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# Inverse limit stability for semiflows

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*Abstract*. On semiflows on Banach manifolds, Quandt (1988 and 1989) stated that some special semiflows were inverse limit stable. However, the concepts and proofs used to show inverse limit stability of semiflows are rough and ambiguous. We consider semiflows on finite-dimensional compact manifolds or on finite-dimensional compact non-singular branched manifolds. In this paper we show that more special semiflows are inverse limit stable. That is, Anosov semiflows are inverse limit stable.

## 1. *Introduction*

For diffeomorphisms (respectively flows) of *closed manifolds*, the theory of structural stability has been established. For a geometric study of evolution equations, Quandt [**15, 16**] *outlined* a corresponding theory for endomorphisms (respectively semiflows) of *Banach manifolds* (or Banach spaces). To extend the stability theory from diffeomorphisms (respectively flows) to endomorphisms (respectively semiflows) structural stability was generalized to inverse limit stability [**10, 13**]. Inverse limit stability does not guarantee preservation of the topological dynamics. However inverse limit stability gives the one-to-one correspondence between all global solutions for any two semiflows near the original semiflow. On semiflows on Banach manifolds, Quandt [**16**] stated that some special semiflows were inverse limit stable. However, the concept and the proof used to show inverse limit stability of semiflows are rough and ambiguous. In order to give a rigorous proof we consider an appropriate setting in this paper. So we consider semiflows on *finite-dimensional* smooth manifolds without boundary or on finite-dimensional compact smooth non-singular branched manifolds without boundary. We call a finite-dimensional *compact* (respectively *non-compact*) smooth manifold without boundary a *closed* (respectively *open*) manifold. Non-singular branched manifolds were introduced by Williams [**20**]. By a *non-singular branched n-dimensional manifold* of class $C^k$ we mean a metrizable space $K$ together with:

(i) a collection $\{U_i\}$ of closed subsets of $K$;
(ii) for each $U_i$ a finite collection $\{D_{ij}\}$ of closed subsets of $U_i$;
(iii) for each $i$, a map $\pi_i : U_i \to D_i^n$, $D_i^n$ a closed $n$-disk of class $C^k$ in $R^n$;

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subject to the following axioms:
(a) $\cup_j D_{ij} = U_i$ and $\cup_i \operatorname{Int} U_i = K$;
(b) $\pi_i | D_{ij}$ is a homeomorphism onto $D_i^n$;
(c) there is a cocycle of diffeomorphisms $\{\alpha_{i'i}\}$ of class $C^k$ such that $\pi_{i'} = \alpha_{i'i} \circ \pi_i$ when defined. The domain of $\alpha_{i'i}$ is $\pi_i(U_i \cap U_{i'})$.

To show the existence of Anosov semiflows which are not flows, we construct a finite-dimensional compact smooth non-singular branched manifold without boundary. In this paper we prove the following theorem.

THEOREM 4.3. *Let $M$ be either a closed manifold or a finite-dimensional compact smooth non-singular branched manifold without boundary. Then Anosov semiflows on $M$ are $C^1$ inverse limit stable in* $\operatorname{Sem}_S^1(M)$.

The content of this paper is as follows. In §2 we give some definitions and fundamental properties for semiflows and inverse limit conjugacy. In §3 we prove the inverse limit conjugacy theorem. In §4 we give the definition of Anosov semiflow and its motivation. Moreover, we prove Theorem 4.3. §5 contains an example of an Anosov semiflow which is not a flow.

## 2. *Preliminaries*

Let $M$ be a finite-dimensional smooth connected boundaryless (non-singular branched) manifold with a Riemannian metric. Let $d$ be a metric on $M$ induced by its Riemannian metric. A $C^r$ *semiflow* $F(t)$ on $M$, $r \geq 1$, is a one-parameter family of $C^r$ maps $\{F(t) : M \to M \mid t \geq 0\}$ such that:
(1) $F(s+t) = F(s)F(t)$ for all $s, t \geq 0$;
(2) $F(0)$ is the identity map of $M$;
(3) $F(\cdot)(x) : [0, \infty) \to M$ is continuous for every $x \in M$.

We shall sometimes use the notation $F(t, x) = F(t)(x)$ for $(t, x) \in [0, +\infty) \times M$. $\operatorname{Sem}^r(M)$ denotes the class of all $C^r$ semiflows on $M$, $r \geq 0$, with the metric $d_r$, where

$$d_r(F(t), G(t)) = \sup\{d(F(t)(x), G(t)(x)), |D^i F(t)(x) - D^i G(t)(x)| \mid t \in [0,1], x \in M \text{ and } i \in \{1, \ldots, r\}\} \quad \text{for } F(t), G(t) \in \operatorname{Sem}^r(M).$$

A *global solution* of a semiflow $F(t)$ is a function $v : \mathbb{R} \to M$ such that $F(t)(v(\tau)) = v(\tau + t)$ for all $t \geq 0$ and $\tau \in \mathbb{R}$. Item (3) of the above definition implies that a global solution is a continuous function of $\mathbb{R}$ to $M$. Let $C(\mathbb{R}, M)$ be the space of continuous functions from $\mathbb{R}$ to $M$. Let $\widetilde{S}(F)$ be the set of all global solutions of $F(t)$. For each $t \in \mathbb{R}$ we define a *shift* $\widetilde{F}(t)$ on $\widetilde{S}(F)$ by $[\widetilde{F}(t)v](\tau) = v(\tau + t)$ for $\tau \in \mathbb{R}$, $v \in \widetilde{S}(F)$. Let $\widetilde{p}_s : C(\mathbb{R}, M) \to M$ be a projection defined by $\widetilde{p}_s(v) = v(s)$ for $v \in C(\mathbb{R}, M)$. Let $A(F) = \bigcap_{t \geq 0} F(t)[M]$ for any semiflow $F(t)$. Note that $A(F) = \widetilde{p}_s(\widetilde{S}(F))$ for every $s \in \mathbb{R}$. We sometimes call $A(F)$ an *attractor* of $F(t)$. A subset $J$ of $M$ is *$\omega$-invariant* for $F(t)$ if $F(t)J \subset J$ for all $t \geq 0$. Then $\widetilde{J}(F)$ or $\widetilde{J}$ denotes the set of all global solutions of $F(t)$ contained in $J$. We will say that a subset $J$ of $M$ is *invariant* for $F(t)$ if $F(t)J = J$ for all $t \geq 0$. We introduce a metric $\widetilde{d}$ on $\widetilde{S}(F)$ (more generally on $C(\mathbb{R}, M)$) by

$$\widetilde{d}(v, w) = \sup_{t \in \mathbb{R}} e^{-|t|} d(v(t), w(t)) \quad \text{for } v, w \in C(\mathbb{R}, M).$$

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Then $\widetilde{S}(F)$ is endowed with a topology induced by the metric $\widetilde{d}$. We shall say that a $C^r$ semiflow $F(t)$ on $M$ is *strong* if $F(t)$ satisfies the following:

(4) every global solution $v(t) \in \widetilde{S}(F)$ is differentiable for $t \in \mathbb{R}$.

Let $\mathrm{Sem}^r_S(M) = \{F(t) \in \mathrm{Sem}^r(M) \mid F(t) \text{ is strong}\}$ be the class of all $C^r$ strong semiflows on $M$, $r \geq 0$, with the metric $d_r$. Let $\mathcal{F}^r(M)$ be the space of all $C^r$ flows with the metric $d_r$. It is obvious that $\mathcal{F}^r(M) \subset \mathrm{Sem}^r_S(M) \subset \mathrm{Sem}^r(M)$.

A global solution $v$ of $F(t)$ is called an *equilibrium solution* if there exists a point $p$ such that $v(t) = p$ for all $t \in \mathbb{R}$, and a *periodic solution* if there exists a constant $T > 0$ such that $v(t + T) = v(t)$ for all $t \in \mathbb{R}$ and $v(t) \neq v(0)$ for $0 < t < T$. The time $T > 0$ is called the *period* of the solution. Thus we distinguish equilibrium solutions from periodic solutions. Let $\mathrm{Per}(F)$ be the set of all points on periodic solutions of $F(t)$. Let $\mathrm{E}(F) = \{v(0) \mid v \text{ is an equilibrium solution}\}$ be the set of all fixed points for $F(t)$. For a global solution $v$ of $F(t)$ define $\mathcal{O}(v) = \{v(t) \mid t \in \mathbb{R}\}$.

PROPOSITION 2.1. *Let $v$ be a global solution of $F(t) \in \mathrm{Sem}^r(M)$:*

(i) *$v$ is an equilibrium solution of $F(t)$ if and only if $\widetilde{F}(t)v = v$ for all $t \in \mathbb{R}$;*

(ii) *$v$ is a periodic solution of $F(t)$ if and only if there exists a constant $T > 0$ such that $\widetilde{F}(T)(v) = v$ and $\widetilde{F}(t)(v) \neq v$ for $0 < t < T$.*

Let $F(t), G(t) \in \mathrm{Sem}^r(M)$. We say that $G(t)$ is *inverse limit semiconjugate* to $F(t)$ if there exists a continuous surjection $H : \widetilde{S}(F) \to \widetilde{S}(G)$ which takes the global solutions of $F(t)$ onto the global solutions of $G(t)$ and preserves the orientation in time; i.e. for each $v \in \widetilde{S}(F)$ there exists a nondecreasing automorphism $\beta_v : \mathbb{R} \to \mathbb{R}$ such that

$$[H \circ \widetilde{F}(t)](v) = [\widetilde{G}(\beta_v(t)) \circ H](v) \quad \text{for all } t \in \mathbb{R}.$$

Furthermore, if $H$ is injective then $F(t)$ and $G(t)$ are *inverse limit conjugate*. A $C^r$ semiflow $F(t)$ is *$C^r$ inverse limit stable* if there exists a neighborhood $\mathcal{U}$ of $F(t)$ in $\mathrm{Sem}^r(M)$ such that, for each $G(t) \in \mathcal{U}$, $F(t)$ and $G(t)$ are inverse limit conjugate. Similarly, $F(t)$ is *$C^r$ inverse limit stable in* $\mathrm{Sem}^r_S(M)$ if there exists a neighborhood $\mathcal{U}$ of $F(t)$ in $\mathrm{Sem}^r_S(M)$ such that, for each $G(t) \in \mathcal{U}$, $F(t)$ and $G(t)$ are inverse limit conjugate. Equilibrium solutions and periodic solutions are preserved by inverse limit conjugacy.

