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Efficient Preconditioner and Iterative Method for Large Complex Symmetric Linear Algebraic Systems

Li Dan Liao and Guo Feng Zhang*

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China.

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Abstract. We discuss an efficient preconditioner and iterative numerical method to solve large complex linear algebraic systems of the form $(W + \mathrm{i}T)u = c$, where W and T are symmetric matrices, and at least one of them is nonsingular. When the real part W is dominantly stronger or weaker than the imaginary part T, we propose a block multiplicative (BM) preconditioner or its variant (VBM), respectively. The BM and VBM preconditioned iteration methods are shown to be parameter-free, in terms of eigenvalue distributions of the preconditioned matrix. Furthermore, when the relationship between W and T is obscure, we propose a new preconditioned BM method (PBM) to overcome this difficulty. Both convergent properties of these new iteration methods and spectral properties of the corresponding preconditioned matrices are discussed. The optimal value of iteration parameter for the PBM method is determined. Numerical experiments involving the Helmholtz equation and some other applications show the effectiveness and robustness of the proposed preconditioners and corresponding iterative methods.

AMS subject classifications: 15A06, 65F10, 65H10

Key words: Preconditioner, complex linear algebraic systems, Krylov subspace method, spectral properties, convergence.

1. Introduction

We consider large and sparse complex symmetric linear algebraic systems of the form

$$\mathbf{A}u = c\,, \quad \mathbf{A} \in \mathbb{C}^{n\times n} \ \text{ and } \ u, c \in \mathbb{C}^n \tag{1.1}$$

where $\mathbf{A} = W + \mathrm{i}T \in \mathbb{C}^{n\times n}$ is a complex nonsingular matrix with W and $T \in \mathbb{R}^{n\times n}$ both symmetric matrices, and at least one of them is nonsingular ($\mathrm{i} = \sqrt{-1}$). Such complex linear algebraic systems arise in many scientific and engineering applications — e.g., wave propagation [28], distributed control problems [22], structural dynamics [16], fast Fourier

*Corresponding author. *Email addresses:* liaold15@lzu.edu.cn (L.D. Liao), gf_zhang@lzu.edu.cn (G.F. Zhang)

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transform solution of time-dependent partial differential equations [13,14], molecular scattering and lattice quantum chromodynamics [20]. More detailed background and additional references can be found in Refs. [4, 10, 11]. Writing $u = x + \mathrm{i}y$ and $c = f + \mathrm{i}\,g$ with $x, y, f, g \in \mathbb{R}^n$, we can re-express the complex linear algebraic system (1.1) as the following equivalent real block two-by-two formulation [2, 15]:

$$\mathscr{A}\tilde{z} \equiv \begin{pmatrix} W & -T \\ T & W \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv b\,. \tag{1.2}$$

Recently, many efficient iteration methods and preconditioning techniques have been developed for solving the complex linear algebraic system (1.1) or its equivalent real two-by-two block formulation (1.2) when W and T are symmetric and semi-positive definite and at least one of them is positive definite. For example, Bai *et al.* [5, 7] proposed a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method and a preconditioned MHSS iteration method for the linear algebraic system (1.1). They also proposed preconditioned MHSS (PMHSS) preconditioner for the linear algebraic system (1.2), which can be expressed as follows [8] :

$$P_{PMHSS} = \frac{1}{2\alpha}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}\begin{pmatrix} \alpha V + W & 0 \\ 0 & \alpha V + W \end{pmatrix}\begin{pmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix}\begin{pmatrix} \alpha V + T & 0 \\ 0 & \alpha V + T \end{pmatrix}.$$

Salkuyeh *et al.* [27] presented a generalised SOR (GSOR) iterative method for (1.2), and to accelerate the convergence rate, Davod *et al.* [21] produced a preconditioned variant of this generalised SOR (PGSOR) iterative method. Their GSOR and PGSOR iterative methods can be associated with the respective preconditioners

$$P_{GSOR} = \frac{1}{\alpha}\begin{pmatrix} W & 0 \\ \alpha T & W \end{pmatrix} \quad \text{and} \quad P_{PGSOR} = \frac{1}{\alpha}\begin{pmatrix} \omega W + T & 0 \\ \alpha(\omega T - W) & \omega W + T \end{pmatrix}.$$

When the matrices W and T are symmetric indefinite and satisfy one of two assumptions $-W \preceq T \prec W$ or $T \preceq W \prec -T$, where $T \prec (\preceq) W$ means the matrix $W - T$ is symmetric positive (semi-) definite, a generalisation of the PMHSS iteration method (GMHSS) was presented by Xu [29]. Recently, Cao *et al.* [12] presented generalised PMHSS (GPMHSS) and generalised LPMHSS (GLPMHSS) iteration methods, and corresponding GPMHSS and GLPMHSS preconditioners for the case when $-T \prec W \preceq T$. Their theoretical results and numerical experiments showed that the GPMHSS iteration method and its preconditioner

$$P_{GPMHSS} = -\frac{1}{2\alpha \mathrm{i}}(\alpha V + W + T)(\alpha V - W + T)\,,$$

where α is a given positive constant and $V \in \mathbb{R}^{n\times n}$ is a prescribed symmetric positive definite matrix, proved more efficient than the proposed GMHSS or GLPMHSS methods with their preconditioners. Other iteration methods and preconditioning techniques to solve the complex symmetric linear algebraic system (1.1) can be found in Refs. [1,6,9,17–19,23–25,30].

Here we discuss effective iterative solution methods of the complex indefinite linear algebraic system (1.1) when T and W are symmetric and at least one of them is nonsingular.

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When W is dominantly stronger than T, i.e. $\|W\| \gg \|T\|$ in some matrix norm sense, we construct a new block multiplicative (BM) preconditioner for the system (1.2) that reflects the matrix structure. It is proved that the BM iteration method converges unconditionally to the unique solution of the complex linear algebraic system, and the spectral properties of the preconditioned matrix are analysed. Furthermore, when W is positive definite, the eigenvalues of the BM preconditioned matrix are all real and positive, and the eigenvalue bounds of the BM preconditioned matrix are presented. When T is strongly dominant, we discuss the equivalent reformulation of (1.2), and the BM method employed to solve this system that we call a variant of the BM (VBM) method. It is not always easy to determine whether W or T is more dominant in applications, so a preconditioned BM (PBM) method for (1.2) is proposed and its convergence properties are analysed, with a quasi-optimal parameter value provided.

