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A rigidity theorem for simply connected manifolds without conjugate points

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Abstract. In this paper we show that manifolds without conjugate points exhibit rigidity phenomena similar to that studied in [**BGS**, Section I.5]. The main theorem is that if X is a complete, simply connected Riemannian manifold without conjugate points, and $M = X \times R$ is given the Riemannian product metric g, then any metric without conjugate points on M which agrees with g outside a compact set is isometric to g.

1. *Introduction*

If (M, g_0) is a complete Riemannian manifold satisfying some property $\mathcal{P}$, then we will say that (M, g_0) is rigid under compactly supported perturbations (respecting the property $\mathcal{P}$) if any metric g_1 on M which satisfies property $\mathcal{P}$ and coincides with g_0 outside a compact set $K \subset M$ is isometric to g_0. Euclidean space was shown to be rigid under compactly supported perturbations with nonpositive (or nonnegative) sectional curvature in [**GrWu**]. In [**BGS**, pp. 69] (see also [**SZ**]), certain classes of simply connected manifolds with nonpositive curvature were shown to be rigid under compactly supported perturbation; for instance, if (M, g) is a symmetric space of noncompact type of rank ≥ 2, or if (M, g) is a Riemannian product of manifolds with nonpositive sectional curvature, then (M, g) is rigid under compactly supported perturbations with nonpositive curvature.

Manifolds without conjugate points enjoy many of the global properties of manifolds of nonpositive curvature. [**GrGu, Cr2, Cr1**] established that metrics without conjugate points on Euclidean space are rigid under compactly supported perturbations having no conjugate points. The purpose of this paper is to extend this last rigidity statement to a much larger class of manifolds without conjugate points.

THEOREM 1.1. *Let X be a 1-connected Riemannian manifold without conjugate points, and let g_0 be the Riemannian product metric on $X \times R$. Then g_0 is rigid under compactly supported deformations having no conjugate points.*

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Given the similarities between manifolds without conjugate points and manifolds with nonpositive curvature, Theorem 1.1 leads us to the following.

CONJECTURE 1.2. *If (M, g_0) is a complete Riemannian manifold without conjugate points, then any metric without conjugate points g on M agreeing with g_0 outside a compact set must satisfy* $\operatorname{rank}(g_1) = \operatorname{rank}(g_0)$. *(Here the rank of (M, g_i) is the largest integer k such that every geodesic $\gamma : R \longrightarrow M$ in (M, g_i) factors through a totally geodesic isometric immersion $i : E^k \longrightarrow M$).*

When M is simply connected, the methods of this paper reduce conjecture 1.2 to the problem of 'lens rigidity' for manifolds with rank ≥ 2 (see Remark 2.2). We should point out that not all such theorems about nonpositive curvature extend to the no conjugate points setting. In particular, the example given in [**Kl**] shows that the main theorem in [**AS**] does not extend to manifolds without conjugate points, while the example in [**Bu**] shows that the flat strip theorem does not hold in manifolds without conjugate points. In fact, almost none of the techniques used to prove rigidity results for manifolds of nonpositive curvature are applicable in our case, and hence the proofs in this paper are completely different.

Outline of proof. The proof of Theorem 1.1 breaks up into three parts. The first part (Proposition 2.1) uses the properties of Busemann functions to show that the two metrics are 'lens equivalent', i.e. if γ^0 is a g_0-geodesic such that $\gamma^0(t) \notin K$ for $|t| > T$ then there is a g_1-geodesic γ^1 such that $\gamma^0(t) = \gamma^1(t)$ for all $|t| > T$. The second part uses this lens equivalence, along with the invariance of the Liouville measures under the geodesic flows, to conclude (Corollary 2.3) that for any domain D with $K \subset D$ we have $\operatorname{Vol}_{g_0}(D) = \operatorname{Vol}_{g_1}(D)$. In the third part we use lens equivalence along with an extremal property of the Jacobi equation in product manifolds to show that for such D we have $\operatorname{Vol}_{g_0}(D) \geq \operatorname{Vol}_{g_1}(D)$ with equality holding if and only if the metrics are isometric.

In a companion paper [**CrKl**], we generalize Theorem 1.1 by allowing certain manifolds X with nontrivial fundamental group. In particular, we show that any noncompact flat manifold is rigid under compactly supported perturbations having no conjugate points. The Hopf conjecture, recently proved by Burago and Ivanov [**BuIv**], proves the compact case. Although the proof in these cases involves analogues of the three steps mentioned above, a number of new ideas are required, leading to considerably longer proofs. The main new complication is that some geodesics remain in a compact region K forever; in particular, it becomes important to control the measure of the set of such geodesics.

