Proc. Camb. Phil. Soc. (1965), **61**, 939
Printed in Great Britain

Magneto-thermo-elastic plane waves

By C. M. PURUSHOTHAMA
Department of Applied Mathematics, Indian Institute of Technology, Kharagpur, India

(*Received* 23 *February* 1965)

Abstract. The combined effects of uniform thermal and magnetic fields on the propagation of plane waves in a homogeneous, initially unstressed, electrically conducting elastic medium have been investigated.

When the magnetic field is parallel to the direction of wave propagation, the compression wave is purely thermo-elastic and the shear wave is purely magneto-elastic in nature. For a transverse magnetic field, the shear waves remain elastic whereas the compression wave assumes magneto-thermo-elastic character due to the coupling of all the three fields—mechanical, magnetic and thermal. In the general case, the waves polarized in the plane of the direction of wave propagation and the magnetic field are not only coupled but are also influenced by the thermal field, once again exhibiting the coupling of the three fields. The shear wave polarized transverse to the plane retains its magneto-elastic character.

Notation.

H_i = primary magnetic field components,
h_i = induced magnetic field components,
T_0 = initial thermal field,
θ = induced thermal field,
C = compression wave velocity.
S = shear wave velocity,
u_i = displacement components,
c_v = specific heat at constant volume,
k = thermal conductivity,
η = magnetic diffusivity,
μ_e = magnetic permeability,
λ, μ = Lamé's constants,
β = ratio of coefficient of volume expansion to isothermal compressibility.

1. *Introduction.* The propagation of plane waves in magneto-elastics was first considered by Knopoff ((3)). He showed that uncoupled waves, which are plane polarized waves whose polarizations are unaltered by propagation through the magnetic field, can arise in two ways: either from the propagation of shear waves polarized transverse to the magnetic field or from the propagation of compression waves through a

https://doi.org/10.1017/S0305004100039335 Published online by Cambridge University Press

transverse magnetic field. He found that if the magnetic induction is cut by the waves at an angle ϕ $(0 < \phi < \frac{1}{2}\pi)$, then the compression wave and the shear wave polarized in the (x, y)-plane, formed by the direction of wave propagation and the magnetic field, will be coupled.

Paria ((4)) considered the propagation of plane elastic waves under the influence of a uniform thermal field and a transverse magnetic field. Willson ((6)) improved on this result by introducing a component of the magnetic field parallel to the direction of propagation. Both of them restricted their analysis by assuming that the field oscillations normal to the (x, y)-plane would vanish.

The energy equation, in which the term corresponding to joule heating is neglected, connects only the mechanical and thermal fields and that through the dilatation term. Since shear waves are characterized by the vanishing of the dilatation term, we may expect the possibility of propagating uncoupled shear waves polarized transverse to the (x, y)-plane, unaffected by the presence of the thermal field. This has led us to reinvestigate the problem of plane wave propagation in the presence of uniform thermal and magnetic fields and for different orientations of the magnetic field.

We have found that if the magnetic field is aligned along the direction of wave propagation, there will arise two types of uncoupled waves: (*a*) a thermo-elastic compression wave unaffected by the presence of magnetic field; and (*b*) a magneto-elastic shear wave unaffected by the thermal field. In the case of transverse magnetic field, we have uncoupled shear waves, which are purely elastic in nature, and magneto-thermo-elastic compression waves. When the magnetic field is inclined at an angle ϕ $(0 < \phi < \frac{1}{2}\pi)$, the shear wave polarized in the (x, y)-plane and the compression wave will be coupled, whereas the shear wave polarized transverse to the plane remains uncoupled and purely magneto-elastic.