PROPOSITION 2.2. *Let $F(t), G(t) \in \mathrm{Sem}^r(M)$ be inverse limit conjugate.*

(i) *If $v$ is an equilibrium solution of $F(t)$, then $H(v)$ is an equilibrium solution of $G(t)$.*

(ii) *If $v$ is a periodic solution of $F(t)$, then $H(v)$ is a periodic solution of $G(t)$.*

*Proof.* (i) Let $v$ be an equilibrium solution of $F(t)$. Then, by Proposition 2.1(i)

$$[\widetilde{G}(\beta_v(t)) \circ H](v) = [H \circ \widetilde{F}(t)](v) = H(v) \quad \text{for all } t \in \mathbb{R}.$$

Hence $\widetilde{G}(\beta_v(t))(H(v)) = H(v)$ for all $t \in \mathbb{R}$. Since $\beta_v(t)$ is a nondecreasing automorphism of $\mathbb{R}$, $\widetilde{G}(\tau)(H(v)) = H(v)$ for all $\tau \in \mathbb{R}$. It follows from Proposition 2.1(i) that $H(v)$ is an equilibrium solution of $G(t)$.

(ii) Let $v$ be a periodic solution of $F(t)$ with period $T$. Then, by Proposition 2.1(ii)

$$[\widetilde{G}(\beta_v(T)) \circ H](v) = [H \circ \widetilde{F}(T)](v) = H(\widetilde{F}(T)(v)) = H(v) \quad \text{and}$$
$$\widetilde{G}(\beta_v(t))(H(v)) = H(\widetilde{F}(t)v) \neq H(v) \quad \text{for } 0 < t < T.$$

It follows from Proposition 2.1(ii) that $H(v)$ is a periodic solution of $G(t)$ with period $\beta_v(T)$. □

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*Definition.* Let $F(t)$ be a $C^r$ semiflow on $M$, $r \geq 1$. We will say that an $\omega$-invariant set $J$ is a *hyperbolic* set for $F(t)$ if every global solution $v \in \widetilde{J}(F)$ is differentiable for $t$ and there exist a Riemannian norm $|\cdot|$ on $M$ and constants $c, \mu > 0$ such that for every $v \in \widetilde{J}(F)$ there exists a continuous splitting of $\bigcup_{t=-\infty}^{\infty} T_{v(t)}M$ into a direct sum $E^s \oplus E^o \oplus E^u$, where

(1) $E^i = \bigcup_{t=-\infty}^{\infty} E^i_{v(t)}$ for $i = s, o, u$,
$E^o_{v(t)} = \mathbb{R} \cdot \frac{d}{dt} v(t)$ for each $t \in \mathbb{R}$,
$(T_{v(t)}F(\tau))[E^i_{v(t)}] = E^i_{v(t+\tau)}$ for $i = s, o, u$, and every $\tau \geq 0$,

(2) $|(T_{v(t)}F(\tau))(w)| \leq ce^{-\mu\tau}|w|$ for $w \in E^s_{v(t)}$, $t \in \mathbb{R}$ and every $\tau \geq 0$,
$|(T_{v(t)}F(\tau))(w)| \geq c^{-1}e^{\mu\tau}|w|$ for $w \in E^u_{v(t)}$, $t \in \mathbb{R}$ and every $\tau \geq 0$.

If $\Lambda$ is a hyperbolic set for $F(t)$ such that $\dim E^s_{v(0)} = j$ for all $v \in \widetilde{\Lambda}(F)$, then we call $j$ the *stable index* of $\Lambda$ for $F(t)$. For convenience we will say that $v \in \widetilde{\Lambda}(F)$ is *hyperbolic* for $F(t)$ if $\Lambda$ is hyperbolic for $F(t)$.

For $v \in \widetilde{S}(F)$ we can define a tangent space $T_vC(\mathbb{R}, M)$ of $C(\mathbb{R}, M)$ at $v$. Let $\Gamma(v^*TM)$ be the space of continuous sections of the pullback bundle of $TM$ by $v$. $\Gamma(v^*TM)$ is endowed with a norm $\|\xi\| = \sup_{t\in\mathbb{R}} |\xi(t)|$, where $|\cdot|$ is a Riemannian norm on $M$. Then we can consider that $T_vC(\mathbb{R}, M) = C(\mathbb{R}, v^*TM) = \Gamma(v^*TM)$ is a Banach space. Thus we can take an open neighborhood $U^*(v) = \Gamma_\varepsilon(v^*TM) = \{\xi \in \Gamma(v^*TM) \mid \|\xi\| < \varepsilon\} \subset C(\mathbb{R}, v^*TM)$ of zero section of $v^*TM$ such that $U^*(v)$ can be identified with a neighborhood $U(v)$ of $v$ in $C(\mathbb{R}, M)$ by a homeomorphism $G : U^*(v) \to U(v)$ defined by $G(\xi)(t) = \exp_{v(t)} \xi(t)$ for all $t \in \mathbb{R}$. Let $C^r(\mathbb{R}, M)$ be the space of $C^r$ functions from $\mathbb{R}$ to $M$, $r \geq 1$. If it is further assumed that every $v \in \widetilde{S}(F)$ is $C^r$, then we can show the similarity to the above and obtain that $T_vC^r(\mathbb{R}, M) = C^r(\mathbb{R}, v^*TM) = \Gamma^r(v^*TM)$.

## 3. *Inverse limit conjugacy theorem*

Now we shall prove the inverse limit conjugacy theorem for hyperbolic sets of semiflows on $M$. In order to prove this theorem, we need the following lemma and several concepts.

*Definition.* A continuous linear map $L$ of a Banach space $X$, $L : X \to X$, is said to be *prehyperbolic* if there exist a splitting of $X = X^s \oplus X^u$, constants $\lambda > 1$, $0 < \mu < 1$, and a norm $|\cdot|$ such that:

(1) $L(X^u) = X^u$, $L(X^s) \subset X^s$, and $X^u$, $X^s$ are closed,

(2) $|L(x^u)| \geq \lambda|x^u|$ for $x^u \in X^u$, $|L(x^s)| \leq \mu|x^s|$ for $x^s \in X^s$.

*Remark.* In general $L|X^s$ is not an isomorphism.

LEMMA 3.1. *Let $E$ be a Banach space and $E(r)$ the closed ball in $E$ about zero of radius $r$. Suppose that $E$ admits a splitting by closed subspaces $E = E_1 \oplus E_2$, and that $E$ has the box norm defined by $\|v\| = \max\{\|v_1\|_E, \|v_2\|_E\}$ for $v = v_1 \oplus v_2 \in E_1 \oplus E_2$, where $\|\cdot\|_E$ is the norm for a Banach space $E$. Let $p_i$ be the projection on $E_i$ for $i = 1, 2$. Let $T$ be a continuous linear map which is prehyperbolic with respect to the splitting $E_1 \oplus E_2$. Let $T_i = T|E_i$ for $i = 1, 2$. Thus there is a positive constant $\lambda < 1$ such that $\|T_1^{-1}\| < \lambda$ and $\|T_2\| < \lambda$. Let $f : E(r) \to E$ be a continuous map close to $T$ with $\mathrm{lip}(f - T) < \varepsilon$ and $\|f(0)\| \leq \delta$.*

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*If $\lambda+\varepsilon < 1$ and $\delta < r(1-\lambda-\varepsilon)$, then $f$ has a unique fixed point $p_f$ in $E(r)$. Moreover,*

$$\|p_f\| < \frac{1}{1-\lambda-\varepsilon}\|f(0)\|$$

*and $p_f$ depends $C^0$ continuously on $f$.*

*Proof.* See Shub [**18**] for the case when $T$ is a hyperbolic isomorphism. The same arguments are valid for the case when $T$ is a prehyperbolic endomorphism. □

THEOREM 3.2. *Let $F(t)$ be a $C^1$ semiflow on $M$, and $I(F)$ an $\omega$-invariant set for $F(t)$. Assume that $I(F)$ is a hyperbolic set for $F(t)$ such that $I(F)\bigcap \mathrm{E}(F) = \phi$ and $\bigcap_{t\geq 0} F(t)[I(F)]$ is compact in $M$. Then there exists a $C^1$ neighborhood $\mathcal{U}$ of $F(t)$ in $\mathrm{Sem}^1(X)$ such that for each $G(t) \in \mathcal{U}$ there is an $\omega$-invariant set $J(G)$ for $G(t)$ such that there exist a homeomorphism $H_G : \widetilde{I}(F) \to \widetilde{J}(G)$ and nondecreasing automorphisms $\beta_v : \mathbb{R} \to \mathbb{R}$ depending on $v$ with $[H_G \circ \widetilde{F}(t)](v) = [\widetilde{G}(\beta_v(t)) \circ H_G](v)$ for all $v \in \widetilde{I}(F)$ and $t \in \mathbb{R}$. By choosing the neighborhood $\mathcal{U}$ sufficiently small, we can get $H_G$ and $\beta_v$ for $v \in \widetilde{I}(F)$ arbitrarily close to the respective identity maps. Subject to these restrictions, these maps are unique. Here, for each $t \in \mathbb{R}$, $\widetilde{F}(t)$ is a shift on $\widetilde{S}(F)$, i.e. $[\widetilde{F}(t)v](\tau) = v(\tau + t)$ for $\tau \in \mathbb{R}$, $v \in \widetilde{S}(F)$.*

*Proof of Theorem 3.2.* Let $A = \bigcap_{t\geq 0} F(t)[I(F)]$. Since $A$ is compact, there exists a positive number $\varepsilon_1 > 0$ such that $\exp_x : T_xM \to M$ is a diffeomorphism on the ball of radius $\varepsilon_1$ for each $x \in U(A)$, where $U(A)$ is some neighborhood of $A$ in $M$. In this setting we can easily prove the following lemma.

LEMMA 3.3. *For all $\eta'$ and $\delta$ with $\eta' > 0$, $0 < \delta < \varepsilon_1$, there is $r = r(\delta, \eta') > 0$ and a $C^1$ neighborhood $V = V(\delta, \eta')$ of $f$ in $\mathrm{End}^1(M)$ such that if $y, z \in U(A)$ satisfy $d(z, f(y)) \leq \delta$ and $g$ belongs to $V$, then we have*

$$g(\exp_y B_r(0_y)) \subset \exp_z(B_{\varepsilon_1}(0_z)),$$
$$\mathrm{lip}[(F_{z,y} - \exp_z^{-1} \circ g \circ \exp_y)|B_r(0)] < \eta'.$$

*Here $F_{z,y} = D(\exp_z^{-1})_{f(y)} Df_y$ and $\mathrm{End}^1(M)$ denotes the space of all $C^1$ maps from $M$ to $M$ endowed with the $C^1$ topology.*

*Proof of Lemma 3.3.* See [**18**, pp. 89, 90]. □

Return to the proof of Theorem 3.2. We shall divide the proof into several steps.