The implementation of our BM preconditioner and BM iteration method are presented in Section 2. In Section 3, the convergence of the BM iteration is analysed, and the spectral properties of the BM preconditioned matrix investigated. In Section 4, we discuss the VBM method and PBM method proposed for the system (1.2), and the optimal value of the accelerated parameter ω for the PBM method. Numerical experiments are presented to compare our BM preconditioner with other preconditioners in Section 5, and our final remarks are made in Section 6.

2. The Block Multiplicative Preconditioner and Iteration Implementation

We split the coefficient matrix $\mathscr{A}$ as

$$\begin{aligned}\mathscr{A} &= \begin{pmatrix} W & -T \\ T & W \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\alpha}W \end{pmatrix}\begin{pmatrix} W & -T \\ \alpha W^{-1}T & \alpha I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\alpha}W \end{pmatrix}\begin{pmatrix} I & -\frac{1}{\alpha}T \\ 0 & I \end{pmatrix}\begin{pmatrix} \widehat{W} & 0 \\ 0 & \alpha I \end{pmatrix}\begin{pmatrix} I & 0 \\ \frac{1}{\alpha}W^{-1}T & I \end{pmatrix}, \end{aligned} \tag{2.1}$$

where $\widehat{W} := W + (1/\alpha)TW^{-1}T$. From Eq. (2.1), we observe that it is difficult to solve the linear algebraic system with coefficient $\mathscr{A}$, as we need to solve two linear sub-systems with the respective coefficient matrices $\widehat{W}$ and W. In order to avoid solving the sub-system with coefficient matrix $\widehat{W}$, we remove the term $(1/\alpha)TW^{-1}T$ in $\widehat{W}$ and replace $(1/\alpha)W^{-1}T$ with $(1/\alpha)T$ to simplify the cost of computation for each iteration. Thus we construct the block multiplicative (BM) preconditioner

$$P_{\mathrm{BM}} := \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\alpha}W \end{pmatrix}\begin{pmatrix} I & -\frac{1}{\alpha}T \\ 0 & I \end{pmatrix}\begin{pmatrix} W & 0 \\ 0 & \alpha I \end{pmatrix}\begin{pmatrix} I & 0 \\ \frac{1}{\alpha}T & I \end{pmatrix} = \begin{pmatrix} W - \frac{1}{\alpha}T^2 & -T \\ \frac{1}{\alpha}WT & W \end{pmatrix}, \tag{2.2}$$

then

$$R_{\mathrm{BM}} := P_{\mathrm{BM}} - \mathscr{A} = \begin{pmatrix} W - \frac{1}{\alpha}T^2 & -T \\ \frac{1}{\alpha}WT & W \end{pmatrix} - \begin{pmatrix} W & -T \\ T & W \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha}T^2 & 0 \\ \frac{1}{\alpha}WT - T & 0 \end{pmatrix},$$

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which can be induced from the matrix splitting

$$\mathscr{A} = P_{\mathrm{BM}} - R_{\mathrm{BM}} = \begin{pmatrix} W - \frac{1}{\alpha}T^2 & -T \\ \frac{1}{\alpha}WT & W \end{pmatrix} - \begin{pmatrix} -\frac{1}{\alpha}T^2 & 0 \\ \frac{1}{\alpha}WT - T & 0 \end{pmatrix}. \tag{2.3}$$

Consequently, we can describe the BM iteration method as follows.

Method 2.1. (The BM iteration method.) Given any arbitrary initial guess $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ and a positive constant α, for $k = 0, 1, 2, \cdots$ until the iteration sequence $\{((x^{(k)})^T, (y^{(k)})^T)^T\}$ converges to the exact solution of the linear algebraic system (1.2), compute the iterate $\{((x^{(k+1)})^T, (y^{(k+1)})^T)^T\}$ according to the procedure

$$\begin{pmatrix} W - \frac{1}{\alpha}T^2 & -T \\ \frac{1}{\alpha}WT & W \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha}T^2 & 0 \\ \frac{1}{\alpha}WT - T & 0 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.4}$$

At each step in the application of the BM preconditioner with a generalised minimal residual method (GMRES), we need to solve a linear algebraic system of the form $P_{\mathrm{BM}} z = r$. Let $r = (r_1^T, r_2^T)^T$ and $z = (z_1^T, z_2^T)^T$ where $r_1, z_1 \in \mathbb{R}^n$ and $r_2, z_2 \in \mathbb{R}^n$, respectively. Then according to the definition (2.2) we have

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\frac{1}{\alpha}T & I \end{pmatrix} \begin{pmatrix} W^{-1} & 0 \\ 0 & \frac{1}{\alpha}I \end{pmatrix} \begin{pmatrix} I & \frac{1}{\alpha}T \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \alpha W^{-1} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix},$$

so we can adopt the following algorithm for the BM preconditioner.

Algorithm 2.1. (The BM preconditioned iteration.) For a given residual vector $r = (r_1^T, r_2^T)^T$, compute the update vector $z = (z_1^T, z_2^T)^T$ for the BM preconditioner in the following steps:

(1) solve $W d_1 = r_2$;
(2) compute $d_2 := T d_1 + r_1$;
(3) solve $W z_1 = d_2$;
(4) compute $z_2 := d_1 - (1/\alpha) T z_1$.

The subsystem with coefficient matrices W must be solved twice at each iteration step. When W is symmetric positive definite, direct methods such as Cholesky factorisation or inexact solvers such as the conjugate gradient (CG) method can be used to solve these linear sub-systems.

3. Convergence of the BM Iteration Method

We now discuss the convergence of the BM iteration method for the linear algebraic system (1.2). Applying straightforward derivations, we may re-express the iteration scheme (2.4) in the fixed point form

$$\tilde{z}^{(k+1)} = \Gamma \tilde{z}^{(k)} + P_{\mathrm{BM}}^{-1} b\,, \tag{3.1}$$

where $\Gamma = P_{\mathrm{BM}}^{-1} R_{\mathrm{BM}}$ is the iteration matrix of the BM iteration method, $\tilde{z}^k = [(x^k)^T, (y^k)^T]^T$ and $b = [f^T, g^T]^T$. The fixed point iteration (3.1) converges to the solution of the linear

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algebraic system (1.1) or (1.2) for an arbitrary initial guess if and only if $\rho(\Gamma) < 1$, where $\rho(\cdot)$ denotes the spectral radius of the corresponding matrix. The following theorem gives the convergence results of the BM iteration method (2.4).