2. *Equation for phase velocity.* The equations governing the propagation of small disturbances in a homogeneous, electrically conducting elastic medium at rest, initially unstressed and maintained at a uniform temperature, in the presence of a uniform external magnetic field are (see (5))

$$\frac{\partial h_j}{\partial x_j} = 0, \tag{2·1}$$

$$\frac{\partial h_i}{\partial t} = \left(H_j \frac{\partial}{\partial x_j}\right) \frac{\partial u_i}{\partial t} - H_i \frac{\partial}{\partial t}\left(\frac{\partial u_j}{\partial x_j}\right) + \eta \nabla^2 h_i, \tag{2·2}$$

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) + \mu \nabla^2 u_i - \beta \frac{\partial \theta}{\partial x_i} + \mu_e H_j \frac{\partial h_i}{\partial x_j} - \mu_e H_j \frac{\partial h_j}{\partial x_i}, \tag{2·3}$$

$$k \nabla^2 \theta = \rho c_v \frac{\partial \theta}{\partial t} + \beta T_0 \frac{\partial}{\partial t}\left(\frac{\partial u_j}{\partial x_j}\right), \tag{2·4}$$

with repeated suffixes indicating summation over the space coordinates.

We choose the direction of wave propagation for x-axis and the primary magnetic field to have components $(H \cos\phi, H \sin\phi, 0)$, ϕ being the angle at which the wave crosses the magnetic field. The field components u_i, h_i and θ are assumed to be proportional to $\exp i(\gamma x - \omega t)$.

https://doi.org/10.1017/S0305004100039335 Published online by Cambridge University Press

Under these assumptions equations (2·1) to (2·4) assume the form

$$h_1 = 0, \tag{2·5}$$

$$\gamma\omega H \sin\phi u_1 - \gamma\omega H \cos\phi u_2 + (\eta\gamma^2 - i\omega) h_2 = 0, \tag{2·6}$$

$$\gamma\omega H \cos\phi u_3 - (\eta\gamma^2 - i\omega) h_3 = 0, \tag{2·7}$$

$$\{(\lambda + 2\mu)\gamma^2 - \rho\omega^2\} u_1 + i\gamma\beta\theta + i\mu_e \gamma H \sin\phi h_2 = 0, \tag{2·8}$$

$$(\mu\gamma^2 - \rho\omega^2) u_2 - i\mu_e \gamma H \cos\phi h_2 = 0, \tag{2·9}$$

$$(\mu\gamma^2 - \rho\omega^2) u_3 - i\mu_e \gamma H \cos\phi h_3 = 0, \tag{2·10}$$

$$\beta T_0 \gamma\omega u_1 + (k\gamma^2 - i\rho c_v \omega)\theta = 0. \tag{2·11}$$

The x-component of the equation (2·2) is identically satisfied in view of (2·5).

Equations (2·6) to (2·11) can be separated into two independent systems, one for the unknowns u_1, u_2, h_2 and the other for u_3 and h_3

$$\begin{bmatrix} (\lambda+2\mu)\gamma^2 - \rho\omega^2 - \dfrac{i\beta^2 T_0 \gamma^2 \omega}{k\gamma^2 - i\rho c_v \omega} & 0 & i\mu_e \gamma H \sin\phi \\ 0 & \mu\gamma^2 - \rho\omega^2 & -i\mu_e \gamma H \cos\phi \\ \gamma\omega H \sin\phi & -\gamma\omega H \cos\phi & \eta\gamma^2 - i\omega \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ h_2 \end{bmatrix} = 0, \tag{2·12}$$

$$\begin{bmatrix} \gamma\omega H \cos\phi & i\omega - \eta\gamma^2 \\ \mu\gamma^2 - \rho\omega^2 & -i\mu_e \gamma H \cos\phi \end{bmatrix} \begin{bmatrix} u_3 \\ h_3 \end{bmatrix} = 0. \tag{2·13}$$

The first row of the matrix equation (2·12) is obtained by eliminating θ from the equations (2·8) and (2·11). The equation (2·5) implies that the magnetic field oscillations remain transverse to the direction of wave propagation for all inclinations ϕ of the magnetic field.