*Step 1. Existence of a semiconjugacy $H_G$.*
First, for each $v \in \widetilde{I}(F)$ we try to find a conjugacy for the time-one map $F(1)$, i.e. we are looking for a continuous function $u^v : \mathcal{O}(v) \to M$ and a continuous function $\alpha_v^1 : \mathbb{R} \to \mathbb{R}$ such that

$$u^v(F(1, v(t))) = G(\alpha_v^1(t), u^v(v(t))) \quad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Let $u^v(v(t)) = \exp_{v(t)} \xi(t)$ where $\xi \in \Gamma_\varepsilon(v^*(E^s \oplus E^u)) = \{\nu \in \Gamma(v^*(E^s \oplus E^u)) \mid \|\nu\| < \varepsilon \leq \varepsilon_1\}$. Here $\|\nu\| = \sup_{t\in\mathbb{R}} |\nu(v(t))|$. We define a map $\alpha_{v,\xi}^1 : \mathbb{R} \to \mathbb{R}$ by $\alpha_{v,\xi}^1(t) =$ the smallest positive number $s$ such that $G(s, \exp_{v(t)} \xi(t))$ hits $\exp_{v(t+1)}(E^s \oplus E^u)_{v(t+1)}$.

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$\alpha^1_{v,\xi}$ is well defined and continuous for $t$ because $G(t)$ is $C^1$ close to $F(t)$ and $\|\xi\|$ is small. Using $\xi$, we can rewrite (3.1) as

$$\xi(t) = \exp^{-1}_{v(t)} \circ G(\alpha^1_{v,\xi}(t-1)) \circ \exp_{v(t-1)} \xi(t-1) \quad \text{for } t \in \mathbb{R}. \tag{3.2}$$

To solve equation (3.2), we define two maps $\mathcal{L}$ and $\mathcal{F}^G_v$;

$$\begin{gathered}
\mathcal{L}_v : \Gamma(v^*(E^s \oplus E^u)) \to \Gamma(v^*(E^s \oplus E^u)) \quad \text{by} \\
[\mathcal{L}_v(\xi)](t) = (D_{v(t-1)}F(1)) \circ \xi \circ S_{-1}(t), \\
\mathcal{F}^G_v : \Gamma_\varepsilon(v^*(E^s \oplus E^u)) \to \Gamma(v^*(E^s \oplus E^u)) \quad \text{by} \\
[\mathcal{F}^G_v(\xi)](t) = \exp^{-1}_{v(t)} \circ G(\alpha^1_{v,\xi}(t-1)) \circ \exp_{v(t-1)} \circ \xi \circ S_{-1}(t) \\
- (D_{v(t-1)}F(1)) \circ \xi \circ S_{-1}(t).
\end{gathered}$$

Here $S_\mu : \mathbb{R} \to \mathbb{R}$ is a shift with $S_\mu(t) = t + \mu$. Equation (3.2) is then equivalent to

$$[(\mathcal{L}_v + \mathcal{F}^G_v)(\xi)](t) = \xi(t) \quad \text{for } t \in \mathbb{R}. \tag{3.3}$$

Note that $\mathcal{L}_v$ is prehyperbolic with respect to the splitting $\Gamma(v^*(E^s \oplus E^u)) = \Gamma(v^*E^s) \oplus \Gamma(v^*E^u)$. Moreover, it is easy to see that there exist a number $0 < \lambda < 1$ and an adapted operator norm $\|\cdot\|$ such that $\|\mathcal{L}|\Gamma(v^*E^s)\| < \lambda$, $\|\mathcal{L}|\Gamma(v^*E^u)\| > \lambda^{-1}$. By Lemma 3.3, for $\eta, \delta > 0$ satisfying $\eta + \lambda < 1$ and $0 < \delta < \varepsilon_1(1 - \lambda - \eta)$, there exist $\varepsilon_2 = \varepsilon_2(\delta, \eta) > 0$ and a neighborhood $V$ of $F(1)$ in $\mathrm{End}^1(M)$ such that:

(i) $\mathrm{lip}[(D_{v(t)}F(1) - \exp^{-1}_{v(t+1)} \circ g \circ \exp_{v(t)})|B_{\varepsilon_2}(0)] < \eta/3$ for each $g \in V$;

(ii) if $G(t)$ is $C^1$ close to $F(t)$ in $\mathrm{Sem}^1(M)$, then $G(1+s)$ is $C^1$ close to $F(1)$ in $\mathrm{End}^1(M)$ for small $|s|$. That is, $G(1+s) \in V$ for small $|s|$.

Then it is easy to see that

$$\mathrm{lip}(\mathcal{F}^G_v) \le \eta \quad \text{for the domain } \Gamma_{\varepsilon_2}(v^*(E^s \oplus E^u)).$$

That is, for $\xi, \xi' \in \Gamma_{\varepsilon_2}(v^*(E^s \oplus E^u))$ with $\|\xi - \xi'\|$ small,

$$\|\mathcal{F}^G_v(\xi) - \mathcal{F}^G_v(\xi')\| = \sup_{t \in \mathbb{R}} |\mathcal{F}^G_v(\xi)(t) - \mathcal{F}^G_v(\xi')(t)| \le \sup_{t \in \mathbb{R}} \eta |\xi(t) - \xi'(t)| = \eta \|\xi - \xi'\|.$$

Now we know that $(\mathcal{L}_v + \mathcal{F}^G_v) : \Gamma_{\varepsilon_2}(v^*(E^s \oplus E^u)) \to \Gamma(v^*(E^s \oplus E^u))$ is a map close to $\mathcal{L}_v$ with $\mathrm{lip}(\mathcal{F}^G_v) < \eta$. Moreover, we take $G(t)$ $C^1$ close to $F(t)$ satisfying $\|(\mathcal{L}_v + \mathcal{F}^G_v)(0)\| < \varepsilon_2(1 - \lambda - \eta)$. Then it follows from Lemma 3.1 that $(\mathcal{L}_v + \mathcal{F}^G_v)$ has a unique fixed point $\xi_v$ in $\Gamma_{\varepsilon_2}(v^*(E^s \oplus E^u))$. Therefore, setting $u^v = \exp \xi_v$, we get from (3.2)

$$u^v(F(1)(v(t))) = G(\alpha^1_{v,\xi_v}(t), u^v(v(t))) \quad \text{for } t \in \mathbb{R}. \tag{3.4}$$

For simplicity we write $\alpha^1_v$ for $\alpha^1_{v,\xi_v}$. Note that $u^v \circ v \in C(\mathbb{R}, M)$. We shall show that $u^v \circ v \in \widetilde{S}(G)$, i.e. $u^v \circ v$ is a global solution of $G(t)$. It is obvious that $\{u^v(v(i)) \mid i \in \mathbb{Z}\}$ belongs to a global solution of $G(t)$. We denote this global solution $w(t)$. We can take

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$w(t)$ satisfying $w(0) = u^v(v(0))$. We define $\widehat{u}^v \in C(\mathcal{O}(v), \mathcal{O}(w))$ by

$$\widehat{u}^v \circ v(t) = \begin{cases} G((t-i)\cdot\alpha_v^1(i) + \sum_{j=0}^{i-1} \alpha_v^1(j), w(0)) & \text{for } i \le t \le i+1, i \in \mathbb{Z}^+, \\ G(t\cdot\alpha_v^1(0), w(0)) & \text{for } 0 \le t \le 1, \\ G(t\cdot\alpha_v^1(-1), w(0)) & \text{for } -1 \le t \le 0, \\ G((t+i)\cdot\alpha_v^1(-i-1) - \sum_{j=-1}^{-i} \alpha_v^1(j), w(0)) & \text{for } -i-1 \le t \le -i, i \in \mathbb{Z}^+. \end{cases}$$

Then $\{\widehat{u}^v \circ v(t) \mid t \in \mathbb{R}\} = \mathcal{O}(w) = \{w(t) \mid t \in \mathbb{R}\}$. Moreover, we can define a continuous map $\widehat{\alpha}_v^1 : \mathbb{R} \to \mathbb{R}$ such that

$$\widehat{u}^v(F(1)(v(t))) = G(\widehat{\alpha}_v^1(t), \widehat{u}^v(v(t))) \quad \text{for } t \in \mathbb{R}.$$

Setting $\widehat{\xi}(t) = \exp_{v(t)}^{-1} \widehat{u}^v(v(t))$, we have $\widehat{\xi} \in \Gamma_{\varepsilon_2}(v^*(E^s \oplus E^u))$ because $G(t)$ is sufficiently $C^1$ close to $F(t)$. By the uniqueness of the fixed point in Lemma 3.1, we have $\widehat{\xi} = \xi$, so $\widehat{\alpha}_v^1 = \alpha_v^1$.

We define a map $H_G : \widetilde{I}(F) \to \widetilde{S}(G)$ by

$$H_G(v) = w \quad \text{for } v \in \widetilde{I}(F), \tag{3.5}$$

where $w$ is a global solution of $G(t)$ with $w(0) = \widehat{u}^v(v(0))$, induced by the above argument.

We define a map $\beta_v : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{u}^v(v(\mu)) = w(\beta_v(\mu)) \quad \text{for } \mu \in \mathbb{R},$$

where $w$ is defined by (3.5). It is obvious that $\beta_v$ is an nondecreasing automorphism of $\mathbb{R}$. Then it is easy to see that

$$[H_G \circ \widetilde{F}(\mu)](v) = [\widetilde{G}(\beta_v(\mu)) \circ H_G](v) \quad \text{for } v \in \widetilde{I}(F) \text{ and } \mu \in \mathbb{R}. \tag{3.6}$$

Let $J(G) = \widetilde{p}_0[H_G(\widetilde{I}(F))]$.

*Step 2. Injectivity of $H_G$.*

Next we shall show that $H_G : \widetilde{I}(F) \to H_G[\widetilde{I}(F)]$ is injective.

By [**5**] there exists a constant $\varepsilon_3 > 0$ such that for every $v \in \widetilde{I}(F)$ there exist a local stable manifold $W_{\varepsilon_3}^s(v)$ and a local unstable manifold $W_{\varepsilon_3}^u(v)$.

Before continuing the proof, we need stable and unstable sets for hyperbolic sets of semiflows. First, we define the following sets.

Let $\Lambda$ be a hyperbolic set for $F(t) \in \mathrm{Sem}^r(M)$, $r \ge 1$ with the stable index $j$. For $x = v(0)$, $v \in \widetilde{\Lambda}(F)$, $\varepsilon > 0$,

$$W^{ss}(x; v) = \{y \in M \mid d(F(t)(x), F(t)(y)) \to 0 \text{ as } t \to +\infty\}$$
$$W_\varepsilon^{ss}(x; v) = \{y \in M \mid d(F(t)(x), F(t)(y))) \le \varepsilon \text{ for } t \ge 0 \text{ and } d(F(t)(x), F(t)(y))) \to 0 \text{ as } t \to +\infty\}$$

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$$W^{uu}(x; v) = \{y \in M \mid \text{there exists a global solution } w \text{ with } w(0) = y \text{ such that } d(S_v(t)(x), S_w(t)(y)) \to 0 \text{ as } t \to -\infty\}$$
$$W^{uu}_\varepsilon(x; v) = \{y \in M \mid \text{there exists a global solution } w \text{ with } w(0) = y \text{ such that } d(S_v(t)(x), S_w(t)(y)) \le \varepsilon \text{ for } t \ge 0 \text{ and } d(S_v(t)(x), S_w(t)(y)) \to 0 \text{ as } t \to -\infty\}.$$

Here $S_v(t)(x) = v(t + \tau)$ for $t \in \mathbb{R}$ where $x = v(\tau)$.

