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# ON UNIQUENESS IN ELECTROMAGNETIC SCATTERING FROM BIPERIODIC STRUCTURES

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**Abstract.** Consider time-harmonic electromagnetic wave scattering from a biperiodic dielectric structure mounted on a perfectly conducting plate in three dimensions. Given that uniqueness of solution holds, existence of solution follows from a well-known Fredholm framework for the variational formulation of the problem in a suitable Sobolev space. In this paper, we derive a Rellich identity for a solution to this variational problem under suitable smoothness conditions on the material parameter. Under additional non-trapping assumptions on the material parameter, this identity allows us to establish uniqueness of solution for all positive wave numbers.

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## 1. Introduction

Scattering of electromagnetic waves from periodic structures is not only an interesting mathematical topic in its own right but also of great interest in applications, *e.g.* for the construction and optimization of optical filters, lenses, and beam-splitters in optics. An overview about this and further topics in applied mathematics related to wave propagation in periodic structures can be found in, *e.g.* [5]. In this paper we consider scattering of time-harmonic electromagnetic waves from a dielectric biperiodic structure mounted on a perfectly conducting plate in three dimensions. By biperiodic, we mean that the structure is periodic in the, say, $x_1$- and $x_2$-direction, while it is bounded in the $x_3$ direction. In contrast to scattering from bounded structures, uniqueness of solution for this scattering problem does in general not hold for all positive wave numbers. Instead, non-trivial solutions to the homogeneous problem might exist for a discrete set of exceptional wave numbers, and these solutions turn out to be exponentially localized surface waves. Our study in the present paper focuses on conditions guaranteeing the well-posedness of the full three-dimensional electromagnetic scattering problem mentioned above. We establish non-trapping and smoothness conditions on the (non-absorbing) dielectric such that uniqueness of solution holds for *all* positive wave numbers. This means that materials satisfying the latter conditions cannot guide surface waves.

Mathematical formulation for the well-posedness of electromagnetic scattering problem for periodic structures has been an active area of research in the last years. For the scalar case, the authors in [2,17] studied uniqueness of

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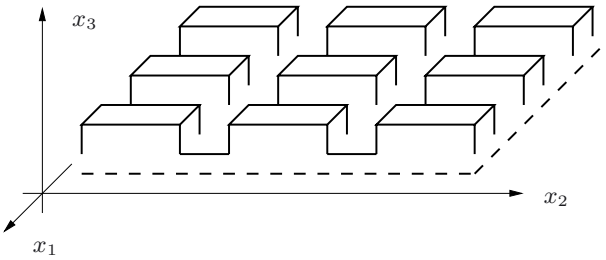



FIGURE 1. Sketch of the biperiodic structure under consideration.

solution for all wave numbers (or, equivalently, all frequencies), under geometrical conditions on the scatterer, for impenetrable structures with Dirichlet and Neumann conditions. Similar results are obtained in the paper [7] for more complicated periodic structures which are constituted of conducting and dielectric materials. The latter paper further gave examples of structures for which non-uniqueness of solution occurs at the so-called singular wave numbers. These wave numbers were shown to be related to guided waves (surface waves) that are exponentially localized along the structure.

For the case of Maxwell's equations, the authors of [10] studied well-posedness of the scattering problem from a medium consisting of two homogeneous materials separated by a smooth biperiodic surface using an integral equation approach. In [4,6,11] the authors studied existence and uniqueness of solution for the scattering problem from penetrable biperiodic structures using a variational approach for the magnetic field. Nevertheless, unlike the scalar cases, the uniqueness results in cited cases of Maxwell's equations were proven for all but possibly a discrete set of wave numbers. Furthermore, all the cited papers above considered the non-magnetic case, *i.e.*, the coefficient magnetic permeability is assumed to be the same constant outside and inside the structure. The case of variable magnetic permeability were investigated in the paper [1] for Maxwell's equations where the biperiodic structure consists of conducting and dielectric materials. That paper studied a variational approach, formulated in terms of the electric field, and showed that the obtained saddle point problem satisfies the Fredholm alternative, and again uniqueness of solution was proven for all but possibly a discrete set of wave numbers. More recently, the paper [16] analyzed the well-posedness of the scattering problem for penetrable anisotropic biperiodic structures with a restriction on the non-magnetic case again. The latter paper also proved that the scattering problem is uniquely solvable for all wave numbers if the structure contains absorbing materials, and if the dielectric tensor is piecewise analytic. Hence, to the best of our knowledge, uniqueness results for all wave numbers for the vectorial scattering problem still remains open if the biperiodic materials is non-absorbing.