Theorem 3.1. *Let $W \in \mathbb{R}^{n\times n}$ be a real symmetric nonsingular matrix, and $T \in \mathbb{R}^{n\times n}$ a symmetric matrix. Then the BM iteration* (2.4) *converges if $\rho(S) < 1$, where $S := W^{-1}T$.*

Proof. We readily obtain

$$\begin{pmatrix} W-\frac{1}{\alpha}T^2 & -T \\ T & \alpha I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -\frac{1}{\alpha}T & I \end{pmatrix}\begin{pmatrix} W^{-1} & 0 \\ 0 & \frac{1}{\alpha}I \end{pmatrix}\begin{pmatrix} I & \frac{1}{\alpha}T \\ 0 & I \end{pmatrix} = \begin{pmatrix} W^{-1} & \frac{1}{\alpha}W^{-1}T \\ -\frac{1}{\alpha}TW^{-1} & I-\frac{1}{\alpha^2}TW^{-1}T \end{pmatrix}$$

and

$$P_{\mathrm{BM}}^{-1} = \begin{pmatrix} W-\frac{1}{\alpha}T^2 & -T \\ T & \alpha I \end{pmatrix}^{-1}\begin{pmatrix} I & 0 \\ 0 & \alpha W^{-1} \end{pmatrix} = \begin{pmatrix} W^{-1} & W^{-1}TW^{-1} \\ -\frac{1}{\alpha}TW^{-1} & \alpha W^{-1}-\frac{1}{\alpha}TW^{-1}TW^{-1} \end{pmatrix},$$

so the iteration matrix of the BM iteration method can be rewritten as

$$\Gamma = P_{\mathrm{BM}}^{-1}R_{\mathrm{BM}} = \begin{pmatrix} -W^{-1}TW^{-1}T & 0 \\ \Theta & 0 \end{pmatrix} = \begin{pmatrix} -S^2 & 0 \\ \Theta & 0 \end{pmatrix} \tag{3.2}$$

with $\Theta := T - \alpha S + (1/\alpha)TS^2$. From Eq. (3.2), the spectral radius of the iteration matrix Γ satisfies $\rho(\Gamma) = \rho(S^2)$, and obviously we have $\rho(S) < 1 \Rightarrow \rho(S^2) = \rho(S)^2 < 1$ such that $\rho(\Gamma) = \rho(S^2) = \rho(S)^2 < 1$. □

For any initial guess, the BM iteration method thus converges unconditionally to the unique solution of the complex symmetric indefinite linear algebraic systems (1.1), and this is parameter independent. Further, if W is strongly dominant over T then the BM method is more efficient, since $S := W^{-1}T$.

Next, we discuss the spectral properties of the preconditioned matrix $P_{\mathrm{BM}}^{-1}\mathscr{A}$. The following theorem describes the eigenvalue distributions and the properties of the minimal polynomial of the preconditioned matrix $P_{\mathrm{BM}}^{-1}\mathscr{A}$.

Theorem 3.2. *Under the conditions of Theorem* 3.1*, the eigenvalues of the preconditioned matrix $P_{BM}^{-1}\mathscr{A}$ are* 1 *with multiplicity at least n, and the remaining eigenvalues are $1+v_i^2$ $(i = 1,2,\cdots,n)$, where the v_i $(i = 1,2,\cdots,n)$ are the eigenvalues of the matrix S. Moreover, the degree of the minimal polynomial of the preconditioned matrix $P_{BM}^{-1}\mathscr{A}$ is at most $n+1$, so the dimension of the Krylov subspace $K(P_{BM}^{-1}\mathscr{A}, b)$ is at most $n+1$.*

Proof. From Fq. (3.2), we have

$$P_{\mathrm{BM}}^{-1}\mathscr{A} = I - \Gamma = \begin{pmatrix} I+S^2 & 0 \\ -\Theta & I \end{pmatrix} = \begin{pmatrix} \Theta_1 & 0 \\ -\Theta & I \end{pmatrix} \tag{3.3}$$

where $\Theta_1 := I + S^2$. The eigenvalues of $P_{\mathrm{BM}}^{-1}\mathscr{A}$ are therefore 1 with multiplicity at least n and the eigenvalues of $I+S^2$, such that $\sigma(I+S^2) = 1+\sigma(S^2) = 1+\sigma^2(S) = 1+v_i^2$, where $\sigma(I+S^2)$ denotes the eigenvalue of the matrix $(I+S^2)$.

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From Eq. (3.3), the characteristic polynomial of $P_{\mathrm{BM}}^{-1}\mathscr{A}$ is

$$\Phi(\lambda) = det(P_{\mathrm{BM}}^{-1}\mathscr{A} - \lambda I) = (\lambda - 1)^n \prod_{i=1}^{n}(\lambda - \lambda_i),$$

where λ_i is the eigenvalue of the matrix $P_{\mathrm{BM}}^{-1}\mathscr{A}$. Let us define $\varphi(\lambda) := (\lambda-1)\prod_{i=1}^{n}(\lambda-\lambda_i)$. Evidently,

$$P_{\mathrm{BM}}^{-1}\mathscr{A} - I = \begin{pmatrix} \Theta_1 - I & 0 \\ -\Theta & 0 \end{pmatrix}$$

and

$$\prod_{i=1}^{n}(P_{\mathrm{BM}}^{-1}\mathscr{A} - \lambda_i I) = \begin{pmatrix} \prod_{i=1}^{n}(\Theta_1 - \lambda_i I) & 0 \\ * & \prod_{i=1}^{n}(I - \lambda_i I) \end{pmatrix},$$

hence

$$\varphi(P_{\mathrm{BM}}^{-1}\mathscr{A}) = (P_{\mathrm{BM}}^{-1}\mathscr{A} - I)\prod_{i=1}^{n}(P_{\mathrm{BM}}^{-1}\mathscr{A} - \lambda_i I) = \begin{pmatrix} (\Theta_1 - I)\prod_{i=1}^{n}(\Theta_1 - \lambda_i I) & 0 \\ \Theta\prod_{i=1}^{n}(\Theta_1 - \lambda_i I) & 0 \end{pmatrix}.$$