By [**5**] the following properties are obtained: for $\varepsilon > 0$ sufficiently small:

(a) $W^{ss}_\varepsilon(x; v)$ is a $C^r$ disk with dimension $j = \dim E^s_{v(0)}$;
(b) $W^{uu}_\varepsilon(x; v)$ is a $C^r$ disk with dimension $= \dim E^u_{v(0)}$;
(c) $W^{ss}_\varepsilon(x; v)$ (respectively $W^{uu}_\varepsilon(x; v)$) is tangent to $E^s_{v(0)}$ (respectively $E^u_{v(0)}$);
(d) $W^{ss}_\varepsilon(x; v) \bigcap W^{uu}_\varepsilon(x; v) = \{x\}$.

There are constants $c, \lambda > 0$ such that for small $\varepsilon > 0$,

(i) $d(F(t)(x), F(t)(y)) \le ce^{-\lambda t} d(x, y)$ for $y \in W^{ss}_\varepsilon(x)$ and $t \ge 0$
(ii) $d(S_v(t)(x), S_w(t)(y)) \le ce^{\lambda t} d(x, y)$ for $y = w(0) \in W^{uu}_\varepsilon(x; v)$, $w \in \widetilde{M}$ and $t \le 0$.

Note that $\widetilde{M}$ denotes the set of all global solutions of $F(t)$ (contained in $M$).

Let

$$W^s_\varepsilon(v) = \bigcup_{t\in\mathbb{R}} W^{ss}_\varepsilon(v(t); v), \quad W^u_\varepsilon(v) = \bigcup_{t\in\mathbb{R}} W^{uu}_\varepsilon(v(t); v).$$

$W^s_\varepsilon(v)$ (respectively $W^u_\varepsilon(v)$) is invariantly fibered by $C^r$ disks $W^{ss}_\varepsilon(v(t); v)$ (respectively $W^{uu}_\varepsilon(v(t); v)$), i.e. for $\tau \ge 0$ and $t \in \mathbb{R}$

$$F(\tau)[W^{ss}_\varepsilon(v(t); v)] \subset W^{ss}_\varepsilon(v(t + \tau); v), \quad W^{uu}_\varepsilon(v(t + \tau); v) \subset F(\tau)[W^{uu}_\varepsilon(v(t); v)].$$

$W^s_\varepsilon(v)$ is tangent at $v(t)$ to $E^s_{v(t)} \oplus E^o_{v(t)}$ for every $t \in \mathbb{R}$. Similarly $W^u_\varepsilon(v)$ is tangent at $v(t)$ to $E^u_{v(t)} \oplus E^o_{v(t)}$ for every $t \in \mathbb{R}$.

For $\varepsilon > 0$,

$$\widetilde{W}^s_\varepsilon(v) = \{w \in \widetilde{S}(F) \mid \widetilde{d}(S(t)(v), S(t)(w)) \le \varepsilon \text{ for all } t \ge 0\},$$
$$\widetilde{W}^u_\varepsilon(v) = \{w \in \widetilde{S}(F) \mid \widetilde{d}(S(-t)(v), S(-t)(w)) \le \varepsilon \text{ for all } t \ge 0\}.$$

Suppose that $w \in \widetilde{W}^s_\varepsilon(v) \cap \widetilde{W}^u_\varepsilon(v)$. Since $w \in \widetilde{W}^s_\varepsilon(v)$, $\widetilde{d}(S(t)(v), S(t)(w)) \le \varepsilon$ for all $t \ge 0$. Then $d(S_v(t)(v(0)), S_w(t)(w(0))) \le \varepsilon$ for all $t \ge 0$. Hence there exists $t' \in \mathbb{R}$ such that $W^{ss}_\varepsilon(v(0); v) \ni S_w(t')(w(0))$ and $|t'|$ is small. By similar argument there exists $t'' \in \mathbb{R}$ such that $W^{uu}_\varepsilon(v(0); v) \ni S_w(t'')(w(0))$ and $|t''|$ is small. By property (d) and continuity of semiflow $F(t)$, we obtain that $t' = t''$, i.e. $w(t') \in W^{uu}_\varepsilon(v(0); v) \bigcap W^{ss}_\varepsilon(v(0); v) = \{v(0)\}$. Hence $w = S(t')(v)$. So in general $\widetilde{W}^s_\varepsilon(v) \cap \widetilde{W}^u_\varepsilon(v) \supseteq \{v\}$ for sufficiently small $\varepsilon$.

Here we adopt the definition of expansivity by Bowen and Walters [**1, 2**].

*Definition.* Let $X$ be a metric space with a metric $d$, and $\Phi$ a continuous semiflow on $X$. $\Phi$ is *expansive* on a subset $Y$ of $X$ if, for small $\varepsilon > 0$, there exists $\delta > 0$ such that if $v, w \in \widetilde{Y}(F)$ and $\widetilde{d}(S(t)(v), S(g(t))(w)) < \delta$ for all $t \in \mathbb{R}$ and a continuous map $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$, then $w = S(\tau)(v)$ for some $\tau$ satisfying $|\tau| < \varepsilon$.

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By similar arguments to those for hyperbolic sets of flows [**5, 14**], we have the following properties of hyperbolic sets for semiflows.

Let $\Lambda$ be a compact hyperbolic set of $F(t) \in \mathrm{Sem}^r(M)$, $r \geq 1$, with the stable index $j > 0$, $\Lambda \cap \mathrm{E}(F) = \phi$, and $F(t)(\Lambda) = \Lambda$ for all $t \geq 0$.

*Property 1.* (Canonical coordinate) For every small $\varepsilon > 0$, there are $\delta > 0$ and $r > 0$ such that if $x, y$ in $\Lambda$ satisfying $d(x, y) < \delta$, then $W^{ss}_{\varepsilon}(S_v(t)(x); v) \cap W^{uu}_{\varepsilon}(y; w) =$ {a single point} for a unique $t \in \mathbb{R}$ with $|t| < \varepsilon$ whenever a pair $v, w \in \widetilde{\Lambda}(F)$ satisfy $v(0) = x$, $w(0) = y$ and $\widetilde{d}(v, w) < r$.

We denote the above single point by $\langle v(0), w(0)\rangle_{\varepsilon,\delta,r}$ and the above unique $t$ by $u(v(0), w(0))_{\varepsilon,\delta,r}$. We adopt not $\langle x, y\rangle_{\varepsilon,\delta,r}$ but $\langle v(0), w(0)\rangle_{\varepsilon,\delta,r}$, because the intersection of $W^{uu}_{\varepsilon}(y; w)$ and $W^{ss}_{\varepsilon}(x; v)$ depends not only on $(x, y)$ but also $(v, w)$. For simplicity, write $\langle v, w\rangle_{\varepsilon,\delta,r}$ for $\langle v(0), w(0)\rangle_{\varepsilon,\delta,r}$, and $u(v, w)_{\varepsilon,\delta,r}$ for $u(v(0), w(0))_{\varepsilon,\delta,r}$.

*Property 2.* $F(t)$ is expansive on $\Lambda$.

In order to prove Property 2 we need the following proposition.

PROPOSITION 3.4. *Let $L > 0$ be sufficiently large and $\gamma > 0$ small. For sufficiently small $\eta > 0$ there is an $\alpha > 0$ with the following property.*

*If $v, w \in \widetilde{\Lambda}$ and $s : \mathbb{R} \to \mathbb{R}$, a continuous function with $s(0) = 0$, satisfy*

$$d(S_w(t + s(t))(w(0)), S_v(t)(v(0))) \leq \alpha \quad \text{for } |t| \leq L,$$

*then*

$$|s(t)| \leq 3\eta \quad \text{for all } |t| \leq L,\ |u(v(0), w(0))_{\eta,\delta,r}| \leq \eta$$

*and*

$$d(S_w(t)(w(0)), S_v(t + u(v, w)_{\eta,\delta,r})(v(0))) \leq c\gamma e^{-\lambda(L-|t|)} \quad \text{for any } |t| \leq L.$$

*In particular,*

$$d(w(0), S_v(u(v, w)_{\eta,\delta,r})(v(0))) \leq c\gamma e^{-\lambda L}.$$

*Here $c, \lambda > 0$ are constants induced by the hyperbolicity of $\Lambda$.*

*Remark.* For sufficiently large $L > 0$ and sufficiently small $\alpha > 0$, if $v, w \in \widetilde{\Lambda}$ satisfy the assumption of Proposition 3.4, then $\widetilde{d}(v, w) < r$ and $d(v(0), w(0)) < \delta$. Here $\delta > 0$ and $r > 0$ are the numbers determined by $\eta > 0$ corresponding to $\varepsilon$ in Property 1. Thus, the appropriate choice of $L$ and $\alpha$ guarantees the existence of $\langle v, w\rangle_{\eta,\delta,r}$ and $u(v, w)_{\eta,\delta,r}$.

