta(\boldsymbol{\sigma}', \ldots, \boldsymbol{\sigma}')\, \partial'\Phi(\boldsymbol{\sigma}', \ldots, \boldsymbol{\sigma}')| < \exp(-\tfrac{1}{4}H^\epsilon).$$

If we use the lower bound for the Θ-function supplied by Lemma 28, the calculations (26) show that

$$|\partial'\Phi(\boldsymbol{\sigma}', \ldots, \boldsymbol{\sigma}')| < \exp(-c_{38}^{-1} l r_i^2). \tag{27}$$

This completes the analytic argument; we must now translate everything into algebraic numbers by using the basic inequality of Lemma 26. We first show that the hypotheses of the present lemma imply that $E(\sigma', \partial'') = 0$ for all $\partial'' < \partial'$. This is by induction on ∂''; suppose $E(\sigma', \partial''') = 0$ for all $\partial''' < \partial''$. Then

$$|u(\partial'')\, E(\sigma', \partial'')| \leqslant \exp(-\tfrac{1}{2}H^\epsilon) + |\partial''\Phi(\boldsymbol{\sigma}', \ldots, \boldsymbol{\sigma}')| \leqslant 2\exp(-\tfrac{1}{2}H^\epsilon) \tag{28}$$

which from Lemma 27 clearly implies that $E(\sigma', \partial'') = 0$.

Next we use the result (27) to deduce that in fact $E(\sigma', \partial') = 0$ as well. For, from (28) with ∂'' replaced by ∂' we see that

$$|u(\partial')\, E(\sigma', \partial')| < \exp(-c_{39}^{-1} l r_i^2)$$

and again Lemma 27 together with (26) forces the vanishing of $E(\sigma', \partial')$.

The final contradiction comes yet again from Lemma 26, since this now yields the estimate

$$|\partial'\Phi(\boldsymbol{\sigma}', \ldots, \boldsymbol{\sigma}')| \leqslant \exp(-\tfrac{1}{2}H^\epsilon)$$

in direct opposition to (25). This completes the proof of the lemma.

It now becomes necessary to extrapolate to a large set of positive measure on the complex diagonal. Let $R = e^g$ and let $\mathscr{Z}$ denote the set of complex vectors $\mathbf{z}$ such that

$$|\mathbf{z}| \leqslant R, \quad |\theta(\mathbf{u}_m \mathbf{z})| \geqslant \exp(-R^3)$$

and such that for $0 \leqslant i \leqslant m-1$ (but not necessarily $i = m$) the points $\mathbf{u}_i \mathbf{z}$ are congruent to points of $\frac{1}{2}\mathscr{B}$. In the next lemma we prove that the derivatives of Φ are small when $\mathbf{z}_1, \ldots, \mathbf{z}_m$ are all equal to a point of $\mathscr{Z}$.

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LEMMA 30. *For* $\mathbf{z}$ *in* $\mathscr{Z}$ *and* ∂ *in* $\mathfrak{D}(L)$ *we have*

$$|\partial\Phi(\mathbf{z},\dots,\mathbf{z})| < \exp(-R^5).$$

Proof. If ∂ is in $\mathfrak{D}(L)$ it is apparent from Lemma 3 that $\partial\Phi(\mathbf{z}_1,\dots,\mathbf{z}_m)$ is a polynomial of degree $c_{40}L$ in the functions $\mathbf{A}(\mathbf{u}_i\mathbf{z}_i)$, $B(\mathbf{u}_i\mathbf{z}_i)$ and $(C(\mathbf{u}_i\mathbf{z}_i))^{-1}$ for $0 \leqslant i \leqslant m$ Hence there is an integer $q \leqslant c_{41}$ such that $\Gamma = \Theta^q\partial\Phi$ is an entire function, and we put

$$f(\mathbf{z}) = \Gamma(\mathbf{z},\dots,\mathbf{z}).$$

Suppose that $j = [g^2]-1$ is the penultimate integer in the range of the extrapolation, and let σ be in $\hat{\mathbf{E}}(r_j)$. Then by calculations analogous to those of the preceding lemma we readily see that