It is obvious that λ_i $(i = 1, \cdots, n)$ are also the eigenvalues of $\Theta_1 \in \mathbb{R}^{n\times n}$. From the Hamilton-Cayley theorem we have $\varphi(P_{\mathrm{BM}}^{-1}\mathscr{A}) = 0$ since $\prod_{i=1}^{n}(\Theta_1 - \lambda_i I) = 0$, so the degree of the minimal polynomial of the preconditioned matrix $P_{\mathrm{BM}}^{-1}\mathscr{A}$ is at most $n+1$. Then since the polynomial degree is equal to the dimension of the corresponding Krylov subspace (cf. Ref. [26]), the dimension of the Krylov subspace $K(P_{\mathrm{BM}}^{-1}\mathscr{A}, b)$ is also at most $n+1$. □

Remark 3.1. When the matrix W is symmetric positive definite and T is symmetric, the eigenvalues of $W^{-1}T$ are all real [27]. From Theorem 3.2, if W is symmetric positive definite, then the eigenvalues of the preconditioned matrix $P_{\mathrm{BM}}^{-1}\mathscr{A}$ are also all real and positive.

4. The VBM and PBM Methods and their Convergence

When T is strongly dominant, an equivalent reformulation of the system (1.2) is

$$\widetilde{\mathscr{A}}\tilde{z} \equiv \begin{pmatrix} T & W \\ -W & T \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g \\ -f \end{pmatrix} \equiv \tilde{b}\,. \tag{4.1}$$

For the linear algebraic system (4.1), we can construct a variant of the BM preconditioner (the VBM preconditioner):

$$P_{\mathrm{VBM}} := \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\alpha}T \end{pmatrix}\begin{pmatrix} T - \frac{1}{\alpha}W^2 & W \\ -W & \alpha I \end{pmatrix} = \begin{pmatrix} T - \frac{1}{\alpha}W^2 & W \\ -\frac{1}{\alpha}TW & T \end{pmatrix}.$$

Similar to the above BM preconditioner iteration method, we obtain the following algorithm for the implementation of the VBM preconditioner.

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Algorithm 4.1. (The VBM iteration method.) For a given residual vector $r = (r_1^T, r_2^T)^T$, compute the update vector $z = (z_1^T, z_2^T)^T$ for the VBM preconditioner in the following steps:

(1) solve $Td_1 = r_2$;
(2) compute $d_2 = -Wd_1 + r_1$;
(3) solve $Tz_1 = d_2$;
(4) compute $z_2 = d_1 + (1/\alpha)Wz_1$.

The convergence properties of the VBM iteration method and spectral properties of the VBM preconditioned matrix are quite similar to those of Theorems 3.1-3.2, so omitted here. In solving the linear algebraic system (1.1), when W is dominantly stronger than T we prefer the BM iteration, and when T is dominantly stronger than W we prefer the VBM iteration. However, often it is not easy to determine which of W and T is the more dominant, and we therefore transform the linear algebraic system (1.2) into the following preconditioned form:

$$\mathscr{A}_\omega = \begin{pmatrix} \omega W + T & W - \omega T \\ \omega T - W & \omega W + T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \omega f + g \\ \omega g - f \end{pmatrix} = \hat{b}, \tag{4.2}$$

where $\omega > 0$. For convenience, we denote $W_\omega := \omega W + T$ and $T_\omega := \omega T - W$.

We now apply the BM preconditioning to the preconditioned system (4.2), which we refer to as the preconditioned BM method (PBM). Thus we construct the preconditioned BM preconditioner

$$P_{\mathrm{PBM}} := \begin{pmatrix} I & 0 \\ 0 & \frac{1}{\alpha}W_\omega \end{pmatrix} \begin{pmatrix} W_\omega - \frac{1}{\alpha}T_\omega^2 & -T_\omega \\ T_\omega & \alpha I \end{pmatrix} = \begin{pmatrix} W_\omega - \frac{1}{\alpha}T_\omega^2 & T_\omega \\ -\frac{1}{\alpha}W_\omega T_\omega & W_\omega \end{pmatrix},$$

and similar to the above BM preconditioner iteration we have the following algorithm for the implementation of the PBM preconditioner.

Algorithm 4.2. (The PBM preconditioned iteration.) For a given residual vector $r = (r_1^T, r_2^T)^T$, compute the update vector $z = (z_1^T, z_2^T)^T$ for the PBM preconditioner in the following steps:

(1) solve $W_\omega d_1 = r_2$;
(2) compute $d_2 := T_\omega d_1 + r_1$;
(3) solve $W_\omega z_1 = d_2$;
(4) compute $z_2 := d_1 - (1/\alpha)T_\omega z_1$.

The PBM preconditioner can be induced by splitting the coefficient matrix — i.e.

$$\begin{aligned} \mathscr{A}_\omega &= P_{\mathrm{PBM}} - R_{\mathrm{PBM}} \\ &= \begin{pmatrix} W_\omega - \frac{1}{\alpha}T_\omega^2 & -T_\omega \\ \frac{1}{\alpha}W_\omega T_\omega & W_\omega \end{pmatrix} - \begin{pmatrix} -\frac{1}{\alpha}T_\omega^2 & 0 \\ \frac{1}{\alpha}W_\omega T_\omega - T_\omega & 0 \end{pmatrix}. \end{aligned}$$

Based on this splitting, we obtain the PBM iteration method as follows.

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Method 4.1. (The PBM iteration method) Given any initial guess $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ and a given positive constant α, for $k = 0, 1, 2, \cdots$ until the iteration sequence $\{((x^{(k)})^T, (y^{(k)})^T)^T\}$ converges to the exact solution of the linear algebraic system (1.2), compute the iterate $\{((x^{(k+1)})^T, (y^{(k+1)})^T)^T\}$ according to the following procedure:

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \Gamma_\omega \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + P_{\text{PBM}}^{-1} \begin{pmatrix} \omega f + g \\ \omega g - f \end{pmatrix}, \tag{4.3}$$

where

$$\Gamma_\omega = \begin{pmatrix} W_\omega - \frac{1}{\alpha} T_\omega^2 & -T_\omega \\ \frac{1}{\alpha} W_\omega T_\omega & W_\omega \end{pmatrix}^{-1} \begin{pmatrix} -\frac{1}{\alpha} T_\omega^2 & 0 \\ \frac{1}{\alpha} W_\omega T_\omega - T_\omega & 0 \end{pmatrix}$$

is the PBM iteration matrix.