We introduce the following terms having the dimensions of a velocity:

$$C = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}}, \quad S = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}, \quad V_h = H\left(\frac{\mu_e}{\rho}\right)^{\frac{1}{2}}, \quad V_T = \frac{\beta}{\rho}\left(\frac{T_0}{c_v}\right)^{\frac{1}{2}},$$

$$V = \frac{\omega}{\gamma}, \quad V_e = (\omega\eta)^{\frac{1}{2}}, \quad V_a = \left(\frac{\omega k}{\rho c_v}\right)^{\frac{1}{2}}.$$

C and S are the velocities of compression and shear waves, respectively, in the absence of thermal and magnetic fields. V_h is the Alfvén velocity and V, the phase velocity. V_e is the velocity of propagation of electrical eddy currents at the frequency ω. In analogy with this we can call V_a the velocity of propagation of 'thermal eddy currents'. V_T is characteristic of the thermal field and the thermal properties of the medium.

The condition that the two matrix equations (2·12) and (2·13) may be individually consistent gives the following two equations for the phase velocities

$$\{(V^2 - C^2)(V^2 + iV_a^2) - V_T^2 V^2\}(V^2 - S^2)(V^2 + iV_e^2) - (V^2 - S^2)(V^2 + iV_a^2) V_h^2 V^2 \sin^2\phi$$
$$- V^2 V_h^2 \cos^2\phi\{(V^2 - C^2)(V^2 + iV_a^2) - V_T^2 V^2\} = 0, \tag{2·14}$$

$$(V^2 - S^2)(V^2 + iV_e^2) - V^2 V_h^2 \cos^2\phi = 0. \tag{2·15}$$

https://doi.org/10.1017/S0305004100039335 Published online by Cambridge University Press

The equation (2·14) for the phase velocity is, but for the notation, the same as that obtained by Willson ((6)). $V_T = 0$ reduces it to that derived by Knopoff ((3)).

3. *Uncoupled waves.* From the equations (2·12) and (2·13), we observe that uncoupled waves can arise in two ways: (*a*) from the propagation of shear waves polarized normal to the magnetic field, and (*b*) from the propagation of compression waves through a magnetic field aligned along either the x or y coordinate axis.

When the magnetic induction is parallel to the direction of wave propagation, the equations for phase velocity reduce to

$$(V^2-C^2)(V^2+iV_a^2)-V^2V_T^2=0, \tag{3·1}$$

$$(V^2-S^2)(V^2+iV_e^2)-V^2V_h^2=0. \tag{3·2}$$

Referring back to the matrix equations (2·12) and (2·13), we notice that the equation (3·1) gives the phase velocity of compression wave which is purely thermo-elastic in nature and the equation (3·2) gives that of shear wave which is magneto-elastic. Equations (3·1) and (3·2) have been separately treated in their non-dimensional form by Chadwick ((2)) and by the author ((5)), respectively, for ranges of small wavelengths and of small frequencies.

When the magnetic induction is transverse, the shear waves remain purely elastic unaffected by the presence of thermal and magnetic fields. But the compression wave is influenced by both the fields and its phase velocity is given by the equation

$$V^2-C^2-\frac{V^2V_h^2}{V^2+iV_e^2}-\frac{V^2V_T^2}{V^2+iV_a^2}=0, \tag{3·3}$$

which gives three different modes of propagation.

For solids S is of the order 10^6, whereas V_T (for T_0 moderately below 100 °C) is of the order 10^5 and V_h is of the order of the magnetic field. Hence, under normal laboratory conditions, it is a reasonable approximation to assume

$$V_T^2 \ll S^2 \leqslant \tfrac{1}{2}C^2, \quad V_h^2 \ll S^2 \leqslant \tfrac{1}{2}C^2.$$