*Proof of Proposition 3.4.* We may assume that $\eta$ is so small that

$$\eta < \gamma/8 \quad \text{and} \quad \sup\{d(S_v(u)(v(0)), v(0)) \mid v \in \widetilde{\Lambda}, |u| \leq 4\eta\} \leq \gamma/8.$$

Let $\delta, r$ be the numbers determined by $\eta$ corresponding to $\varepsilon$ in Property 1. Consider $v, w$ as above. Since $s(0) = 0$, $d(w(0), v(0)) \leq \delta$. Define

$$x = \langle v(0), w(0)\rangle_{\eta,\delta,r} = W^{ss}_{\eta}(S_v(u(v, w)_{\eta,\delta,r})(v(0)); v) \bigcap W^{uu}_{\eta}(w(0); w)$$

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and the sets

$$A = \{t \in [0, L] \mid |s(t)| \geq 3\eta \text{ or } d(S_w(t)(w(0)), F_t(x)) \geq \gamma/2\}$$
$$B = \{t \in [0, L] \mid |s(-t)| \geq 3\eta \text{ or } d(S_v(-t)(v(0)), \widehat{F}_{-t}(x)) \geq \gamma/2\},$$

where $\widehat{F}_{-t}(x)$ is a backward orbit of $x$ contained in $W^u_\eta(w)$. The uniqueness of the existence of $\widehat{F}_{-t}(x)$ is guaranteed by $x \in W^{uu}_\eta(w(0); w)$. Let $z$ be the global solution such that $z(0) = x$ and $z(t) \in W^{uu}_\eta(w(t); w)$ for $t \leq 0$. Suppose that $A \neq \phi$. Let $t_1 = \inf A \in A$. Then $d(S_w(t_1 - b)(w(0)), S_z(t_1 - b)(x)) \leq (1/2)\gamma$ for all $b \geq 0$. Since $|s(t_1)| \leq 3\eta$, using the triangle inequality we get

$$d(S_w(t_1 + s(t_1) - b)(w(0)), S_z(t_1 + s(t_1) - b)(x)) \leq (3\gamma)/4 \quad \text{for } b \geq 0;$$

hence

$$S_z(t_1 + s(t_1))(x) \in W^{uu}_\gamma(S_w(t_1 + s(t_1))(w(0)); w).$$

Now

$$d(F_b(x), F_b(S_v(u(v, w)_{\eta,\delta,r})(v(0))) \leq \eta < \gamma/8 \quad \text{for } b \geq 0.$$

Since $|s(t_1)| \leq 3\eta$, the triangle inequality gives us

$$d(S_z(t_1 + s(t_1) + p)(x), S_v(t_1 + s(t_1) + u + p)(v(0))) \leq (3\gamma)/8 \quad \text{for } p \geq 0;$$

hence

$$S_z(t_1 + s(t_1))(x) \in W^{ss}_\gamma(S_v(s(t_1) + u)(S_v(t_1)(v(0))); v).$$

Now we obtain that

$$S_z(t_1 + s(t_1))(x) \in W^{ss}_\gamma(S_v(u + s(t_1))(S_v(t_1)(v(0))); v) \cap W^{uu}_\gamma(S_w(t_1 + s(t_1))(w(0)); w),$$

where $u = u(v, w)_{\eta,\delta,r}$.

Since

$$|s(t_1) + u| \leq |s(t_1)| + |u| \leq 4\eta < \gamma$$

and

$$d(S_w(t_1 + s(t_1))(w(0)), S_v(t_1)(v(0))) \leq \alpha,$$

we have

$$u(S_v(t_1)(v(0)), S_w(t_1 + s(t_1))(w(0))) = s(t_1) + u(v(0), w(0))$$

and

$$S_z(t_1 + s(t_1))(x) = \langle S_v(t_1)(v(0)), S_w(t_1 + s(t_1))(w(0))\rangle.$$

Then we obtain $|s(t_1) + u| \leq \eta$ by Property 1. Moreover, it follows from Property 1 that $d(S_w(t_1 + s(t_1))(w(0)), S_z(t_1)(x)) \leq \alpha$ implies

$$\langle S_z(t_1)(x), S_w(t_1 + s(t_1))(w(0))\rangle \in W^{ss}_\eta(S_z(\bar{u})(S_z(t_1)(x)); z) \bigcap W^{uu}_\eta(S_w(t_1 + s(t_1))(w(0)); w) \qquad \text{(a)}$$

where $\bar{u} = u(S_z(t_1)(x), S_w(t_1 + s(t_1))(w(0)))$ and $|\bar{u}| \leq \eta$.

Then (a) implies

$$d(S_z(t_1 + s(t_1))(x), S_w(t_1 + s(t_1))(w(0))) \leq \eta.$$

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$|s(t_1) + u(v(0), w(0))| \leq |\bar{u}| \leq \eta$ implies $|s(t_1)| \leq 2\eta$. Using $|s(t_1)| \leq 2\eta$, we get

$$d(S_z(t_1)(x), S_w(t_1)(w(0))) < (3\gamma)/8.$$

The above results contradict $t_1 \in A$. Hence $A = \phi$. We can show that $B = \phi$ by the similar arguments above. It is easy to see that $A = \phi$ implies $S_z(L)(x) \in W^{uu}_{\frac{1}{2}\gamma}(S_w(L)(w(0)); w)$. For $|s| \leq L$,

$$d(S_z(s)(x), S_w(s)(w(0))) \leq \tfrac{1}{2}\gamma c e^{-\lambda(L-|s|)}.$$

Similarly, $B = \phi$ implies $S_z(-L)(x) \in W^{ss}_{\frac{1}{2}\gamma}(S_v(-L+u)(v(0)); v)$ and

$$d(S_z(s)(x), S_v(s+u)(v(0))) \leq \tfrac{1}{2}\gamma c e^{-\lambda(L-|s|)}.$$

These two inequalities combine:

$$d(S_w(s)(w(0)), S_v(s+u)(v(0))) \leq \gamma c e^{-\lambda(L-|s|)}.$$

$A \bigcup B = \phi$ also gives $|s(t)| < 3\eta$ for $t \in [-L, L]$. □

*Proof of Property 2.* Since $\Lambda$ is compact and $\Lambda \cap \mathrm{E}(F) = \phi$, there exists a constant $T > 0$ such that if $v \in \widetilde{\Lambda}$ is a periodic solution then the period of $v$ is larger than $T$. Let $\eta > 0$ be sufficiently small such that $6\eta < T$. Then we obtain $\alpha > 0$ corresponding to $\eta$ by Proposition 3.4. It is easy to see that if $v, w \in \widetilde{\Lambda}$ and $\widetilde{d}(S(t)(v), S(g(t))(w)) \leq \alpha$ for all $t \in \mathbb{R}$ and a continuous map $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$, then $w(0) = S_v(\tau)(v(0))$ for some $\tau$ with $|\tau| < \eta$. Hence $w(s) = v(s + \tau)$ for all $s \geq 0$. However, we do not know whether $w(s) = v(s + \tau)$ for all $s < 0$. Suppose that there exists $\tilde{s} \leq 0$ such that for some $\bar{\tau}$ with $|\bar{\tau}| < 4\eta$, $w(s) = v(s + \bar{\tau})$ for all $s \geq \tilde{s}$. Then $v(\tau) = w(0) = v(\bar{\tau})$. Since $v \notin \mathrm{E}(F)$ and $|\tau - \bar{\tau}| < T$, $\bar{\tau} = \tau$. Suppose that there is a real number $s_o = \inf\{s \in \mathbb{R} \mid v(s+\tau) = w(s)\}$. Let $s_1$ be arbitrary real number such that $s_o - s_1 > 7\eta$. Let $\bar{v} = S(s_1)(v)$ and $\bar{w} = S(g(s_1))(w)$. Define a continuous map $\bar{g} : \mathbb{R} \to \mathbb{R}$ by

$$\bar{g}(t) = g(t + s_1) - g(s_1) \quad \text{for } t \in \mathbb{R}.$$

Then

$$\begin{aligned} d(\bar{v}(t), \bar{w}(\bar{g}(t))) &= d(v(t + s_1), w(\bar{g}(t) + g(s_1))) \\ &= d(v(t + s_1), w(g(t + s_1))) \leq \alpha \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

By Proposition 3.4, there exists a unique $\tau_1 \in \mathbb{R}$ such that $\bar{w}(0) = S_{\bar{v}}(\tau_1)(\bar{v}(0))$ and $|\tau_1| < \eta$. Thus $w(g(s_1)) = v(s_1 + \tau_1)$ and $s_1 + \tau_1 < s_o$ and $g(s_1) < g(s_0)$. Then $w(t) = v(t + s_1 + \tau_1 - g(s_1))$ for all $t \geq g(s_1)$. Moreover, it is easy to see that $|s_1 + \tau_1 - g(s_1)| < 4\eta$. From the above argument and $g(s_1) < s_o \leq 0$, we obtain that $\tau = s_1 + \tau_1 - g(s_1)$. Hence $w(t) = v(t + \tau)$ for all $t \geq g(s_1)$. This result contradicts the existence of $s_o$. Therefore $w = S(\tau)(v)$. □

Now return to the proof of Step 2. Let $\varepsilon_4 = \min\{\varepsilon_2, \varepsilon_3\}$. Let $\varepsilon_5 > 0$ be the number obtained by $\eta = \varepsilon_4$ in Proposition 3.4. By taking $G(t)$ sufficiently $C^1$ close to $F(t)$, we

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see that $H_G$ is within $\varepsilon_5/2$ of the identity map of $\widetilde{I}(F)$. Suppose that there are $v, w \in \widetilde{I}(F)$ such that $H_G(v) = H_G(w)$. We define a map $g : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}[H_G \circ \widetilde{F}(t)](v) &= \widetilde{G}(\beta_v(t)) \circ H_G(v) \\ &= \widetilde{G}(\beta_v(t)) \circ H_G(w) \\ &= \widetilde{G}(\beta_w(g(t))) \circ H_G(w).\end{aligned}$$

Then it is easy to see that $g$ is continuous and nondecreasing with $g(0) = 0$. Moreover, for each $t \in \mathbb{R}$

$$\begin{aligned}\|\widetilde{F}(t)(v) - \widetilde{F}(g(t))(w)\| &\le \|\widetilde{F}(t)(v) - H_G\widetilde{F}(t)(v)\| + \|H_G\widetilde{F}(t)(v) - H_G\widetilde{F}(g(t))(w)\| \\ &\quad + \|H_G\widetilde{F}(g(t))(w) - \widetilde{F}(g(t))(w)\| \\ &\le \tfrac{1}{2}\varepsilon_5 + 0 + \tfrac{1}{2}\varepsilon_5 \\ &= \varepsilon_5.\end{aligned}$$

Then it is easy to see that $d(v(t), w(g(t))) \le \varepsilon_5$ for all $t \in \mathbb{R}$. Hence expansiveness of $A = \bigcap_{t\ge 0} F(t)[I(F)]$ implies that $w = S(\tau)(v)$ for some $\tau$ satisfying $|\tau| \le \varepsilon_4$. By the construction of $H_G$, $\tau = 0$, that is, $w = v$.

*Step 3. Continuity of $H_G$ and other properties.*
By the construction of $H_G$ above, the continuity of $H_G$ and the last statement of Theorem 3.2 are obvious. □

*Remark.* Quandt stated similar results to the above theorem. However, the author cannot understand the rough proofs of [**15**, pp. 199, 200] and [**16**, p. 335]. In Robinson's framework [**17**] to prove Theorem 2 [**15**, p. 199], we cannot use the implicit function theorem because the differentiability of global solutions with respect to $t$ is not guaranteed by his setting. In the proof of Theorem 7.1 [**16**, p. 335] it is not known whether the disk family $\{\exp_{v(t)}(E^s \oplus E^u)(\varepsilon) | t \in \mathbb{R}\}$ has the following properties:

(1) $\exp_{v(t)}(E^s \oplus E^u)(\varepsilon) \cap \exp_{v(s)}(E^s \oplus E^u)(\varepsilon) = \phi$ for $t \ne s$ and $|t - s|$ small;

(2) $\bigcup_{t\in\mathbb{R}} \exp_{v(t)}(E^s \oplus E^u)(\varepsilon)$ is a tubular neighborhood of a global solution $v(t)$.