The aim of the present work is to prove that the electromagnetic scattering problem for non-absorbing biperiodic dielectric structures mounted on a perfectly conducting plate is uniquely solvable for all positive wave numbers if the material parameter satisfies non-trapping and smoothness conditions. We formulate the Maxwell's equations variationally in terms of the magnetic field in a suitable Sobolev space. We further restrict ourselves to the case of non-magnetic and isotropic materials. The variational problem is well-known to fit into a Fredholm framework, see, *e.g.* [4,11,16]. (These papers deal with periodic scattering in the full space, but can be adapted to the half-space setting that we consider here). As mentioned in the paper [7] on the corresponding scalar scattering problems, non-uniqueness phenomena indeed arise at certain singular wave numbers if the non-absorbing material parameter satisfies suitable trapping conditions. In this paper we show a converse result for the full three-dimensional periodic Maxwell equations: uniqueness of solution holds for all positive wave numbers if the material parameter is non-absorbing and satisfies suitable non-trapping and smoothness conditions. To prove the uniqueness result we derive a so-called Rellich identity for a solution to the homogeneous variational problem. The solution estimates resulting from this integral identity allow us to show that the homogeneous variational problem has only the trivial solution for *all* positive wave numbers.

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Our analysis extends the approach in [12] that was motivated by an existence and uniqueness proof for solutions to rough surface scattering problems via Rellich identities in [8]. For scalar periodic problems, a related technique has been used in [7]. The paper [12] studied electromagnetic scattering from rough, unbounded penetrable layers. Such scattering problems are considered to be more complicated than those for periodic structures since the problem to find the scattered field cannot reduced, *e.g.* to a bounded domain. The applications of rough scattering problems include for instance outdoor noise propagation, oceanography or even optical technologies when the dielectric lacks periodicity. The authors in [12] formulated the latter scattering problem in terms of the electric field. We will instead choose a formulation in terms of the magnetic field, which somewhat changes the role of the dielectric material parameter in the integral identities since the material is non-magnetic. The paper [12] establishes existence and uniqueness of solution under non-trapping and smoothness conditions on the material parameter. While *a priori* estimates resulting from the Rellich identity allowed the authors in [12] to deduce uniqueness of solution, existence of solution has been obtained using a limiting absorption argument. The approach studied in the present paper is, from the technical point of view, somewhat similar to the one introduced in [12]. However, the analysis of the biperiodic case is definitely simpler since uniqueness of solution directly implies existence. Therefore, one only needs to investigate the Rellich identity and estimates for solutions to the homogeneous problem. It turns out also that this procedure produces weaker assumptions on the material parameter than those found in [12]. More precisely, uniqueness and existence of solution for all wave numbers are obtained under the following (non-trapping and smoothness) assumptions on the biperiodic relative material parameter $\varepsilon_{\mathrm{r}} : \mathbb{R}^3_+ := \{x \in \mathbb{R}^3,\, x_3 > 0\} \to \mathbb{R}$. First, we assume that $\varepsilon_{\mathrm{r}}^{-1} \in L^\infty(\mathbb{R}^3_+)$ equals one in $\{x_3 > h\}$ for some $h > 0$ and possesses essentially bounded and measurable first weak derivatives. Second, we require that

$$(a) \quad \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \leq 0 \text{ in } \mathbb{R}^3_+,$$

$$(b) \quad \text{It holds that } \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} < 0 \text{ in some non-empty open subset of } \mathbb{R}^3_+,$$

$$(c) \quad \text{There exists } \delta > 1/2 \text{ such that } \frac{\delta}{2}\|\nabla_T \varepsilon_{\mathrm{r}}^{-1}\|^2_{L^\infty(\mathbb{R}^3_+)^3} + \frac{\sqrt{2}}{h}\left\|\frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\right\|_{L^\infty(\mathbb{R}^3_+)} < \frac{2}{h^2},$$

where $\nabla_T \varepsilon_{\mathrm{r}}^{-1} := (\partial \varepsilon_{\mathrm{r}}^{-1}/\partial x_1, \partial \varepsilon_{\mathrm{r}}^{-1}/\partial x_2, 0)^\top$. Under these conditions, the existence of surface waves is automatically ruled out. While conditions (a) and (c) are similar to conditions (a) and (d) in [12], Equation (7.2), condition (b) is weaker and clearly simpler than the corresponding conditions (b) and (c) in [12], Equation (7.2).