Similar to the BM iterative method, the convergence properties of the PBM iteration method and the spectral properties of the preconditioned matrix can be obtained. The necessary and sufficient conditions to achieve these desired results are formally stated in the following lemma and theorem.

Theorem 4.1. *Let $W \in \mathbb{R}^{n\times n}$ and $W + T \in \mathbb{R}^{n\times n}$ be real symmetric nonsingular matrices, $T \in \mathbb{R}^{n\times n}$ a symmetric matrix, and $\omega > 0$. Assume that λ is an eigenvalue of the matrix $S_\omega := W_\omega^{-1} T_\omega$. Then there is an eigenvalue v of the matrix $S = W^{-1}T$ that satisfies*

$$\lambda = \frac{\omega v - 1}{\omega + v}. \tag{4.4}$$

Moreover, if $v = re^{i\theta}$ and $0 \le |\theta| \le \theta_0 < \pi/2$, then the PBM iteration (4.3) *is convergent if and only if one of the following conditions is satisfied:*

$$\begin{aligned} &(i) \quad r_{\min} > 1 \quad \text{and} \quad 0 < \omega < \sqrt{r_{\min}}\,, \\ \text{or } &(ii) \quad r_{\max} < 1 \quad \text{and} \quad \sqrt{r_{\max}} < \omega\,, \end{aligned}$$

where $r_{\min}$ and $r_{\max}$ are the respective smallest and largest amplitudes of the eigenvalues of S.

Proof. Let (λ, x) be an eigenpair of S_ω — i.e. $S_\omega x = \lambda x$, which is equivalent to

$$(\omega T - W)x = \lambda(\omega W + T)x\,.$$

Multiplying both sides of this equation by W_ω, we obtain

$$(\omega - \lambda)Sx = \lambda(\omega\lambda + 1)x\,,$$

hence there is an eigenvalue

$$v = \frac{\omega\lambda - 1}{\omega - \lambda}$$

of the matrix $S = W^{-1}T$ such that we readily obtain Eq. (4.4).

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To prove the second part, we assume that $\tilde{\lambda}$ is an eigenvalue of the PBM iteration matrix Γ_ω. From Theorem 4.1 and the proof of Theorem 3.1, we obtain

$$\tilde{\lambda} = -\frac{(\omega v - 1)^2}{(\omega + v)^2},$$

where v denotes an eigenvalue of S, so obviously

$$\rho(\Gamma_\omega) = \rho(S_\omega^2) = \rho(S_\omega)^2.$$

Hence

$$\rho(\Gamma_\omega) < 1 \Leftrightarrow \rho(S_\omega) < 1,$$

and replacing v by $re^{i\theta}$ in Eq. (4.4) we have

$$|\lambda| = \sqrt{\frac{r^2(\omega^4 + 1 - 2\omega^2\cos 2\theta)}{(r^2 + \omega^2 + 2\omega r\cos\theta)^2}} \leq \frac{r(\omega^2+1)}{\omega^2 + r^2},$$

such that $|\lambda| < 1$ if and only if

$$\omega^2(r-1) + r - r^2 < 0. \tag{4.5}$$

Consequently, from inequality (4.5):

$$\begin{aligned} &\omega > \sqrt{r_{\max}} && \text{when } r_{\max} < 1, \\ &0 < \omega < \sqrt{r_{\min}} && \text{when } r_{\min} > 1; \end{aligned}$$

and when $0 < r_{\min} \leq 1 \leq r_{\max}$ the inequality (4.5) does not hold for any ω. □

Theorem 4.2. *Assume that the conditions of Theorem* 4.1 *are satisfied. If* $\omega = 1$*, then the eigenvalues of the preconditioned matrix* $P_{PBM}^{-1}\mathscr{A}$ *are* 1 *with multiplicity at least n, and the others are located in a disc centred at* $1+\delta$ *with radius* $1-\delta$*, where*

$$\delta := \frac{\cos\theta_0}{1 + \cos\theta_0}.$$

Proof. Assume that λ is an eigenvalue of the matrix $P_{\mathrm{BM}}^{-1}\mathscr{A}$. From Eq. (4.4), we have

$$\lambda = 1 + \frac{(\nu-1)^2}{(\nu+1)^2} = 2 - \frac{4\nu}{(\nu+1)^2} = 2 - \frac{2}{1 + \frac{1}{2}(\frac{1}{\nu} + \nu)}, \tag{4.6}$$

and simplifying Eq. (4.6) we obtain

$$\lambda = 2 - \frac{2(1 + k\cos\theta - il\sin\theta)}{(1 + k\cos\theta)^2 + (l\sin\theta)^2}$$

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where $k := (1/2)(r + (1/r))$, $l := (1/2)(r - (1/r))$. Obviously, $k \geq 1$ and $|l| < k$, so for the real part of λ

$$2 > \Re(\lambda) = 2 - \frac{2(1 + k\cos\theta)}{(1 + k\cos\theta)^2 + (l\sin\theta)^2} \geq 2 - \frac{2}{1 + k\cos\theta} \geq \frac{2k\cos\theta_0}{1 + k\cos\theta_0} \geq \frac{2\cos\theta_0}{1 + \cos\theta_0} > 0\,.$$

For the imaginary part of λ, we have

$$\begin{aligned}\Im(\lambda) &= \frac{2|l\sin\theta|}{(1 + k\cos\theta)^2 + (l\sin\theta)^2} \\ &\leq \frac{2|l\sin\theta_0|}{(1 + k\cos\theta_0)^2 + (l\sin\theta_0)^2} \leq \frac{1 + k\cos\theta_0}{(1 + k\cos\theta_0)^2} \leq 1 - \frac{\cos\theta_0}{1 + \cos\theta_0} = 1 - \delta\,. \qquad \square\end{aligned}$$

Remark 4.1. When the matrices W and T are respectively symmetric positive definite and symmetric positive semi-definite, the matrices W_ω and T_ω are symmetric positive definite and symmetric, respectively.