If we restrict objects to strong semiflows, then we have following consideration. Each global solution $w(t)$ in $\widetilde{J}(G)$ is differentiable for $t \in \mathbb{R}$. Moreover, for each $w(t) \in \widetilde{J}(G)$ there is a global solution $v(t)$ in $\widetilde{I}(F)$ which is close to $w(t)$ (that is, trace $w(t)$). Since $\widetilde{I}(F)$ is hyperbolic, we can show that $\widetilde{J}(G)$ is hyperbolic for $G(t)$ by the similar argument of [**7**, pp. 2196–2198].

## 4. *Anosov semiflows*

In this section we will give the definition of Anosov semiflows.

*Definition.* We say that a $C^1$ semiflow $F(t)$ is an *Anosov semiflow* if $F(t)$ satisfies the following:

(i) $A(F)$ is a compact hyperbolic set for $F(t)$ with stable index $j = \text{constant} > 0$;

(ii) $A(F)$ contains no equilibrium solutions.

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Intentionally we do not refer to manifolds in the definition, because we consider the general case in consideration of the extension to the infinite-dimensional case. We shall describe the motivation of the above definition. First of all, remember the definition of Anosov flows. If the entire manifold $M$ possesses a hyperbolic structure then the flow is called an *Anosov flow*. In the case of semiflows, it is appropriate that only the attractor possesses a hyperbolic structure (i.e. condition (i)). In the case of Anosov flows on a compact manifold $M$, hyperbolicity of $M$ implies that $M$ has constant stable index. In the finite-dimensional case of semiflows in this paper, we easily have that $A(F)$ is equal to $M$ and has constant stable index. However, in the infinite-dimensional case or special finite-dimensional case (i.e. open manifolds) we cannot guarantee that $A(F) = M$. So a constant stable index of $A(F)$ is not guaranteed by only hyperbolicity of $A(F)$. If $A(F) \subsetneqq M$ then there exists the possibility of decomposition of $A(F)$ into disjoint hyperbolic sets which have different stable index. Therefore, from intuition of original Anosov flows or diffeomorphisms it is natural to require a constant stable index of $A(F)$ and no equilibrium solutions.

In the case of flows the following result, essentially due to Mañé [**9**], is obtained.

THEOREM 4.1. [**4**] *Let $M$ be a closed manifold. Necessary and sufficient conditions for a flow $\psi_t$ on $M$ to be Anosov are:*

(1) *$\psi_t$ has hyperbolic chain recurrent set;*
(2) *the weak stable and unstable manifolds $W^s(\gamma)$ and $W^u(\gamma)$ intersect transversally for each orbit $\gamma$ in $M$;*
(3) *the dimension of $W^s(\gamma)$ and $W^u(\gamma)$ is constant respectively, i.e. independent of $\gamma$.*

Then it is easy to see that an Anosov flow satisfies (1), (2), and (3) of Theorem 4.1. In the case of semiflows we need condition (ii) in order to guarantee properties corresponding to (2), (3) in Theorem 4.1. In fact, Anosov semiflows satisfy properties corresponding to (1), (2) and (3) of Theorem 4.1.

*Remark 4.1.* Quandt [**15, 16**] gave definition of an Anosov semiflow which is weaker than the above definition. He required only that $A(F)$ is a compact hyperbolic set for $F(t)$.

If $F(t)$ is an Anosov semiflow, then $A(F)$ is the maximal compact $F(t)$-invariant set. Observe that

$$\bigcap_{t\geq 0} F(t)[M] = \bigcap_{n\in\mathbb{Z}^+} F(n)[M] = \bigcap_{n\in\mathbb{Z}^+} F(1)^n[M].$$

By compactness of $A(F)$, there exists a compact neighborhood $U$ of $A(F)$ such that $A(G) \subset U$ for $G(t)$ sufficiently $C^1$-close to $F(t)$. Thus perturbed $G(t)$ has no explosion of the attractor $A(G)$.

If a semiflow $F(t)$ satisfies only the condition (i) of the above definition, then we find the existence of one-to-one correspondence between all elements in $\widetilde{S}(F)$ and elements of a subset of $\widetilde{S}(G)$ for $G(t)$ $C^1$-near $F(t)$ by Theorem 3.2. However, we do not know whether this correspondence is a homeomorphism from $\widetilde{S}(F)$ onto $\widetilde{S}(G)$. To prove the existence of a homeomorphism from $\widetilde{S}(F)$ onto $\widetilde{S}(G)$ we will use an infinite chain (i.e. pseudo-orbit).

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*Definition.* Let $F(t)$ be a semiflow on $M$ and let $d$ be a distance on $M$. If $Y$ is a subset of $M$ and $J$ is a subset of $\mathbb{R}$, then define $Y \cdot J = \{F(t)(x) \mid x \in Y \text{ and } t \in J\}$. Given $x, y \in M$ and $\varepsilon, T > 0$, an $(\varepsilon, T)$-*chain* from $x$ to $y$ is a collection $\{x_0 = x, x_1, \ldots, x_k = y; t_0, t_1, \ldots, t_{k-1}\}$ such that for $0 \le i \le k-1$ we have $t_i \ge T$ and $d(x_i \cdot t_i, x_{i+1}) < \varepsilon$. A point $x$ is *chain equivalent* to $y$ (written $x \sim y$) if for every $\varepsilon, T > 0$ there is an $(\varepsilon, T)$-chain from $x$ to $y$ and one from $y$ to $x$. The *chain recurrent set* of $F(t)$, $\mathcal{CR}(F)$, is $\{x \in M \mid x \sim x\}$. An *infinite* $(\varepsilon, T)$-*chain* is a pair of doubly infinite sequences $\{\ldots, x_{-2}, x_{-1}, x_0 = x, x_1, \ldots; \ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots\}$ such that $t_i \ge T$ and $d(x_i \cdot t_i, x_{i+1}) < \varepsilon$ for all $i \in \mathbb{Z}$. Let $x_0 * t$ denote the point on this chain starting from $x_0$ as follows. If $t \ge 0$ then

$$x_0 * t = \begin{cases} x_0 \cdot t, & \text{if } 0 \le t < t_0, \\ x_i \cdot \left(t - \sum_{n=0}^{i-1} t_n\right), & \text{if } \sum_{n=0}^{i-1} t_n \le t < \sum_{n=0}^{i} t_n \text{ and } i \ge 1. \end{cases}$$

If $t < 0$ then

$$x_0 * t = \begin{cases} x_{-1} \cdot (t + t_{-1}), & \text{if } 0 > t \ge -t_{-1}, \\ x_i \cdot \left(t + \sum_{n=i}^{-1} t_n\right), & \text{if } -\sum_{n=i+1}^{-1} t_n > t \ge -\sum_{n=i}^{-1} t_n \text{ and } i \le -2. \end{cases}$$

Define $x_0 * [a, b]$ to be $\bigcup_{t \in [a,b]} \{x_0 * t\}$. A *reparameterization* of a global solution is an orientation preserving homeomorphism of $\mathbb{R}$ fixing the origin. A global solution $v(t)$ is said to $\delta$-*trace* an infinite $(\varepsilon, T)$-chain $x_0 * \mathbb{R}$ if there is a reparameterization $g$ such that $d(x_0 * t, v(g(t))) < \delta$ for all $t \in \mathbb{R}$. Remember that the chain recurrent set $\mathcal{CR}(F)$ of a semiflow $F(t)$ coincides with the chain recurrent set $\mathcal{CR}(F_1)$ of the time-one map $F_1$ of $F(t)$ (see [**6**] for the definitions and details).

In the proof of Theorem 4.3 we will use the following lemma. This lemma is the shadowing lemma. By using Properties 1 and 2 of §3, we can develop similar arguments to those for flows by Franke and Selgrade [**3**].

LEMMA 4.2. *Let $\Lambda$ be a hyperbolic compact invariant set for a semiflow $F(t)$ on $M$. For each $\delta > 0$ there is an $\varepsilon > 0$ such that each infinite $(\varepsilon, 1)$-chain in $\Lambda$ is $\delta$-traced by some global solution $v(t) \in \widetilde{\Lambda}'(F)$. Here $\Lambda'$ is a maximal compact invariant hyperbolic set for $F(t)$ contained in some compact neighborhood $N$ of $\Lambda$.*

*Remark 4.2.* In the proof of Theorem 4.3 we consider the case that $\Lambda$ is equal to $M$. In general, as we take $\delta > 0$ smaller, we can take $\varepsilon > 0$ smaller.

THEOREM 4.3. *Let $M$ be either a closed manifold or a finite-dimensional compact smooth non-singular branched manifold without boundary. Then Anosov semiflows on $M$ are $C^1$ inverse limit stable in* $\mathrm{Sem}^1_S(M)$.

*Proof of Theorem 4.3.* Since $F(t)$ is an Anosov semiflow, there exists a $C^1$ neighborhood $\mathcal{U}$ of $F(t)$ in $\mathrm{Sem}^1_S(M)$ such that for each $G(t) \in \mathcal{U}$ there exist a homeomorphism $H_G$ from $\widetilde{S}(F)$ onto its image and a nondecreasing automorphism $\beta_v : \mathbb{R} \to \mathbb{R}$ depending on $v$

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with $[H_G \circ \widetilde{F}(t)](v) = [\widetilde{G}(\beta_v(t)) \circ H_G](v)$ for all $t \in \mathbb{R}$. We only need to prove that there exists a neighborhood $\widetilde{\mathcal{U}} \subset \mathcal{U}$ of $F(t)$ such that $H_G[\widetilde{S}(F)] = \widetilde{S}(G)$ for each $G(t) \in \widetilde{\mathcal{U}}$. Since $F(0) = \mathrm{id}$ is a $C^1$ diffeomorphism of $M$, it is easy to see that $F(t)$ is surjective and has no singularities for $0 \leq t \leq 1$. Furthermore, since $G(t)$ is $C^1$ close to $F(t)$ in $\mathrm{Sem}^1(M)$, $G(t)$ is also surjective and has no singularities for $0 \leq t \leq 1$. By the semigroup property, each $G(t)$ is surjective for all $t \geq 0$. Hence $A(G) = M$ for each $G(t) \in \mathcal{U}$ if necessary retaking $\mathcal{U}$ smaller. Let $\varepsilon > 0$ be small. Let $\mathcal{U}_\varepsilon \subset \mathcal{U}$ be a neighborhood of $F(t)$ such that $d_1(F(t), G(t)) < \varepsilon$ for every $G(t) \in \mathcal{U}_\varepsilon$. At first we shall prove that $A(G) = M$ is hyperbolic for each $G(t) \in \mathcal{U}_\varepsilon$. Let $G(t) \in \mathcal{U}_\varepsilon$ and $v(t) \in \widetilde{S}(G)$. Then we can take an infinite $(\varepsilon, 1)$-chain $\{x_n; t_n\}$ for $F(t)$ such that:

(i) $x_n = v(n)$, for every $n \in \mathbb{Z}$;
(ii) $d(F(t)(x_n), G(t)(v(n))) < \varepsilon$, for $0 \leq t \leq 1 = t_n$ and every $n \in \mathbb{Z}$.