The half-space setting that we consider in this paper is somewhat special, and it seems worth to mention that the Rellich identity itself generalizes to a corresponding periodic scattering problem in full space. The resulting estimate for a solution $H$ to the scattering problem has a similar structure to the estimate in Lemma 5.4. However, in the half-space setting, the term $2\mathrm{Re}\int_\Omega (\partial \varepsilon_{\mathrm{r}}^{-1}/\partial x_3)\,(\partial H_3/\partial x_3)\overline{H_3}\,\mathrm{d}x$ can be treated without integration by parts using a Poincaré lemma. In contrast, in the full-space setting the only obvious way of treating this term is to integrate by parts. Since we seek for solution estimates, this introduces the condition that $x_3 \mapsto \varepsilon_{\mathrm{r}}^{-1}(x_1, x_2, x_3)$ needs to be concave to conclude. Since this is a somewhat unnatural condition, we do not present this result in more detail.

One can further generalize the results presented here to certain anisotropic structures. However, already for the simpler case of isotropic coefficients the derivation of the Rellich identity is a technical matter. Again, we have opted to try to keep the presentation simple instead of treating the most general setting that could be considered.

The paper is organized as follows: In Section 2 we present setting of the problem. Section 3 is dedicated to a variational formulation and to the Fredholm property of the latter. Section 4 contains a couple of technical lemmas. We derive the integral inequalities resulting from the Rellich identity in Section 5. Finally, the uniqueness of the variational problem for all wave numbers is proven in Section 6.

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**Notation.** We denote by $H^s(\mathbb{R}^d)^3$, $d = 2, 3$, the usual $L^2$-based Sobolev space of vector-valued functions in $\mathbb{R}^d$. Moreover, $H^s_{\mathrm{loc}}(\mathbb{R}^3)^3 = \{v \in H^s(B)^3 \text{ for all balls } B \subset \mathbb{R}^3\}$, and $W^{1,\infty}(\mathbb{R}^3) = \{v \in L^\infty(\mathbb{R}^3) : \nabla v \in L^\infty(\mathbb{R}^3)^3\}$.

## 2. Problem setting

We consider scattering of time-harmonic electromagnetic waves from a biperiodic structure which models a dielectric layer mounted on a perfectly conducting plate. The electric field $E$ and the magnetic field $H$ are governed by the time-harmonic Maxwell equations at frequency $\omega > 0$ in $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$,

$$\operatorname{curl} H + i\omega\varepsilon E = 0 \qquad \text{in} \quad \mathbb{R}^3_+, \tag{2.1}$$

$$\operatorname{curl} E - i\omega\mu H = 0 \qquad \text{in} \quad \mathbb{R}^3_+, \tag{2.2}$$

$$e_3 \times E = 0 \qquad \text{on} \quad \{x_3 = 0\}, \tag{2.3}$$

where $e_3 = (0, 0, 1)^\top$. The electric permittivity $\varepsilon$ is a bounded measurable function that is $2\pi$-periodic in $x_1$ and $x_2$. Further, we assume that $\varepsilon$ equals $\varepsilon_0 > 0$ outside the biperiodic structure, that is, for $x_3 \geq h$ where $h > 0$ is chosen larger than $\sup\{x_3 : (x_1, x_2, x_3)^\top \in \operatorname{supp}(\varepsilon - \varepsilon_0)\}$. The magnetic permeability $\mu = \mu_0$ is assumed to be a positive constant and the conductivity is assumed to vanish. As usual, the problem (2.1)–(2.3) has to be completed by a radiation condition that we set up using Fourier series.