2.

2.1. *Notation.* We will always assume that geodesics are parametrized by arclength unless otherwise stated. We will be considering two metrics g_i, $i = 0, 1$, on a given manifold M. $S(M, g_i)$ will denote the unit sphere bundle of (M, g_i), $S_p(M, g_i)$ the fiber of $S(M, g_i)$ at $p \in M$. If $v \in S(M, g_i)$, then γ_v^i will denote the g_i-geodesic with initial velocity v, $\exp^i : TM \longrightarrow M$ is the exponential map for g_i, $\Phi_t^i : S(M, g_i) \longrightarrow S(M, g_i)$ is the geodesic flow after time t. $d^i : M \times M \longrightarrow R$ is the distance function associated with g_i, and if $x \in M$ then $d_x^i : M \longrightarrow R$ is the distance from x.

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When $\pi_1(M) = \{e\}$ and g_i has no conjugate points, then $\gamma_v^i|_{[0,\infty]}$ is a ray for every $v \in S(M, g_i)$. The Busemann function b_v^i associated with $v \in S(M, g_i)$ is given by $b_v^i(x) = \lim_{t\to\infty}(d^i(x, \gamma_v^i(t)) - t)$. The convergence in this limit is in the C^1 topology while the functions b_v^i are $C^{1,1}$ (see [**E**]). In particular we see that $(\nabla b_v^0)(p) = \lim_{t\to\infty}(\nabla d^i_{\gamma_v^i(t)})(x)$. We define horoballs by $B_v^i(x) = (b_v^i)^{-1}((-\infty, x])$. The g_i-horoball (respectively g_i-horosphere) generated by $v \in S(M, g_i)$ is $B_v^i \equiv B_v^i(0)$, (respectively $\partial B_v^i = (b_v^i)^{-1}(0)$). If B is a g_i-horoball and w is an outward normal at some $p \in \partial B$, then by the triangle inequality B_w^i is disjoint from Interior(B). See [**E**] for a discussion of horospheres in manifolds without conjugate points.

If $K \subset M$ is a closed set, we let Chord$^i(K)$ be the union of the g_i-geodesic segments with endpoints in K; if $g_1|_{M-K} = g_0|_{M-K}$ then Chord$^0(K) =$ Chord$^1(K) \equiv$ Chord(K).

2.2. *Lens rigidity.* The first step in the proof of Theorem 1.1 is to show that if a manifold without conjugate points, M, has rank$(M) = k \geq 2$, then the geodesic flow of M is 'rigid' under compactly supported perturbations without conjugate points.

PROPOSITION 2.1. *Let (M, g_0) be a 1-connected manifold without conjugate points, and assume that* rank$(M, g_0) = k \geq 2$ *in the sense that every $v \in S(M, g_0)$ is tangent to a k-flat (i.e. a totally geodesic copy of E^k). Let g_1 be a compactly supported perturbation of g_0 having no conjugate points, and let $K \subset M$ be a compact subset such that $g_1|_{M-K} = g_0|_{M-K}$. If $x, y \in M -$* Chord(K)*, then:*

1. $d^1(x, y) = d^0(x, y)$;
2. $(\exp_x^1)^{-1}(y) = (\exp_x^0)^{-1}(y)$.

Proof. Our first goal is to show that $b_v^1(y) = b_v^0(y)$ for all $y \notin$ Chord(K) provided $\gamma_v^i([0, \infty)) \cap K = \emptyset$.

If $v \in S(M, g_i)$, then we define Shadow$^i(K, v)$ to be the set of $x \in M$ such that the g_i-gradient line of $-b_v^i$ starting at x hits K. In particular, if $y \notin$ Shadow$^i(K, v)$ then for all sufficiently large t the g_i-geodesic segment from y to $\gamma_v^i(t)$ is disjoint from K. If $\gamma_v^i([0, \infty)) \cap K = \emptyset$ then Shadow$^0(K, v) =$ Shadow$^1(K, v) \equiv$ Shadow(K, v), and for all $y \notin$ Shadow(K, v), $b_v^0(y) = b_v^1(y)$.

By definition, a $(k-1)$-flat F in (M, g_0) is a v-admissible $(k-1)$-flat if v is in the unit normal bundle of F, and there is a k-flat $\bar{F} \subset M$ containing F which is tangent to v.