$$|\Delta f(\boldsymbol{\sigma})| < \exp(-\tfrac{1}{4}H^\epsilon)$$

for any operator Δ on $\mathbf{z}$ with $|\Delta| \leqslant \frac{1}{2}k$. For the derivatives of Θ^q can be estimated by writing it as a product; and since $L+\frac{1}{2}k < t_j$ the derivatives of Φ all satisfy Lemma 29. With r', $\mathscr{S}$ as in the proof of that lemma we derive the upper bound

$$\mathfrak{M}(f, r'\mathscr{B}) \leqslant \exp(-\tfrac{1}{8}H^\epsilon) + \exp(-c_{42}^{-1}kr_j^2)\,\mathfrak{M}(f, c_{42}r'\mathscr{B}).$$

By writing Γ as a polynomial in the entire functions θ, $\theta\mathbf{A}$, θB, and θC^{-1} we find that

$$\mathfrak{M}(f, c_{42}r'\mathscr{B}) < \exp(c_{43}Lr_{j+1}^2\log r_{j+1})$$

which is clearly dominated by the exponential factor as before. The subsequent upper bound for $\mathfrak{M}(f, r'\mathscr{B})$ also holds for $\mathfrak{M}(f, \mathscr{Z})$, since

$$r' > r_j > e^{g^2} > R^3 > R.$$

Hence for any $\mathbf{z}$ in $\mathscr{Z}$ we have

$$|f(\mathbf{z})| < \exp(-c_{44}^{-1}kr_j^2).$$

Since for $0 \leqslant i \leqslant m-1$ the points $\mathbf{u}_i\mathbf{z}$ are congruent to points of $\frac{1}{2}\mathscr{B}$, the corresponding factors of $(\Theta(\mathbf{z},\dots,\mathbf{z}))^q$ are relatively harmless, and the remaining factor is directly so by hypothesis; it follows that

$$|\partial\Phi(\mathbf{z},\dots,\mathbf{z})| < \exp(-c_{45}^{-1}kr_j^2) < \exp(-R^5).$$

This proves the lemma and completes the extrapolation procedure. We now use the information of this lemma to obtain a contradiction that will establish the theorem. In the next short section we do this for $m = 1$; i.e. the case of two algebraic points. This will in fact represent the initial step of an induction on the number of algebraic points involved.

8. *Proof of the theorem for two algebraic points.* For this we need hardly any of Lemma 30. We have

$$\Phi(\mathbf{z}_1) = \Sigma\, p(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1)\, A(\boldsymbol{\sigma}_1\mathbf{u}_1\mathbf{z}_1, \boldsymbol{\lambda}_0)\, A(\mathbf{u}_1\mathbf{z}_1, \boldsymbol{\lambda}_1)$$

and this is a polynomial of degree at most $2L$ to which Lemma 15 can be applied immediately, because $H > \exp(k^\delta)$ for some fixed $\delta > 0$. If $\mathbf{u}_1\mathbf{z}$ is in the set $\mathscr{S}(\sigma_1, m_1)$ with $m_1 = \max(2L, (\log H)^3) < k$ then $\mathbf{z}$ lies in $\mathscr{Z}$; for the congruence condition on $\mathbf{u}_1\mathbf{z}$

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implies the required lower bound for the absolute value of $\theta(\mathbf{u}_1\mathbf{z})$. Thus for all $\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1$ we have

$$|p(\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1)| \leqslant e^{2k^2\mathfrak{M}(\Phi, \mathscr{Z})} < 1.$$

This means that all the coefficients vanish, which contradicts the assertion of Lemma 25.

9. *Proof of the general theorem.* We shall assume that $m \geqslant 2$ and that the theorem holds for m algebraic points. Thus a contradiction arising from Lemma 30 will establish the theorem by induction.