Under these assumptions, the three roots of the equations (3·3) are

$$V_1^L \approx C\left\{1+\frac{1}{2}\frac{V_h^2}{C^2+iV_e^2}+\frac{1}{2}\frac{V_T^2}{C^2+iV_a^2}\right\}, \tag{3·4}$$

$$V_2^L \approx i^{\frac{3}{2}}V_e\left\{1-\frac{1}{2}\frac{V_h^2}{C^2+iV_e^2}\right\}, \tag{3·5}$$

$$V_3^L \approx i^{\frac{3}{2}}V_a\left\{1-\frac{1}{2}\frac{V_T^2}{C^2+iV_a^2}\right\}. \tag{3·6}$$

The first mode corresponds to perturbed elastic wave motion and the second and third modes are perturbations of the electrical and thermal eddy current waves.

When $0 < \phi < \frac{1}{2}\pi$, the only uncoupled wave is the shear wave polarized normal

https://doi.org/10.1017/S0305004100039335 Published online by Cambridge University Press

to the magnetic field, whose phase velocity is given by the equation (2·15). Under the usual assumptions, the two modes of propagation are given by

$$V_1^S \approx S\left\{1+\frac{1}{2}\frac{V_h^2\cos^2\phi}{S^2+iV_e^2}\right\}, \tag{3·7}$$

$$V_2^S \approx i^{\frac{3}{2}}V_e\left\{1-\frac{1}{2}\frac{V_h^2\cos^2\phi}{S^2+iV_e^2}\right\}, \tag{3·8}$$

so that the general motion comprises of a perturbed shear elastic mode and a perturbed and attenuated eddy current.

4. *Coupled waves.* When the magnetic field is not parallel to any of the coordinate axes, the longitudinal and transverse field oscillations in the (x, y)-plane are coupled, implying that they have no independent existence. In this case, the phase velocity satisfies the equation (2·14). With the usual assumptions made in the case of weak magnetic fields and moderate thermal fields, the four modes of propagation are given by

$$V_1^C \approx C\left\{1+\frac{1}{2}\frac{V_h^2\sin^2\phi}{C^2+iV_e^2}+\frac{1}{2}\frac{V_T^2}{C^2+iV_a^2}\right\}, \tag{4·1}$$

$$V_2^C \approx S\left\{1+\frac{1}{2}\frac{V_h^2\cos^2\phi}{S^2+iV_e^2}\right\}, \tag{4·2}$$

$$V_3^C \approx i^{\frac{3}{2}}V_e\left\{1-\frac{1}{2}V_h^2\frac{C^2\cos^2\phi+S^2\sin^2\phi+iV_e^2}{(S^2+iV_e^2)(C^2+iV_e^2)}\right\}, \tag{4·3}$$

$$V_4^C \approx i^{\frac{3}{2}}V_a\left\{1-\frac{1}{2}\frac{V_T^2}{C^2+iV_a^2}\right\}. \tag{4·4}$$

Thus, in addition to three modes similar to those given in (3·4)–(3·6) for the uncoupled case $\phi = \frac{1}{2}\pi$, we have one more mode of propagation which is a perturbed shear elastic wave motion. The motion which in the field-free case would be purely transverse and propagating with the velocity S is now a mixed wave of four velocities; one is close to the velocity of compression wave, the second one is close to that of shear wave and the other two are of attenuated waves. Similarly, in the case of motion which will be longitudinal in the field-free case. If, initially, a pure plane-polarized wave, of either longitudinal type or shear type is produced at some coordinate x, then as this wave propagates, the components of motion which are absent at x are induced. The remaining component of the displacement will also be generated.