Pick $v \in S(M, g_0)$, and let $\bar{F}$ be a k-flat tangent to v. Then $\gamma_v^0 \subset \bar{F}$, and $b_v^0|_{\bar{F}}$ is a Busemann function on $\bar{F} \simeq E^k$. Hence $F = (b_v^0)^{-1}(0) \cap \bar{F}$ is a hyperplane in $\bar{F}$, and a v-admissible $(k-1)$-flat in (M, g_0). If $w = -(\nabla b_v^0)(p)$ for some $p \in \bar{F}$, then $b_w^0 = b_v^0 + c$. This follows since clearly $b_w^0 = b_v^0 + c$ on $\bar{F}$, hence on the geodesics γ_v and γ_w, and an easy triangle inequality argument (see, for example, the proof of Proposition 2.5 of [**CS**]) then shows $b_w^0 = b_v^0 + c$ everywhere.

We now consider $v \in S(M, g_0)$ such that $\gamma_v^0([0, \infty)) \cap K = \emptyset$. Pick r_1, r_2 such that for the horoballs $B_v^0(r_1)$ and $B_v^1(r_2)$ we have $K \subset B_v^0(r_1) \subset B_v^1(r_2)$; this is possible because K and $B_v^0(r) \cap$ Shadow(K, v) is compact for every r. We claim that $B_v^1(r_2) = B_v^0(r_2)$. To see this note that if w is an outward normal to $\partial(B_v^1(r_2))$ then B_w^1 is disjoint from Interior$(B_v^1(r_2))$, and since $K \subset B_v^1(r_2)$ we have $B_w^1 = B_w^0$. Further, $B_v^1(r_2)$ is supported at $\pi(w)$ by any w-admissible $(k-1)$-flat F (since $F \subset \partial B_w^1$), while

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$F \subset \partial B_v^0(r_2)$. If we let w run over all outward normals to $B_v^1(r_2)$ based at points in $(M - \text{Shadow}(K, v))$ then the union of the w-admissible $(k-1)$-flats is disjoint from $\text{Interior}(B_v^1(r_2))$ and it coincides with $\partial B_v^0(r_2)$. This implies that $B_v^1(r_2) \subseteq B_v^0(r_2)$. Let $r_3 = \text{Max}\{r \mid B_v^0(r) \subseteq B_v^1(r_2)\} = \text{Min}(b_v^0|_{\partial(B_v^1(r_2))})$. Then $B_v^0(r_3) \subseteq B_v^1(r_2)$ is tangent to $\partial B_v^1(r_2)$ at some point p. Let w be the common outward normal to $\partial B_v^1(r_2)$ and $\partial B_v^0(r_3)$ at p. Then any w-admissible $(k-1)$-flat F is contained in $\partial B_v^0(r_3)$ and supports $\partial B_v^1(r_2)$ from the outside, forcing $F \subset \partial B_v^1(r_2)$. But since $b_v^0(x) = r_3$ and $b_v^1(x) = r_2$ for $x \in F$ and $b_v^0(x) = b_v^1(x)$ for $x \in F - \text{Shadow}(K, v) \neq \emptyset$, we conclude that $r_3 = r_2$ and $B_v^1(r_2) = B_v^0(r_2)$.

We now know that $b_v^0(x) = b_v^1(x)$ when $x \notin \text{Shadow}(K, v)$ or when $b_v^0(x) \geq \text{Max}(b_v^0|_K)$. Pick $x \notin \text{Chord}(K)$, and let $w = (\nabla b_v^0)(x)$. If $\gamma_w^0([0, \infty)) \cap K = \emptyset$, then for large enough t, $\gamma_w^0(t)$ will lie in the region where $b_v^1 = b_v^0$, so $(\nabla b_v^0)(\gamma_w^0(t)) = (\nabla b_v^1)(\gamma_w^0(t))$, which implies that $\gamma_w^0|_{[0,\infty)}$ is a gradient line for both b_v^0 and b_v^1, which implies that $b_v^1(x) = b_v^0(x)$. On the other hand, if $\gamma_w^0((-\infty, 0]) \cap K = \emptyset$ then for large enough t, $\gamma_w^0(-t)$ will lie in the region where $b_v^1 = b_v^0$ and similar reasoning gives $b_v^0(x) = b_v^1(x)$ as desired.