Theorem 4.3. *Let $W \in \mathbb{R}^{n\times n}$ be real symmetric positive definite, $T \in \mathbb{R}^{n\times n}$ be symmetric semi-positive definite, and $\omega > 0$. Then the PBM iteration* (4.3) *is convergent if and only if one of the following conditions is satisfied:*

$$\begin{aligned}&\textit{(i)}\quad v_{\max} > 1 \quad \textit{and} \quad \frac{1 - v_{\min}}{1 + v_{\min}} < \omega < \frac{v_{\max} + 1}{v_{\max} - 1}\,,\\ &\textit{(ii)}\quad v_{\max} < 1 \quad \textit{and} \quad \frac{1 - v_{\min}}{1 + v_{\min}} < \omega\,,\end{aligned}$$

where $v_{\min}$ and $v_{\max}$ are the respective smallest and largest eigenvalues of S.

Proof. Assume that $\tilde{\lambda}$ is an eigenvalue of the PBM iteration matrix Γ_ω. From Theorem 4.3 and the proof of Theorem 3.1, we have

$$\tilde{\lambda} = -\frac{(\omega v - 1)^2}{(\omega + v)^2}\,,$$

where v is an eigenvalue of S. Obviously, $|\tilde{\lambda}| < 1$ if and only if

$$\frac{(\omega v - 1)^2}{(\omega + v)^2} < 1\,, \tag{4.7}$$

and Eq. (4.7) is equivalent to

$$\begin{cases}\omega(v + 1) > 1 - v\,,\\ \omega(v - 1) < v + 1\,.\end{cases} \tag{4.8}$$

If $v_{\min} - 1 \geq 0$, then $v + 1 > v - 1 > 0$ for all $v \in \sigma(S)$, hence the inequality (4.8) holds if and only if

$$\frac{1 - v_{\min}}{1 + v_{\min}} < \omega < \frac{v_{\max} + 1}{v_{\max} - 1}\,.$$

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If $1 > v_{\max} > v_{\min} \geq 0$, then $v + 1 > 0 > v - 1$ for all $v \in \sigma(S)$, hence the inequality (4.8) holds if and only if

$$\frac{1 - v_{\min}}{1 + v_{\min}} < \omega \, .$$

If $v_{\max} > 1 > v_{\min} > 0$, then $v + 1 > 0$ for all $v \in \sigma(S)$, hence the inequality (4.8) holds if and only if

$$\frac{1 - v_{\min}}{1 + v_{\min}} < \omega < \frac{v_{\max} + 1}{v_{\max} - 1} \, .$$

Combining the above three cases completes the proof. □

Lemma 4.1 (cf. Ref. [21]). *Let $W \in \mathbb{R}^{n \times n}$ be a real symmetric positive definite matrix, $T \in \mathbb{R}^{n \times n}$ be a symmetric semi-positive definite matrix, and $\omega > 0$. Assume that λ is an eigenvalue of the matrix $S_\omega := W_\omega^{-1} T_\omega$. Then*

$$\rho(S_\omega) = \max\left\{ \frac{\omega v_{\max} - 1}{\omega + v_{\max}} \, , \, \frac{1 - \omega v_{\min}}{\omega + v_{\min}} \right\} ,$$

where $v_{\min}$ and $v_{\max}$ are the smallest and largest eigenvalues of S, respectively.

We now provide a quasi-optimal value ω^* by minimising the spectral radius of the iterative matrix of the PBM method — i.e.

$$\rho(\Gamma_{\omega^*}) = \min_{\omega} \rho(\Gamma_\omega) \, .$$

Based on Lemma 4.1 and Theorem 3.1, we obtain the following results.

Lemma 4.2. *Let $W \in \mathbb{R}^{n \times n}$ be a real symmetric positive definite matrix, $T \in \mathbb{R}^{n \times n}$ be a symmetric semi-positive definite matrix, and $\omega > 0$. Assume that λ is an eigenvalue of the matrix $S_\omega := W_\omega^{-1} T_\omega$. Then the optimal value of the parameter ω that minimises the spectral radius $\rho(\Gamma_\omega)$ is given by*

$$\omega^* = \frac{1 - v_{\min} v_{\max} + \sqrt{1 + v_{\min}^2}\sqrt{1 + v_{\max}^2}}{v_{\max} + v_{\min}} \, , \tag{4.9}$$

and the corresponding optimal convergence factor by

$$\rho(\Gamma_{\omega^*}) = \frac{\left(\omega^* v_{\max} - 1\right)^2}{\left(\omega^* + v_{\max}\right)^2},$$

where $v_{\min}$ and $v_{\max}$ are the respective smallest and largest eigenvalues of S.

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5. Numerical Experiments

Numerical examples involving two-dimensional Helmholtz equations and some other applications are now considered, to examine the effectiveness of the BM and VBM iteration as solver and also preconditioner for the GMRES in solving the complex symmetric linear algebraic systems (1.1). We compare the BM or VBM, PBM iteration with the PMHSS [8], GSOR [27], PGSOR [21], AMG [3] and GPMHSS [12] as solvers and preconditioners. All tests were performed in MATLAB R2013a, with machine precision 10^{-16} on a personal computer with 3.20 GHz CPU (Intel(R) Core(TM) i5-3470), 8.00 GB RAM . All tests started with the zero vector, and the iteration was terminated when the current residual in the first stage satisfied RES $:= \|r_k\|/\|r_0\| < 10^{-7}$ or the prescribed iteration $k_{\max} = 1000$ was exceeded. In the following tables, we denote the number of iteration steps as "IT", the elapsed CPU time in seconds as "CPU", and the mark "–" means the iterative method did not converge in the prescribed maximum number of iterations.

Example 5.1 (cf. Refs. [3,27])**.** Consider the linear algebraic system $(W + \mathrm{i}T)u = c$ where

$$T = I \otimes V + V \otimes I \quad \text{and} \quad W = 10(I \otimes V_c + V_c \otimes I) + 9(e_1 e_m^T + e_m e_1^T) \otimes I \,,$$

with $V = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}$, $V_c = V - e_1 e_m^T - e_m e_1^T$ and e_1 and e_m the first and last unit vectors in $\mathbb{R}^m$, respectively. We took the right-hand side vector c to be $c = (1 + \mathrm{i})\mathbf{A1}$, where $\mathbf{1}$ denotes the vector with all entries 1. Here T and W correspond to the five-point centred difference matrices that approximate the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions respectively, on a uniform mesh in the unit square $[0,1]\times[0,1]$ with mesh-size $h = 1/(m+1)$. We also took $m = 64, 128, 256$. For this example, by simple computation we find that the matrix W is strongly dominant over T, so we used the BM rather than the VBM iteration.