We are going to use generalized Wronskians in our argument just as for elliptic functions. The technique is somewhat similar to that appearing in Roth's theorem (8); in particular lemma 2 of (8) indicates the kind of decomposition we shall make.

We define the functions $F(\boldsymbol{\lambda}_m) = F(\mathbf{z}_1, \ldots, \mathbf{z}_m, \boldsymbol{\lambda}_m)$ by the sum

$$F(\boldsymbol{\lambda}_m) = \Sigma p A(\mathbf{u}_0\mathbf{z}_0, \boldsymbol{\lambda}_0) \ldots A(\mathbf{u}_{m-1}\mathbf{z}_{m-1}, \boldsymbol{\lambda}_{m-1})$$

taken over all $\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{m-1}$, so that

$$\Phi = \sum_{\boldsymbol{\lambda}_m} F(\boldsymbol{\lambda}_m) A(\mathbf{u}_m\mathbf{z}_m, \boldsymbol{\lambda}_m). \tag{29}$$

Since $\sigma_m \neq 0$ the abelian functions appearing in $F(\boldsymbol{\lambda}_m)$ are algebraically independent. Also because not all of the coefficients p are zero, it follows that at least one of the functions $F(\boldsymbol{\lambda}_m)$ is not identically zero. Suppose that these functions span a vector space of dimension $M+1$ over $\mathbf{C}$, where $M \geqslant 0$. Then by our distinguished choice of L_m we have

$$M \leqslant (L_m+1)^n \leqslant L.$$

Let $G(\mu)$ for $0 \leqslant \mu \leqslant M$ be a subset of the $F(\boldsymbol{\lambda}_m)$ that is a basis for this vector space. Then there are complex numbers $a(\boldsymbol{\lambda}_m, \mu)$ such that

$$F(\boldsymbol{\lambda}_m) = \sum_{\mu} a(\boldsymbol{\lambda}_m, \mu) G(\mu). \tag{30}$$

For $M+1$ differential operators ∂_ν in $\mathfrak{D}(\nu)$ for $0 \leqslant \nu \leqslant M$ we construct the generalized Wronskian determinant of the basis functions; it is given by

$$W(\mathbf{z}_1, \ldots, \mathbf{z}_m) = \det_{\nu,\mu} (\partial_\nu G(\mu)) \quad (0 \leqslant \mu, \nu \leqslant M).$$

Finally we define

$$H(\mu) = H(\mathbf{z}_m, \mu) = \sum_{\boldsymbol{\lambda}_m} a(\boldsymbol{\lambda}_m, \mu) A(\mathbf{u}_m\mathbf{z}_m, \boldsymbol{\lambda}_m) \tag{31}$$

and we prove the following consequence of Lemma 30.

LEMMA 31. *Suppose the differential operators $\partial_0, \ldots, \partial_M$ do not involve the variables in $\mathbf{z}_m$. Then for $\mathbf{z}$ in $\mathscr{Z}$ we have*

$$|H(\mathbf{z}, 0)\, W(\mathbf{z}, \ldots, \mathbf{z})| < \exp(-R^4).$$

Proof. Since the operators do not involve $\mathbf{z}_m$, we have from (29), (30) and (31)

$$\partial_\nu \Phi = \sum_{\mu} H(\mu) \partial_\nu G(\mu).$$

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Hence the function $H(\mathbf{z}, 0)\, W(\mathbf{z}, \ldots, \mathbf{z})$ can be expressed as the Wronskian determinant with the functions $\partial_\nu \Phi(\mathbf{z}, \ldots, \mathbf{z})$ replacing the first column. If $\mathbf{z}$ is in $\mathscr{Z}$ these take small values by Lemma 30, and the remaining entries of the form $\partial_\nu G(\mathbf{z}, \ldots, \mathbf{z}, \mu)$ can be estimated by the technique of Lemma 28. In fact for $|\mathbf{z}_i - \mathbf{z}| \leqslant H^{-2}$ it is clear from the congruence conditions on $\mathbf{u}_i \mathbf{z}$ for $0 \leqslant i \leqslant m-1$ that

$$|G(\mathbf{z}_1, \ldots, \mathbf{z}_m, \mu)| < h^{c_1 L h^2},$$

and the Cauchy integral provides a similar upper bound for the derivatives at the point $(\mathbf{z}, \ldots, \mathbf{z})$. The inequality of the lemma is now apparent.