If $x, y \notin \text{Chord}(K)$, consider the g_0-geodesic segment η from x to y. If η lies in $M - K$, then $d^0(x, y) = \text{length}_{g_0}(\eta) = \text{length}_{g_1}(\eta) = d^1(x, y)$. Otherwise, letting $v = -\eta'(0) \in S_x(M, g_0)$, we have $\gamma_v^0([0, \infty)) \cap K = \emptyset$, so $b_v^1(y) = b_v^0(y)$ by the arguments of the previous paragraphs. Hence

$$\begin{aligned} d^0(x, y) &= b_v^0(y) - b_v^0(x) = b_v^1(y) - b_v^1(x) \\ &\leq d^1(x, y) \end{aligned}$$

since b_v^1 has Lipschitz constant 1. Symmetric reasoning gives $d^1(x, y) \leq d^0(x, y)$, so we have part 1 of Proposition 2.1. The second part of Proposition 2.1 follows immediately from the first, since $(\exp_x^i)^{-1}(y) = -d^i(x, y)(\nabla d_y^i)(x)$. □

Remark 2.2. Proposition 2.1 reduces Conjecture 1.2—in the simply connected case—to the 'lens rigidity' question posed in [**Cr2**].

By the proposition, we may define a diffeomorphism $F : S(M, g_0) \longrightarrow S(M, g_1)$ by $F(v) = \Phi_t^1(\Phi_{-t}^0(v))$, where $t \in R$ is chosen so that $\Phi_s^0(v) \in S(M - K, g_0) = S(M - K, g_1)$ for all $s \leq -t$. Using Proposition 2.1 and the fact that the geodesic flow preserves the canonical measure on $S(M, g_i)$, it is easy to verify that F satisfies:

1. F is smooth;
2. $F(-v) = -F(v)$;
3. If $\pi(v) \notin \text{Chord}(K)$, then $F(v) = v$;
4. F is a measure-preserving map.

Hence we have the following.

COROLLARY 2.2. *If $D \subset M$ is a domain containing K, then* $\text{Vol}_{g_0}(D) = \text{Vol}_{g_1}(D)$.

Proof. The triangle inequality implies that $\text{Chord}(K)$ is compact, so we first assume that $\text{Chord}(K) \subset D$.

$F(S(M - D, g_0)) = S(M - D, g_1)$ by property 3, so $F(S(D, g_0)) = S(D, g_1)$. Hence

$$\text{Vol}_{g_0}(D) = \frac{1}{c_n} \text{Vol}_{g_0}(S(D, g_0)) = \frac{1}{c_n} \text{Vol}_{g_1}(S(D, g_1)) = \text{Vol}_{g_1}(D),$$

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where c_n is the volume of the standard unit sphere in E^n.

We now know that $\mathrm{Vol}_{g_0}(K) = \mathrm{Vol}_{g_1}(K)$, so the corollary follows immediately. □

2.3. *The proof of Theorem 1.1.* Let $M = X \times R$, and let g_1 be a metric without conjugate points on M such that $g_1|_{M-K} = g_0|_{M-K}$, where $K = K_1 \times [a, b]$, and $K_1 \subset X$ is a compact domain with smooth boundary. Since g_0 has rank ≥ 2, Proposition 2.1 applies, and we conclude that g_1-geodesics coincide with g_0-geodesics outside the slab $X \times [a, b]$. For each $x \in K_1$ let γ_x^i be the g_i-geodesic such that $\gamma_x^i(t) = (x, t)$ for all $t \notin [a, b]$. Thus we have a well-defined map $G : K \longrightarrow K$ given by $G(x, t) = \gamma_x^1(t)$.

For fixed $x \in K$ and parallel orthonormal frames along the γ^i, using arguments from [**Cr2**, Section 6], we see that the differential $D(G)(x, t)$ represents a nonsingular Jacobi tensor along $\gamma_x^1(t)$ that is orthogonal for $t \notin [a, b]$, and that

$$\int_a^b D(G)(x,t)^{-1} D(G)(x,t)^{-1*}\, dt = (b-a)Id.$$

Using the fact (as in [**Cr2**]) that $\mathrm{Det}^{-1/2}$ is strictly convex on the space of positive definite symmetric matrices we see that $\int_a^b \mathrm{Jac}(G)(x,t)\, dt \geq b - a \ \forall x \in K_1$, with equality holding if and only if with respect to parallel frames along the γ_x^i, $D(G)(x, t)$ is a fixed orthogonal matrix. (A complete argument, in a slightly more general context than we have here, can also be found in the last section of [**CrKl**].)

Hence we estimate

$$\begin{aligned} \mathrm{Vol}_{g_1}(K) &= \int_K \mathrm{Jac}(G)\, \mathrm{Vol}_{g_0} = \int_{K_1} \left(\int_a^b \mathrm{Jac}(G)(x,t)\, dt \right) dx \geq \int_{K_1} (b-a)\, dx \\ &= \mathrm{Vol}_{g_0}(K). \end{aligned}$$

By Corollary 2.3 equality holds in the above inequality, so

$$\int_a^b \mathrm{Jac}(G)(x,t)\, dt = b - a \quad \forall x \in K_1,$$

which forces G to be an isometry. □

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