Example 5.2 (cf. Refs. [3, 11, 12, 27])**.** Consider the linear algebraic system

$$\big[\big(K - \gamma^2 M\big) + i\big(\gamma C_V + C_H\big)\big]u = c \,,$$

where M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, and γ is the driving circular frequency. We took $C_H = \mu K$ with μ a damping coefficient, $M = I$, $C_V = 10I$, and K the five-point centred difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions, on a uniform mesh in the unit square $[0,1]\times[0,1]$ with mesh-size $h = 1/(m+1)$. The matrix $K \in \mathbb{R}^{n\times n}$ has the tensor-product form

$$K = I \otimes V_m + V_m \otimes I, \quad \text{with} \quad V_m = h^{-2}\text{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m} \,.$$

In this example, we took the right-hand side vector $c = (1 + i)\mathbf{A1}$, where $\mathbf{1}$ denotes the vector with all entries 1. In the implementation, we took $m = 64, 128, 256$, and adopted the parameter values $\gamma = 10$ and $\mu = 500, 1800, 7000$ for three test sizes. As before, we normalised the system by multiplying both sides by h^2. For this example, by simple computation we find that the matrix T is strongly dominant over W, so we used the VBM rather than the BM iteration.

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Example 5.3. Consider the linear algebraic system

$$\left[\left(K+\frac{(3-\sqrt{3})}{\tau}I\right)+i\left(K+\frac{(3+\sqrt{3})}{\hat{\omega}}I\right)\right]u=c\ ,$$

where the parameters τ and $\hat{\omega}$ are both positive, and K is the five-point centred difference matrix that approximates the negative Laplacian operator $L=-\Delta$ with homogeneous Dirichlet boundary conditions on a uniform mesh in the unit square $[0,1]\times[0,1]$ with mesh-size $h=1/(m+1)$. The matrix $K\in\mathbb{R}^{n\times n}$ has the tensor-product form

$$K=I\otimes V_m+V_m\otimes I,\quad \text{with}\quad V_m=h^{-2}\text{tridiag}(-1,2,-1)\in\mathbb{R}^{m\times m}\ ,$$

so K is an $n\times n$ block-tridiagonal matrix with $n=m^2$. As for Example 4.2, we took the right-hand side vector $c=(1+i)\mathbf{Al}$ where $\mathbf{l}$ is the vector with all entries 1, the grid sizes $m=64,128,256$, and chose the parameter $\widehat{\omega}=\tau=h$.

Example 5.4 (cf. Refs. [21,27])**.** Consider the complex Helmholtz equation

$$-\Delta p+\sigma_1 p+i\sigma_2 p=f\ ,$$

where Δ is the Laplacian operator, σ_1 and σ_2 are real functions, and p satisfies Dirichlet boundary conditions in $D=[0,1]\times[0,1]$. We adopted finite differences on a $m\times m$ grid with mesh size $h=1/(m+1)$, which produced the system of linear equations

$$\big((K+\sigma_1 I)+i\sigma_2 I\big)u=c\ ,$$

where $K=I\otimes V_m+V_m\otimes I$ is the discretisation of $-\Delta$ involving centred differences, and $V_m=h^{-2}\text{tridiag}(-1,2,-1)\in\mathbb{R}^{m\times m}$. The right-hand side vector c was taken to be $c=(1+\text{i})\mathbf{A1}$, where $\mathbf{1}$ denotes the vector with all entries 1. Furthermore, before solving the system we normalised the coefficient matrix and the right-hand side vector by multiplying both by h^2. For this example, we took $m=64,128,256$ for three test sizes, and the parameters $\sigma_1=100$ and $\sigma_2=100$.

In the implementation, the parameter ω^* was chosen for the PGSOR and the PBM method using (4.9), while for the PMHSS method and GPMHSS method we let $V=W$ and $V=T$. Further, the optimal parameter values of α for the GSOR, PGSOR, PMHSS and GPMHSS methods were chosen based on Refs. [21,27]. From the discussion of Section 3, our method is independent of the parameter α, so in principle we can arbitrarily select α, e.g., where we choose $\alpha=10$.

Numerical results for Example 5.1 are listed in Table 1, showing the IT and CPU for the GSOR, PGSOR, BM, PMHSS, GPMHSS and AMG methods and corresponding results for their preconditioned Krylov subspace methods, such as preconditioned GMRES methods. We see that the iteration counts with the GSOR, PGSOR, BM, PBM, PMHSS and AMG methods grow slowly with problem size, while the GPMHSS method does not converge as a solver. In terms of iteration numbers, all of these preconditioners are almost constant and small. However, the BM methods considerably outperform the GSOR, PGSOR, PMHSS

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Table 1: Numerical results for Example 5.1.

	Alg	GSOR	PGSOR	BM	PBM	PMHSS	GPMHSS	AMG
	IT	6	6	5	6	32	–	8
m=64	CPU	0.0342	0.0356	0.0322	0.0341	0.0661	–	0.0459
	IT	6	6	5	6	32	–	8
$m = 128$	CPU	0.2421	0.2552	0.2466	0.2201	0.3892	–	0.2675
	IT	6	6	5	6	32	–	8
$m = 256$	CPU	2.1685	2.0572	1.9673	2.0375	2.7338	–	2.0359
	Pre	P_{GSOR}	P_{PGSOR}	P_{BM}	P_{PBM}	P_{PMHSS}	P_{GPMHSS}	P_{AMG}
	IT	2	2	3	2	3	2	2
$m = 64$	CPU	0.0340	0.0315	0.0313	0.0308	0.0390	0.0608	0.0345
	IT	2	2	2	2	2	2	2
$m = 128$	CPU	0.2263	0.2168	0.2004	0.2316	0.2387	0.4407	0.2190
	IT	2	2	2	2	2	1	2
$m = 256$	CPU	1.8889	1.8304	1.8990	1.9156	1.9687	3.7079	1.9055

Table 2: Numerical results for Example 5.2.