To remove the function $H(\mathbf{z}, 0)$ from this lemma we shall use the polynomial result proved as Lemma 22. We retain the hypothesis concerning the derivatives $\partial_0, \ldots, \partial_M$, and we recall Lemma 11 and the ball $\mathscr{A}$ defined there.

LEMMA 32. *Suppose $\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{m-1}$ are points of $\mathscr{A}$. Then there is a point $\mathbf{z}$ in $\mathscr{Z}$ such that*

$$|\mathbf{A}(\mathbf{u}_i \mathbf{z}) - \boldsymbol{\xi}_i| < k^{-1} \quad (0 \leqslant i \leqslant m-1)$$

and

$$|W(\mathbf{z}, \ldots, \mathbf{z})| < \exp(-R^3).$$

Proof. By supposition the theorem is true for m algebraic points, and so we may apply Lemma 11 to the points $\mathbf{u}_0, \ldots, \mathbf{u}_{m-1}$. This gives a ball $\mathscr{R}$ of radius k^{-2} such that for all $\mathbf{z}$ in $\mathscr{R}$ the points $\mathbf{u}_i \mathbf{z}$ are congruent to points of $\frac{1}{2}\mathscr{B}$ and

$$|\mathbf{A}(\mathbf{u}_i \mathbf{z}) - \boldsymbol{\xi}_i| < k^{-1}, \quad |\mathbf{z}| < e^{k^\eta} = R. \tag{32}$$

Let $\mathscr{R}_0$ be the translation of $\mathscr{R}$ nearest the origin by periods of $\mathbf{A}(\mathbf{u}_m \mathbf{z})$. By Lemma 21 this ball contains a subball $\mathscr{R}_0'$ of radius k^{-4} on which

$$|\theta(\mathbf{u}_m \mathbf{z})| > \exp(-k^{c_2}),$$

and by translating $\mathscr{R}_0'$ back into $\mathscr{R}$ it becomes a ball $\mathscr{R}'$ on which

$$|\theta(\mathbf{u}_m \mathbf{z})| > \exp(-k^{c_2})\, c_3^{-|\mathbf{z}|^2} > \exp(-R^3). \tag{33}$$

Hence $\mathbf{A}(\mathbf{u}_m \mathbf{z})$ is regular on $\mathscr{R}'$, and we can apply Lemma 22 to the function

$$\phi(\mathbf{z}) = H(\mathbf{z}, 0).$$

This is a polynomial of degree at most L_m in $\mathbf{A}(\mathbf{u}_m \mathbf{z})$ and one of its coefficients is unity. For an index $\boldsymbol{\lambda}_m$ exists with $G(0) = F(\boldsymbol{\lambda}_m)$, and so (30) implies that $a(\boldsymbol{\lambda}_m, 0) = 1$ for this index. It follows that

$$M(\phi, \mathscr{R}') \geqslant \exp(-k^{c_4})$$

and so there is a point $\mathbf{z}$ of $\mathscr{R}'$ such that

$$|H(\mathbf{z}, 0)| \geqslant \exp(-k^{c_4}). \tag{34}$$

From (32), (33) and the congruence conditions, this point is also a point of $\mathscr{Z}$. The inequality for $W(\mathbf{z}, \ldots, \mathbf{z})$ now follows from (34) and the preceding lemma.

This result gives enough information about $W(\mathbf{z}_1, \ldots, \mathbf{z}_m)$ to ensure, with the help of Lemma 13, its identical vanishing.