The biperiodic structure is illuminated by an electromagnetic plane wave with wave vector $d = (d_1, d_2, d_3) \in \mathbb{R}^3$, $d_3 < 0$, such that $d \cdot d = \omega^2\varepsilon_0\mu_0$. The polarizations $p, q \in \mathbb{R}^3$ of the incident wave satisfy $p \cdot d = 0$ and $q = 1/(\omega\varepsilon_0)(p \times d)$. With these definitions, the incident plane waves $E^i$ and $H^i$ are given by

$$E^i := q\mathrm{e}^{id\cdot x}, \quad H^i := p\mathrm{e}^{id\cdot x}, \quad x \in \mathbb{R}^3_+.$$

In the following we will exploit that one can explicitly compute the corresponding reflected field at $\{x_3 = 0\}$. To this end, we introduce the notation $\tilde{a} = (a_1, a_2, -a_3)^\top$ for $a = (a_1, a_2, a_3)^\top \in \mathbb{R}^3$. The reflected waves at the plane $\{x_3 = 0\}$ are

$$E^r(x) := -\tilde{q}\mathrm{e}^{i\tilde{d}\cdot x}, \quad H^r(x) := \tilde{p}\mathrm{e}^{i\tilde{d}\cdot x}, \quad x \in \mathbb{R}^3_+,$$

since $\operatorname{div} E^r = 0$, $\operatorname{div} H^r = 0$, and $e_3 \times (E^i + E^r) = 0$, $e_3 \cdot (H^i + H^r) = 0$ on $\{x_3 = 0\}$. From now on, we denote the sum of the incident and reflected plane waves by

$$E^{ir} := E^i + E^r \quad \text{and} \quad H^{ir} := H^i + H^r.$$

Set

$$\alpha = (\alpha_1, \alpha_2, \alpha_3)^\top := (d_1, d_2, 0)^\top$$

and define $E^{ir}_\alpha$ and $H^{ir}_\alpha$ by

$$E^{ir}_\alpha := \mathrm{e}^{-i\alpha\cdot x}E^{ir}(x), \quad H^{ir}_\alpha := \mathrm{e}^{-i\alpha\cdot x}H^{ir}(x), \quad x \in \mathbb{R}^3_+,$$

such that $E^{ir}_\alpha$ and $H^{ir}_\alpha$ are $2\pi$-periodic in $x_1$ and $x_2$. If we apply the same phase shift to solutions $E$ and $H$ of the Maxwell equations (2.1)–(2.3),

$$E_\alpha = \mathrm{e}^{-i\alpha\cdot x}E(x), \quad H_\alpha = \mathrm{e}^{-i\alpha\cdot x}H(x),$$

and if we denote

$$\nabla_\alpha f = \nabla f + i\alpha f, \quad \operatorname{curl}_\alpha F = \operatorname{curl} F + i\alpha \times F, \quad \operatorname{div}_\alpha F = \operatorname{div} F + i\alpha \cdot F$$

for scalar functions $f$ and vector fields $F$, then $E_\alpha$ and $H_\alpha$ satisfy

$$\operatorname{curl}_\alpha H_\alpha + i\omega\varepsilon E_\alpha = 0 \qquad \text{in} \quad \mathbb{R}^3_+, \tag{2.4}$$

$$\operatorname{curl}_\alpha E_\alpha - i\omega\mu_0 H_\alpha = 0 \qquad \text{in} \quad \mathbb{R}^3_+, \tag{2.5}$$

$$e_3 \times E_\alpha = 0 \qquad \text{on} \quad \{x_3 = 0\}. \tag{2.6}$$

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Note that we still have $\operatorname{div}_\alpha \operatorname{curl}_\alpha = 0$ and $\operatorname{curl}_\alpha \nabla_\alpha = 0$. Let us denote the relative material parameter by

$$\varepsilon_{\mathrm{r}} := \frac{\varepsilon}{\varepsilon_0}.$$

Obviously, $\varepsilon_{\mathrm{r}}$ equals one outside the biperiodic dielectric structure. Recall that the magnetic permeability $\mu_0$ is constant which motivates us to work with the divergence-free magnetic field, that is, $\operatorname{div}_\alpha H_\alpha = 0$.

Note that (2.4) plugged in into (2.6) implies that $e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H_\alpha) = 0$ on $\{x_3 = 0\}$ and that the condition $e_3 \cdot H_\alpha = 0$ on $\{x_3 = 0\}$ can be derived by plugging (2.6) into (2.5). Hence, introducing the wave number $k = \omega(\epsilon_0\mu_0)^{1/2}$, and eliminating the electric field $E_\alpha$ from (2.4)–(2.6), we find that

$$\operatorname{curl}_\alpha \left(\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H_\alpha\right) - k^2 H_\alpha = 0 \qquad \text{in} \quad \mathbb{R}^3_+, \tag{2.7}$$