By Lemma 4.2, for small $\delta > 0$ there is an $\widehat{\varepsilon} > 0$ such that each infinite $(\widehat{\varepsilon}, 1)$-chain is $\delta$-traced by some $b(t) \in \widetilde{S}(F)$. If $\varepsilon > \widehat{\varepsilon}$ then we retake $\varepsilon$ and $\mathcal{U}_\varepsilon$ satisfying $\varepsilon \leq \widehat{\varepsilon}$. Then there exists a global solution $w(t)$ of $F(t)$ such that the above infinite $(\varepsilon, 1)$-chain $\{x_n; t_n\}$ is $\delta$-traced by $w(t)$. That is, there exists a reparameterization $g$ such that $d(x_0 * t, w(0) \cdot g(t)) < \delta$ for all $t \in \mathbb{R}$. By (ii),

$$d(v(t), w(0) \cdot g(t)) \leq d(v(t), x_0 * t) + d(x_0 * t, w(0) \cdot g(t)) < \varepsilon + \delta$$

for all $t \in \mathbb{R}$. Then there exists a reparameterization $h$ such that

$$d(v(0) \cdot h(t), w(t)) < \varepsilon + \delta \quad \text{for all } t \in \mathbb{R}.$$

Since $M = A(F)$ is hyperbolic, then there is a continuous splitting of $\bigcup_{t=-\infty}^{\infty} T_{w(t)}M$ into a direct sum $E^s \oplus E^o \oplus E^u$, where

(1) $E^i = \bigcup_{t=-\infty}^{\infty} E^i_{w(t)}$ for $i = s, o, u$,
$E^o_{w(t)} = \mathbb{R} \cdot \frac{d}{dt} w(t)$ for each $t \in \mathbb{R}$,
$(T_{w(t)}F(\tau))[E^i_{w(t)}] = E^i_{w(t+\tau)}$ for $i = s, o, u$, and every $\tau \geq 0$,
(2) $|(T_{w(t)}F(\tau))(b)| \leq ce^{-\mu\tau}|b|$ for $b \in E^s_{w(t)}$, $t \in \mathbb{R}$ and every $\tau \geq 0$,
$|(T_{w(t)}F(\tau))(b)| \geq c^{-1}e^{\mu\tau}|b|$ for $b \in E^u_{w(t)}$, $t \in \mathbb{R}$ and every $\tau \geq 0$.

Let $\widehat{E}^i_t = \varepsilon_{v(0)\cdot h(t), w(t)} E^i_{w(t)}$ for every $t \in \mathbb{R}$ and $i = s, u$, where $\varepsilon_{x,y}$ is the parallel translation from $T_yM$ to $T_xM$ along the unique shortest geodesic joining $y$ to $x$. Let $\widehat{E}^o_t = \mathbb{R} \cdot \frac{d}{ds} v(0) \cdot h(s)|_{s=t}$ for every $t \in \mathbb{R}$. Then

$$\bigcup_{t=-\infty}^{\infty} T_{v(0)\cdot h(t)}M = \bigcup_{t=-\infty}^{\infty} \widehat{E}^s_t \oplus \widehat{E}^o_t \oplus \widehat{E}^u_t$$

is a continuous splitting. Let $T_nM = T_{v(0)\cdot h(n)}M = \widehat{E}^s_n \oplus \widehat{E}^o_n \oplus \widehat{E}^u_n$ and $G_n = T_{v(0)\cdot h(n)}G(h(n+1) - h(n))$ for every $n \in \mathbb{Z}$. If $G(t)$ is sufficiently $C^1$ close to $F(t)$, we can take $\varepsilon + \delta > 0$ above sufficiently small. Then, since $v(t)$ is $(\varepsilon + \delta)$-traced by $w(t)$, there exist a constant $\lambda > 1$ and a family of cones $H_n \supset \widehat{E}^u_n$ associated with the decomposition $\widehat{E}^u_n \oplus (\widehat{E}^o_n \oplus \widehat{E}^s_n)$, and a family of cones $S_n \supset \widehat{E}^s_n$ associated with the

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decomposition $\widehat{E}^s_n \oplus (\widehat{E}^o_n \oplus \widehat{E}^u_n)$ such that for every $n \in \mathbb{Z}$

$$G_n H_n \subset \mathrm{Int}(H_{n+1})$$
$$(G_{n-1})^{-1} S_n \subset \mathrm{Int}(S_{n-1})$$
$$\|G_n(u,v)\| > \lambda \|(u,v)\| \quad \text{for } (u,v) \in H_n$$
$$\|G_n(u,v)\| < \lambda^{-1} \|(u,v)\| \quad \text{for } (u,v) \in (G_n)^{-1} S_{n+1}.$$

By the Hadamard–Perron theorem there exist a sequence of $(\dim M - j - 1)$-dimensional subspaces $\widetilde{E}^u_n \subset T_n M$, $n \in \mathbb{Z}$, and a sequence of $j$-dimensional subspaces $\widetilde{E}^s_n \subset T_n M$, $n \in \mathbb{Z}$, such that for every $n \in \mathbb{Z}$

(a) $\widetilde{E}^u_n = \bigcap_{i=0}^{\infty} G_{n-1} \circ G_{n-2} \circ \cdots \circ G_{n-i}(H_{n-i})$,

(b) $\widetilde{E}^s_n = \bigcap_{i=0}^{\infty} G_n^{-1} \circ G_{n+1}^{-1} \circ \cdots \circ G_{n+i}^{-1}(S_{n+i+1})$,

(c) $G_n \widetilde{E}^u_n = \widetilde{E}^u_{n+1}$, $G_n \widetilde{E}^s_n = \widetilde{E}^s_{n+1}$,

(d) $\|G_n b\| \geq \lambda' \|b\|$ for $b \in \widetilde{E}^u_n$,

(e) $\|G_n b\| \leq (\lambda')^{-1} \|b\|$ for $b \in \widetilde{E}^s_n$,

where $\lambda' > 1$ is a constant satisfying $\lambda > \lambda' > 1$. By setting $T_{n+t}M = T_{v(0)\cdot h(n+t)}M = \widehat{E}^s_{n+t} \oplus \widehat{E}^o_{n+t} \oplus \widehat{E}^u_{n+t}$ and $\widehat{E}^\sigma_{n+t} = \widehat{E}^\sigma_{v(0)\cdot h(n+t)}$ for $\sigma = s, o, u$, and $G_{n+t} = T_{v(0)\cdot h(n+t)} G(h(n+1+t) - h(n+t))$ for every $n \in \mathbb{Z}$ where some $t$ with $0 < t < 1$, we can show that there exist a sequence of $(\dim M - j - 1)$-dimensional subspaces $\widetilde{E}^u_{n+t} \subset T_{n+t}M$ and a sequence of $j$-dimensional subspaces $\widetilde{E}^s_{n+t} \subset T_{n+t}M$ satisfying properties corresponding to (a), (b), (c), (d), and (e). Then it is easy to see that $\bigcup_{t=-\infty}^{\infty} T_t M = \bigcup_{t=-\infty}^{\infty} \widetilde{E}^s_t \oplus \widetilde{E}^o_t \oplus \widetilde{E}^u_t$ is a continuous splitting with respect to $t$ such that each $\widetilde{E}^\sigma_t$ is $TG(h(t))$-invariant for $\sigma = s, o, u$. Moreover, $v(h(t))$ is hyperbolic for $G(h(t))$. By [**8**, Proposition 17.4.5], we can check that $v(t)$ is hyperbolic for $G(t)$. Hence $M = A(G)$ is hyperbolic for $G(t)$ sufficiently $C^1$ close to $F(t)$. By Theorem 3.2, there exist a homeomorphism $H_F : \widetilde{S}(G) \to \widetilde{S}(F)$ and nondecreasing automorphisms $\beta^G_w : \mathbb{R} \to \mathbb{R}$ depending on $w \in \widetilde{S}(G)$ satisfying

$$[H_F \circ \widetilde{G}(t)](w) = [\widetilde{F}(\beta^G_w(t)) \circ H_F](w) \quad \text{for all } w \in \widetilde{S}(G) \text{ and } t \in \mathbb{R}.$$

Then $H_F \circ H_G$ is a homeomorphism from $\widetilde{S}(F)$ into itself. In fact,

$$\begin{aligned}
[(H_F \circ H_G) \circ \widetilde{F}(t)](v) &= H_F \circ [H_G \circ \widetilde{F}(t)](v) \\
&= H_F \circ [\widetilde{G}(\beta_v(t)) \circ H_G](v) \\
&= [H_F \circ \widetilde{G}(\beta_v(t))](H_G(v)) \\
&= [\widetilde{F}(\beta^G_{H_G(v)}(\beta_v(t))) \circ H_F](H_G(v)) \\
&= [\widetilde{F}(\beta^G_{H_G(v)}(\beta_v(t))) \circ (H_F \circ H_G)](v).
\end{aligned}$$

Here $\beta^G_{H_G(v)}(\beta_v(t))$ is obviously a nondecreasing automorphism of $\mathbb{R}$. For $G(t)$ sufficiently $C^1$ near $F(t)$, it follows from Theorem 3.2 that $H_F \circ H_G$ is the identity map of $\widetilde{S}(F)$. Hence $H_G$ is surjective so $H_G$ is a homeomorphism from $\widetilde{S}(F)$ onto $\widetilde{S}(G)$. □

*Remark 4.3.* If $M$ is infinite-dimensional (i.e. a Banach manifold), then the attractor of semiflow on $M$ is *not* equal to $M$ in general. So we cannot use the argument in the proof of Theorem 4.3, because we do not know whether an infinite $(\varepsilon, 1)$-chain for $F(t)$ $\delta$-traced by a global solution $w(t)$ for $G(t)$ is contained in $A(F) \subsetneqq M$.

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*Remark 4.4.* If $M$ is a closed manifold, then it follows from Proposition 2.8 in [**11**] and the semigroup property that $F(t)$ is a diffeomorphism of $M$ for each $t \in \mathbb{R}$. That is, if $M$ is a closed manifold then Anosov semiflows on $M$ are Anosov flows.

## 5. *Examples*

In this section we construct examples of $C^1$ Anosov semiflows which are not flows.