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LEMMA 33. *If $\partial_0, \ldots, \partial_M$ are independent of $\mathbf{z}_m$ the corresponding Wronskian determinant $W(\mathbf{z}_1, \ldots, \mathbf{z}_m)$ is identically zero.*

Proof. This would be quite rapid but for the fact that differentiation has introduced the functions B and C^{-1} into W. So we start by eliminating these. By Lemma 3 the functions $(u(\partial_\nu))^{-1}\partial_\nu G(\mu)$ are polynomials of degree at most $c_5 L$ in the functions

$$\mathbf{A}(\mathbf{u}_i\mathbf{z}_i), \quad B(\mathbf{u}_i\mathbf{z}_i), \quad (C(\mathbf{u}_i\mathbf{z}_i))^{-1} \quad (0 \leqslant i \leqslant m-1)$$

whose coefficients are numbers of I with sizes at most $h^{c_6 L h^2}$. It follows that

$$U(\mathbf{z}_1, \ldots, \mathbf{z}_m) = (u(\partial_0) \ldots u(\partial_M))^{-1} W(\mathbf{z}_1, \ldots, \mathbf{z}_m)$$

is a polynomial of degree $N \leqslant c_7 L^2$ in the same functions with coefficients in I of sizes at most $c_7^{k^3}$. Hence the function

$$V(\mathbf{z}_1, \ldots, \mathbf{z}_m) = (C(\mathbf{u}_0\mathbf{z}_0) \ldots C(\mathbf{u}_{m-1}\mathbf{z}_{m-1}))^N U(\mathbf{z}_1, \ldots, \mathbf{z}_m)$$

can be written as a polynomial P in

$$\mathbf{A}(\mathbf{u}_i\mathbf{z}_i), B(\mathbf{u}_i\mathbf{z}_i) \quad (0 \leqslant i \leqslant m-1); \tag{35}$$

this polynomial lies in $\mathsf{I}[\mathbf{X}_0, \ldots, \mathbf{X}_{m-1}, Y_0, \ldots, Y_{m-1}]$ and is of degree at most $c_8 L^2$. Its coefficients are of sizes at most $c_8^{k^3}$ and it is equal to $V(\mathbf{z}_1, \ldots, \mathbf{z}_m)$ at the specializations

$$\mathbf{X}_i = \mathbf{A}(\mathbf{u}_i\mathbf{z}_i), \quad Y_i = B(\mathbf{u}_i\mathbf{z}_i). \tag{36}$$

We re-introduce the field $\mathsf{P}' = \mathsf{C}(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}, y_0, \ldots, y_{m-1})$ and we let $Q(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1})$ be the norm of $\pi = P(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}, y_0, \ldots, y_{m-1})$ from P' to P. Thus from Lemma 2 the degree of Q is at most $c_9 L^2$ and its coefficients are in I with sizes at most $c_9^{k^3}$. Finally we write $T(\mathbf{z}_1, \ldots, \mathbf{z}_m)$ for the specialization of Q at (36).

We now observe that T inherits the smallness of W at the points $(\mathbf{z}, \ldots, \mathbf{z})$ for $\mathbf{z}$ in $\mathscr{Z}$. For by the congruence conditions on $\mathbf{u}_i\mathbf{z}$ for $0 \leqslant i \leqslant m-1$, the absolute values of B and its conjugate functions at the point $\mathbf{u}_i\mathbf{z}$ do not exceed c_{10}. Thus the absolute value of the corresponding conjugates of $V(\mathbf{z}, \ldots, \mathbf{z})$ are at most $c_{11}^{k^3}$, and so

$$|T(\mathbf{z}, \ldots, \mathbf{z})| < c_{12}^{k^3} \exp(-R^3) < \exp(-R^2). \tag{37}$$