$$e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H_\alpha) = 0 \qquad \text{on} \quad \{x_3 = 0\}, \tag{2.8}$$

$$e_3 \cdot H_\alpha = 0 \qquad \text{on} \quad \{x_3 = 0\}. \tag{2.9}$$

We now formally reformulate the last three equations in terms of the scattered field $H^s_\alpha$, defined by $H^s_\alpha := H_\alpha - H^{ir}_\alpha$. Since, by construction, $\operatorname{curl}_\alpha \operatorname{curl}_\alpha H^{ir}_\alpha - k^2 H^{ir}_\alpha = 0$ in $\mathbb{R}^3_+$, $H^{ir}_\alpha \cdot e_3 = 0$ and $e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H^{ir}_\alpha) = 0$ on $\{x_3 = 0\}$, a simple computation shows that

$$\begin{aligned} \operatorname{curl}_\alpha \left(\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H^s_\alpha\right) - k^2 H^s_\alpha &= -\operatorname{curl}_\alpha \left((\varepsilon_{\mathrm{r}}^{-1} - 1) \operatorname{curl}_\alpha H^{ir}_\alpha\right) \qquad \text{in} \quad \mathbb{R}^3_+, \\ e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H^s_\alpha) &= 0 \qquad \text{on} \quad \{x_3 = 0\}, \\ e_3 \cdot H^s_\alpha &= 0 \qquad \text{on} \quad \{x_3 = 0\}. \end{aligned} \tag{2.10}$$

Due to the biperiodicity of the right-hand side and of $\varepsilon_{\mathrm{r}}$, we seek for a biperiodic solution $H^s_\alpha$, and reduce the problem to the domain $(0, 2\pi)^2 \times (0, \infty)$. We complement this boundary value problem by a radiation condition, see also in [6, 11], that we set up using Fourier series. The scattered field $H^s_\alpha$ is $2\pi$-periodic in $x_1$ and $x_2$ and can hence be expanded as

$$H^s_\alpha(x) = \sum_{n \in \Lambda} \hat{H}_n(x_3) \mathrm{e}^{in\cdot x}, \quad x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3_+, \ \Lambda = \mathbb{Z}^2 \times \{0\}, \tag{2.11}$$

where the Fourier coefficients $\hat{H}_n(x_3)$ are defined by

$$\hat{H}_n(x_3) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} H^s_\alpha(x_1, x_2, x_3) \mathrm{e}^{-in\cdot x} \, \mathrm{d}x_1 \, \mathrm{d}x_2, \quad n \in \Lambda. \tag{2.12}$$

Define

$$\beta_n := \begin{cases} \sqrt{k^2 - |n + \alpha|^2}, & k^2 \geq |n + \alpha|^2, \\ i\sqrt{|n + \alpha|^2 - k^2}, & k^2 < |n + \alpha|^2, \end{cases} \quad n \in \Lambda.$$

Since $\varepsilon_{\mathrm{r}}^{-1}$ equals one for $x_3 > h$ it holds that $\operatorname{div}_\alpha H^s_\alpha$ vanishes for $x_3 > h$, and equation (2.10) becomes $(\Delta_\alpha + k^2) H^s_\alpha = 0$ in $\{x_3 > h\}$, where $\Delta_\alpha = \Delta + 2i\alpha \cdot \nabla - |\alpha|^2$. Using separation of variables, and choosing the upward propagating solution, we set up a radiation condition in form of a Rayleigh expansion condition, prescribing that $H^s_\alpha$ can be written as

$$H^s_\alpha(x) = \sum_{n \in \Lambda} \hat{H}_n \mathrm{e}^{i\beta_n(x_3 - h) + in\cdot x} \qquad \text{for} \quad \{x_3 > h\}, \quad \text{where } \hat{H}_n := \hat{H}_n(h), \tag{2.13}$$

and that the series converges uniformly in compact subsets of $\{x_3 > h\}$.

The scattering problem to find a scattered field $H^s_\alpha$ that satisfies the boundary value problem (2.10) and the expansion (2.13) is in the following section reformulated variationally in a suitable Sobolev space.