	Alg	GSOR	PGSOR	VBM	PBM	PMHSS	GPMHSS	AMG
	IT	–	4	3	3	41	20	5
m=64	CPU	–	0.0290	0.0256	0.0194	0.0513	0.0459	0.0258
	IT	–	4	3	3	41	20	4
$m = 128$	CPU	–	0.1329	0.1110	0.1237	0.2844	0.2759	0.1120
	IT	–	4	3	3	41	20	4
$m = 256$	CPU	–	0.8625	0.7137	0.7989	1.5597	1.7859	0.7055
	Pre	P_{GSOR}	P_{PGSOR}	P_{VBM}	P_{PBM}	P_{PMHSS}	P_{GPMHSS}	P_{AMG}
	IT	8	2	3	2	5	2	2
$m = 64$	CPU	0.0546	0.0552	0.0369	0.0252	0.0246	0.0340	0.0395
	IT	8	2	2	2	4	2	2
$m = 128$	CPU	0.3072	0.1787	0.1547	0.1618	0.1283	0.1981	0.1881
	IT	8	2	2	2	4	2	2
$m = 256$	CPU	1.7829	0.7874	0.7612	0.7751	0.8802	1.3786	0.7717

methods in terms of iteration count and CPU time, as both solver and preconditioner for Krylov subspace methods such as GMRES.

Numerical results for Example 5.2 are listed in Table 2, showing the IT and CPU for the GSOR, PGSOR, VBM, PBM, PMHSS, GPMHSS and AMG iterative methods, and results for these preconditioned GMRES methods. We see that the iteration steps for all methods are essentially constant after a slight decrease in going from $m = 64$ to $m = 256$. Further, the VBM iterative methods considerably outperform the GSOR, PMHSS, AMG and GPMHSS methods in terms of iteration counts and CPU times, while the GSOR method does not converge as a solver. As preconditioners for Krylov subspace methods such as GMRES, all

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Table 3: Numerical results for Example 5.3.

	Alg	GSOR	PGSOR	PBM	PMHSS	GPMHSS	AMG
	IT	27	8	7	23	33	17
m=64	CPU	0.412	0.0287	0.0266	0.0373	0.0596	0.0377
	IT	27	8	7	23	33	16
$m = 128$	CPU	0.2248	0.1323	0.1237	0.1995	0.3415	0.1751
	IT	28	8	8	23	33	16
$m = 256$	CPU	1.3702	0.8898	0.7989	1.1633	2.3271	0.9826
	Pre	P_{GSOR}	P_{PGSOR}	P_{PBM}	P_{PMHSS}	P_{GPMHSS}	P_{AMG}
	IT	8	5	4	9	7	5
$m = 64$	CPU	0.0434	0.0241	0.0226	0.0314	0.0455	0.0460
	IT	12	5	5	9	7	5
$m = 128$	CPU	0.2912	0.1360	0.1320	0.1676	0.2495	0.1460
	IT	14	5	5	10	7	6
$m = 256$	CPU	1.3398	0.8684	0.8667	1.0609	1.1518	0.9055

of these methods perform efficiently in both iteration count and CPU time, especially when the mesh-size h becomes small. Moreover, the iteration steps of all of these preconditioners remain nearly constant, and the PBM as a preconditioner for a Krylov subspace method costs less than the other methods.

Numerical results for Example 5.3 are listed in Table 3, showing the IT and CPU for the GSOR, PGSOR, PBM, PMHSS, AMG and GPMHSS methods and results for the corresponding preconditioned GMRES. We see that the iteration counts for the PGSOR, PBM, PMHSS, AMG and GPMHSS methods are essentially constant, and the PBM, AMG and PGSOR methods vastly outperform the GSOR, PMHSS and GPMHSS methods in terms of both iteration count and CPU time. When the GSOR, PGSOR, PBM, PMHSS, AMG and GPMHSS iteration act as preconditioner for Krylov subspace methods such as GMRES, the PBM method requires less iteration steps than all other iteration methods, while the GSOR method does not converge as a solver nor as a preconditioner. The PGSOR, AMG and PMHSS methods take less time than the GPMHSS method, and the PBM method less time than all of the other methods, so the PBM iteration is competitive and feasible as a solver or a preconditioner for Krylov subspace methods.

Numerical results for Example 5.4 are listed in Table 4, where we see that the iteration counts for the GSOR, PGSOR, PBM, AMG and PMHSS methods are essentially constant and converge quickly, whereas the GPMHSS method is not convergent. Moreover, the PBM method vastly outperforms the GSOR, PMHSS methods in terms of both iteration counts and CPU times. The results of the GMRES preconditioned methods with the GSOR, PGSOR, PMHSS, AMG and GPMHSS show that the PBM method is better as a solver than as a preconditioner, and the PBM preconditioner is more efficient than the GSOR, PMHSS and GPMHSS preconditioners.

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Table 4: Numerical results for Example 5.4.

	Alg	GSOR	PGSOR	PBM	PMHSS	GPMHSS	AMG
	IT	9	5	7	33	–	17
m=64	CPU	0.0292	0.0199	0.0208	0.0406	–	0.0358
	IT	9	5	7	34	–	17
$m = 128$	CPU	0.1352	0.1217	0.1254	0.2371	–	0.1702
	IT	9	5	7	35	–	17
$m = 256$	CPU	0.9702	0.8116	0.8898	1.3633	–	0.9887
	Pre	P_{GSOR}	P_{PGSOR}	P_{PBM}	P_{PMHSS}	P_{GPMHSS}	P_{AMG}
	IT	7	4	4	14	21	7
$m = 64$	CPU	0.0264	0.0277	0.0216	0.0450	0.1404	0.0546
	IT	6	4	4	14	21	7
$m = 128$	CPU	0.1295	0.1332	0.1316	0.1676	0.6068	0.1750
	IT	7	4	4	14	21	7
$m = 256$	CPU	0.8502	0.8209	0.6967	1.1262	3.0918	0.9324

6. Concluding Remarks

We have considered preconditioned iteration methods to solve a class of complex symmetric linear algebraic systems. Some new preconditioners and their corresponding iteration methods are proposed, and conditions to guarantee convergence of these methods are derived. Spectral properties of the preconditioned matrix are discussed, and it is shown that the new methods are independent of the parameter α. Both theoretical analysis and numerical experiments show that these iteration methods and preconditioners are feasible and effective.

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