But now the polynomial Q is small at many well-distributed points; for suppose $\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{m-1}$ are arbitrary points of $\mathscr{A}$ and let $\boldsymbol{\zeta}$ be the corresponding point of $\mathscr{Z}$ whose existence is asserted by Lemma 32. Then if $\boldsymbol{\alpha}_i = \mathbf{A}(\mathbf{u}_i\boldsymbol{\zeta})$ we have

$$|\boldsymbol{\alpha}_i - \boldsymbol{\xi}_i| < k^{-1} \quad (0 \leqslant i \leqslant m-1)$$

and

$$|Q(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_{m-1})| = |T(\boldsymbol{\zeta}, \ldots, \boldsymbol{\zeta})| < \exp(-R^2).$$

It is now clear that there is a set $\mathscr{S}$, contained in the closed unit ball centred at the origin of C^{mn}, consisting of at least $c_{13}^{-1}k^{2mn}$ points with separation at least $c_{13}^{-1}k^{-1}$, on which the absolute value of Q does not exceed $\exp(-R^2)$. Since $k^2 > k^\delta L^2$ for a fixed $\delta > 0$, we can use Lemma 13 to deduce that

$$|Q| < \exp(-R).$$

This clearly implies the identical vanishing of $Q(\mathbf{X}_0, \ldots, \mathbf{X}_{m-1})$.

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Therefore $Q(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}) = \operatorname{norm} \pi$ is zero, which means that π must vanish. If $P_0(\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}, y_0, \ldots, y_{m-1})$ is the canonical form (see the proof of Lemma 2) of π, it too must be zero and so the polynomial P_0 must vanish identically. In particular it vanishes at (36), when its value is $V(\mathbf{z}_1, \ldots, \mathbf{z}_m)$; the lemma is now immediate.

By the fundamental theorem of generalized Wronskians there are functions $K(\mu)$ $(0 \leqslant \mu \leqslant M)$, independent of $\mathbf{z}_1, \ldots, \mathbf{z}_{m-1}$ and not all zero, such that

$$K(0)\,G(0) + \ldots + K(M)\,G(M) = 0. \tag{38}$$

Now Lemma 7.3 of (7) tells us more about these functions; they are rational functions of degree at most M in the entries of the Wronskian matrices involved. Therefore they are rational functions of degree at most $c_{14}L^2$ in the functions (35). Since H exceeds $\exp(k^\delta)$ for some fixed $\delta > 0$, Lemma 17 forces the functions $K(\mu)$ to be constants, and now the relation (38) contradicts the linear independence of the basis

$$G(0), \ldots, G(M).$$

This completes the proof of the theorem.

REFERENCES

(1) BAKER, A. Linear forms in the logarithms of algebraic numbers. *Mathematika* **13** (1966), 204–16.
(2) BOMBIERI, E. Algebraic values of meromorphic maps. *Inventiones math.* **10** (1970), 267–87.
(3) CASSELS, J. W. S. *An introduction to diophantine approximation.* Cambridge Tracts no. 45. (Cambridge University Press 1957.)
(4) GELFOND, A. O. *Transcendental and algebraic numbers* (New York; Dover, 1960).
(5) LANDAU, E. *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig; Teubner, 1918).
(6) LANG, S. *Introduction to transcendental numbers* (Reading; Addison-Wesley, 1966).
(7) MASSER, D. W. Elliptic functions and transcendence. Ph.D. thesis, University of Cambridge, 1974. (Lecture Notes in Mathematics, 437, Springer, 1975.)
(8) ROTH, K. F. Rational approximations to algebraic numbers. *Mathematika* **2** (1955), 1–20.
(9) SCHNEIDER, T. *Einführung in die transzendenten Zahlen* (Berlin; Springer-Verlag, 1957).
(10) SIEGEL, C. L. *Topics in Complex Function Theory*, vol. III (New York; Wiley-Interscience, 1973).
(11) MASSER, D. W. Linear forms in algebraic points of Abelian functions. I. *Proc. Cambridge Philos. Soc.* **77** (1975), 499–513.
(12) BOMBIERI, E. and LANG, S. Analytic subgroups of group varieties. *Inventiones math.* **11** (1970), 1–14.

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