If the magnetic field is very strong (of the order 10^8 Oe) then V_h can be assumed to be very large compared to C, S and V_T. Then the roots of the equation (2·14) will be modified to

$$(V_1^C)^2 \approx V_h^2+C^2\sin^2\phi+S^2\cos^2\phi-iV_e^2+V_T^2\sin^2\phi-\frac{iV_a^2V_T^2\cos^2\phi}{C^2\cos^2\phi+S^2\sin^2\phi+iV_a^2}, \tag{4·5}$$

$$(V_2^C)^2 \approx (C^2\cos^2\phi+S^2\sin^2\phi)\left\{1+\frac{V_T^2\cos^2\phi}{C^2\cos^2\phi+S^2\sin^2\phi+iV_a^2}\right\}, \tag{4·6}$$

$$(V_3^C)^2 \approx -iV_a^2\left\{1-\frac{V_T^2\cos^2\phi}{C^2\cos^2\phi+S^2\sin^2\phi+iV_a^2}\right\}, \tag{4·7}$$

the fourth being of the order of $1/V_h^2$. The first mode propagates with almost the Alfvén velocity, the influence of the thermal field being negligible. The second mode, which, in

https://doi.org/10.1017/S0305004100039335 Published online by Cambridge University Press

the absence of the thermal field, propagates at a velocity in between those of the shear and compression elastic waves, is influenced by the presence of thermal field.

5. *Conclusions.* The introduction of an initially uniform thermal field has varied effects on the propagation of magneto-elastic waves. In the case of coupled waves, which occur when a longitudinal wave or transverse wave polarized in the plane formed by the direction of wave propagation and the magnetic field is propagated, the effect is not only to perturb the modes of propagation given by Knopoff, but also to introduce another attenuated mode which, in analogy with electrical eddy current wave, can be called thermal eddy current wave.

It is to be noted that for most of the metals the temperature coefficient of sound velocity is of the order of several hundred parts per million per degree Kelvin, whereas the application of a 10 kOe magnetic field may change the sound velocity by only a few parts in a million. Therefore, in the case of weak magnetic fields, the thermal field may influence the wave propagation more than the magnetic field and this influence is, in the first approximation, confined only to that mode which corresponds to perturbed compressional elastic waves.

In the case of strong magnetic fields, the modified elastic wave, whose velocity lies between those of shear and compressional elastic waves, will move faster than it would, in the absence of thermal field. This change in the velocity will not be more than a few parts in a thousand.

It may be mentioned here that Alers and Fleury ((1)) conducted experiments to study the effects of magnetic fields in the range of 10 to 20 kOe on the velocity of sound in single crystals of Cu, Ag, Al, Ta and V using a 10 Mc/sec ultrasonic pulse technique and found that the results in majority of the cases (for metals of high conductivity) were in excellent agreement with the theoretical prediction of the dependence of the velocity on the applied field and the angle between the propagation direction and the applied field. In the case of very strong fields also, perhaps by using similar techniques it will be possible to verify experimentally the approximate formulation given in equation (4·6).

The author is highly thankful to the referee for all his comments on the original text of the paper. He also acknowledges with thanks the financial help received from the Council of Scientific and Industrial Research, New Delhi, in the form of a Senior Fellowship.

REFERENCES

(1) ALERS, G. A. & FLEURY, P. A. Modification of the velocity of sound in metals by magnetic fields. *Phys. Rev.* **129** (1963), 2425–2429.

(2) CHADWICK, P. *Thermo-elasticity, the Dynamical Theory.* Progress in Solid Mechanics, vol. 1 (ed. I. N. Sneddon & R. Hill) (North Holland Co. (1960)), 263–328.

(3) KNOPOFF, L. The interaction between elastic wave motion and a magnetic field in electrical conductors. *J. Geophys. Res.* **60** (1955), 441–456.

(4) PARIA, G. On magneto-thermo-elastic plane waves. *Proc. Cambridge Philos. Soc.* **58** (1962), 527–531.

(5) PURUSHOTHAMA, C. M. Propagation of small disturbances in magneto-elastics. *Proc. Indian Acad. Sci. Sect. A* **62** (1965) (to appear).

(6) WILLSON, A. J. The propagation of magneto-thermo-elastic plane waves. *Proc. Cambridge Philos. Soc.* **59** (1963), 483–488.

https://doi.org/10.1017/S0305004100039335 Published online by Cambridge University Press