Consider a (2,2)-matrix $A_n = \left(\begin{smallmatrix} n & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with $n \in \mathbb{Z}^+$. Then $A_n$ can be thought of as a linear transformation of the plane $R^2$ which preserves the lattice $L$ of points with integer coordinates. There is an induced endomorphism $\widetilde{A}_n$ of the quotient $R^2/L = T^2$, two-dimensional torus, onto itself for $n \geq 3$. For $n = 2$ it is well known that $\widetilde{A}_n$ is an Anosov diffeomorphism [**19**]. It is easy to see that this endomorphism is an Anosov endomorphism (see [**13**] for the definition).

To construct Anosov semiflows we use this Anosov endomorphism $\widetilde{A}_n$. Note that $\widetilde{A}_n$ contains no singularities and is a $C^\infty$ covering map. The degree of $\widetilde{A}_n$, $n \geq 2$, is equal to $n - 1$. Then we can define the suspension of $T^2$ by the same method as in the diffeomorphisms case [**11, 12, 19**].

Let $\widetilde{T}^2 = T^2 \times [0, 1]/\sim$, where $\sim$ denotes the identification of $(x, 1)$ with $(\widetilde{A}_n(x), 0)$ for each $x \in T^2$. We define a semiflow $F(t)$ on $\widetilde{T}^2$ as follows.

For each $x$ in $\widetilde{T}^2$ we can present $x = (x_o, s)$ with $0 \leq s < 1$, so we set

$$F(t)(x) = \begin{cases} (x_o, s + t) & \text{for } 0 \leq t < 1 - s, \\ (\widetilde{A}_n(x_o), t + s - 1) & \text{for } 1 - s \leq t \leq 2 - s, \\ (\widetilde{A}_n^2(x_o), t + s - 2) & \text{for } 2 - s \leq t \leq 3 - s, \\ \text{etc.} \end{cases}$$

To check that $F(t)$ is an Anosov semiflow, we have to see that there exists a $C^\infty$ structure on $\widetilde{T}^2$ such that $F(t)$ is a $C^1$ strong semiflow. Obviously, $\widetilde{T}^2$ is not a usual manifold. However we can introduce $\widetilde{T}^2$ to a natural differential structure. That is, $\widetilde{T}^2$ is a compact smooth non-singular branched manifold without boundary [**20**]. We shall show the following.

THEOREM 5.1. *For $\widetilde{T}^2$ and $F(t)$ as above, there is a $C^\infty$ structure on $\widetilde{T}^2$ such that $F(t)$ is a $C^1$ strong semiflow.*

For the proof we use the following.

LEMMA 5.2. [**12**] *Let $M$ be a smooth manifold. Then there is a $C^1$ neighborhood $\mathcal{U}$ of the identity in* $\mathrm{Diff}^1(M)$ *(with respect to the strong Whitney topology) such that, for every $\Psi \in \mathcal{U}$, there is a $C^1$ vector field $Z_\Psi$ on $M \times [0, 1]$ such that:*

(1) $Z_\Psi(x, t) = (\partial/\partial t)$ *for* $t \in [0, 1/4]$ *and* $t \in [3/4, 1]$*;*

(2) *the time one map of $Z_\Psi$ maps $(x, 0)$ to $(\Psi(x), 1)$.*

*Proof of Theorem 5.1.* Let $(U_i, \varphi_i)$, $i = 1, \ldots, s$, be local charts on $T^2$ such that

(1) $\bigcup_{i=1}^s U_i = T^2$;

(2) for each $i$, $\widetilde{A}_n|U_i$ is a $C^\infty$ diffeomorphism from $U_i$ to its image.

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Then it is obvious that $\{(U_i \times (-\frac{1}{4}, \frac{1}{4}), \varphi_i \times \mathrm{id})\}$ is a $C^\infty$ atlas on $T^2 \times (-\frac{1}{4}, \frac{1}{4})$. Let $\mathfrak{A}$ be the $C^\infty$ structure on $T^2 \times (-\frac{1}{4}, \frac{1}{4})$ induced by $\{(U_i \times (-\frac{1}{4}, \frac{1}{4}), \varphi_i \times \mathrm{id})\}$. Let $\widehat{T} = T^2 \times ([0, \frac{1}{4}) \cup (\frac{3}{4}, 1])/\sim \subset \widetilde{T}^2$. Define a map $\Sigma_1 : T^2 \times (-\frac{1}{4}, \frac{1}{4}) \to \widehat{T} \subset \widetilde{T}^2$ by:

$$(x, t) \mapsto \begin{cases} (\widetilde{A}_n(x), t), & \text{for } t \in [0, \frac{1}{4}), \\ (x, t+1), & \text{for } t \in (-\frac{1}{4}, 0]. \end{cases}$$

We claim that $\{(\Sigma_1(U_i \times (-\frac{1}{4}, \frac{1}{4})), (\varphi_i \times \mathrm{id}) \circ (\Sigma_1 | U_i \times (-\frac{1}{4}, \frac{1}{4}))^{-1})\}$ is a $C^\infty$ atlas on $\widehat{T}$. Let $\psi_i = (\varphi_i \times \mathrm{id}) \circ (\Sigma_1 | U_i \times (-\frac{1}{4}, \frac{1}{4}))^{-1}$. If $\Sigma_1(U_i \times (-\frac{1}{4}, \frac{1}{4})) \cap \Sigma_1(U_j \times (-\frac{1}{4}, \frac{1}{4})) \neq \phi$ and $U_i \cap U_j \neq \phi$, then $\psi_j \circ \psi_i^{-1}(x, t) = (\varphi_j \circ \varphi_i^{-1}(x), t)$ is a $C^\infty$ diffeomorphism. If $\Sigma_1(U_i \times (-\frac{1}{4}, \frac{1}{4})) \cap \Sigma_1(U_j \times (-\frac{1}{4}, \frac{1}{4})) \neq \phi$ and $U_i \cap U_j = \phi$, then $\psi_j \circ \psi_i^{-1}(x, t) = (\varphi_j \circ (\widetilde{A}_n | U_j)^{-1} \circ (\widetilde{A}_n | U_i) \circ \varphi_i^{-1}(x), t)$ is a $C^\infty$ diffeomorphism (because $\widetilde{A}_n | U_i$ is a $C^\infty$ diffeomorphism for each $i$).

Let $\sigma_1$ be the $C^\infty$ structure on $\widehat{T} \subset \widetilde{T}^2$ induced by the $C^\infty$ structure $\mathfrak{A}$ on $T^2 \times (-\frac{1}{4}, \frac{1}{4})$ through the map $\Sigma_1 : T^2 \times (-\frac{1}{4}, \frac{1}{4}) \to \widetilde{T}^2$.

Let $\check{T} = T^2 \times (0, 1) \subset \widetilde{T}^2$. Obviously $\{(U_i \times (0, 1), \varphi_i \times \mathrm{id})\}$ is a $C^\infty$ atlas on $T^2 \times (0, 1)$. Let $\mathfrak{B}$ be the $C^\infty$ structure on $T^2 \times (0, 1)$ induced by $\{(U_i \times (0, 1), \varphi_i \times \mathrm{id})\}$. Define a map $\Sigma_2 : T^2 \times (0, 1) \to \widetilde{T}^2$, defined by $(x, t) \mapsto (\widetilde{A}_n(x), t)$.

Then $\{(\Sigma_2(U_i \times (0, 1)), (\varphi_i \times \mathrm{id}) \circ (\Sigma_2 | U_i \times (0, 1))^{-1})\}$ is a $C^\infty$ atlas on $\check{T}$. Let $\rho_i = (\varphi_i \times \mathrm{id}) \circ (\Sigma_2 | U_i \times (0, 1))^{-1}$. If $\Sigma_2(U_i \times (0, 1)) \cap \Sigma_2(U_j \times (0, 1)) \neq \phi$ and $U_i \cap U_j \neq \phi$, then $\rho_j \circ \rho_i^{-1}(x, t) = (\varphi_j \circ \varphi_i^{-1}(x), t)$ is a $C^\infty$ diffeomorphism. If $\Sigma_2(U_i \times (0, 1)) \cap \Sigma_2(U_j \times (0, 1)) \neq \phi$ and $U_i \cap U_j = \phi$, then $\rho_j \circ \rho_i^{-1}(x, t) = (\varphi_j \circ (\widetilde{A}_n | U_j)^{-1} \circ (\widetilde{A}_n | U_i) \circ \varphi_i^{-1}(x), t)$ is a $C^\infty$ diffeomorphism. Let $\sigma_2$ be the $C^\infty$ structure on $\check{T} \subset \widetilde{T}^2$ induced by the $C^\infty$ structure $\mathfrak{B}$ on $T^2 \times (0, 1)$ through the map $\Sigma_2 : T^2 \times (0, 1) \to \widetilde{T}^2$. Let $Z_{\mathrm{id}}$ be a $C^1$ vector field on $T^2 \times (0, 1)$ such that $Z_{\mathrm{id}}(x, t) = (\frac{\partial}{\partial t})$ for $t \in (0, 1)$, obtained by Lemma 5.2. Let $\Psi_t$ be a $C^1$ flow on $T^2 \times (0, 1)$ generated by $Z_{\mathrm{id}}$. Then $\Sigma_2 \circ \Psi_t = F(t) \circ \Sigma_2$ on $T^2 \times (0, 1)$. Hence $\sigma_1$ and $\sigma_2$ agree on their common domain $T^2 \times [(0, \frac{1}{4}) \cup (\frac{3}{4}, 1)] \subset \widetilde{T}^2$ and hence define a $C^\infty$ structure on $\widetilde{T}^2$ which makes $F(t)(x)$ a $C^1$ function with respect to $t$. □

*Remark.* In fact $F(t) : \widetilde{T}^2 \hookleftarrow$ is $C^\infty$. Moreover, $F(\cdot)(x) : \mathbb{R}^+ \to \widetilde{T}^2$ is $C^\infty$.

Now we know that $F(t)$ is a $C^1$ strong semiflow on $\widetilde{T}^2$. It is easy to see that $F(t)$ is an Anosov semiflow by the same argument in which Anosov flows are induced by the suspension of Anosov diffeomorphisms.

It is easy to see that the periodic points of $\widetilde{A}_n : T^2 \hookleftarrow$ are dense in $T^2$ (for instance, see [**11**, p. 157]). Hence $\widetilde{T}^2 = \bigcap_{t \geq 0} F(t)(\widetilde{T}^2) = A(F)$ has the dense subset of the points on periodic solutions for $F(t)$, i.e. $\widetilde{T}^2 = \mathrm{Cl}(\mathrm{Per}(F))$. So it is obvious that $\widetilde{T}^2 = \Omega(F)$. Furthermore, by using Corollary 1.22 from [**13**] we obtain that the property that the nonwandering set is coincident with the whole maniflod $\widetilde{T}^2$ is persistent, i.e. $\widetilde{T}^2 = \Omega(G)$ for $G(t)$ sufficiently $C^1$ close to $F(t)$.

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