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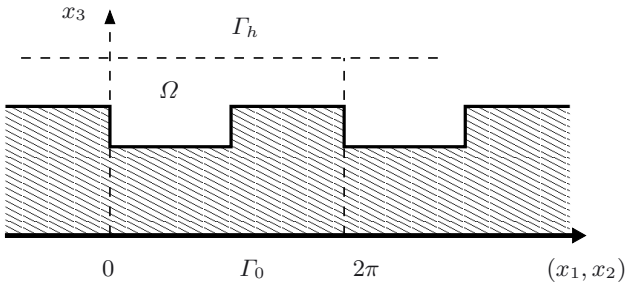



FIGURE 2. Geometric setting for electromagnetic scattering problem from a biperiodic dielectric structure mounted on a perfectly conducting plate (in two dimensions, for simplicity).

## 3. Variational formulation

We solve the scattering problem presented in the last section variationally, and briefly recall in this section a variational formulation of the problem in a suitable Sobolev space. Our framework is an adaption of the results from [16] to our half-space setting. In contrast to the variational formulation in $H(\mathrm{curl})$ in [1], the papers [4, 6, 11, 16] set up a variational formulation in $H^1$ for the magnetic field. Indeed, since the latter is divergence-free, any solution that is locally $H(\mathrm{curl})$ indeed belongs locally to $H^1$. For our purposes, the $H^1$ formulation has the additional advantage that it is well-defined at Rayleigh–Wood frequencies, as it was noted in [16]. We define a bounded domain

$$\Omega = (0, 2\pi)^2 \times (0, h) \qquad \text{for } h > \sup\{x_3 : (x_1, x_2, x_3)^\top \in \mathrm{supp}(\varepsilon_\mathrm{r} - 1)\},$$

with boundaries $\Gamma_0 := (0, 2\pi)^2 \times \{0\}$ and $\Gamma_h := (0, 2\pi)^2 \times \{h\}$, and Sobolev spaces

$$H^\ell_\mathrm{p}(\Omega)^3 := \{F \in H^\ell(\Omega)^3 : F = \tilde{F}|_\Omega \text{ for some } 2\pi\text{-biperiodic } \tilde{F} \in H^\ell_\mathrm{loc}(\mathbb{R}^3)^3\}, \quad \ell \in \mathbb{N},$$
$$H^1_{\mathrm{p,T}}(\Omega)^3 := \{F = (F_1, F_2, F_3)^\top \in H^1_\mathrm{p}(\Omega)^3 : F_3 = 0 \text{ on } \Gamma_0\},$$

equipped with the usual integral norm, *e.g.*,

$$\|F\|^2_{H^1_\mathrm{p}(\Omega)^3} = \|F\|^2_{L^2(\Omega)^3} + \|\nabla_\alpha F\|^2_{L^2(\Omega)^3}.$$

The space $H^1_{\mathrm{p,T}}(\Omega)^3$ of periodic vector fields that are tangential on $\Gamma_0$ is well-defined due to the standard trace theorem in $H^1$. We also define periodic Sobolev spaces of functions with $d = 1, 2, 3$ components on $\Gamma_h$: for $s \in \mathbb{R}$,

$$H^s_\mathrm{p}(\Gamma_h)^d := \{F \in H^s(\Gamma_h)^d : F = \tilde{F}|_{\Gamma_h} \text{ for some } 2\pi\text{-biperiodic } \tilde{F} \in H^s_\mathrm{loc}(\{x_3 = h\})^d\}.$$

A periodic vector field $F \in H^s(\Gamma_h)^d$ can be developed in a Fourier series, $F(x) = \sum_{n\in\Lambda} \hat{F}_n \exp(in \cdot x)$, and $\|F\|_{H^s_\mathrm{p}(\Gamma_h)^d} = (\sum_{n\in\Lambda}(1 + n^2)^s |\hat{F}_n|^2)^{1/2}$ defines a norm on $H^s_\mathrm{p}(\Gamma_h)^d$.

We define a non-local boundary operator $T_\alpha$ (the exterior Dirichlet–Neumann operator) by

$$(T_\alpha f)(x) = \sum_{n\in\Lambda} i\beta_n \hat{f}_n \mathrm{e}^{in\cdot x}, \quad \text{for } f = \sum_{n\in\Lambda} \hat{f}_n \exp(in \cdot x) \in H^{1/2}_\mathrm{p}(\Gamma_h).$$

It is a classical result that $T_\alpha$ is bounded from $H^{1/2}_\mathrm{p}(\Gamma_h)$ into $H^{-1/2}_\mathrm{p}(\Gamma_h)$, see, *e.g.* [3]. Using $T_\alpha$, we define a vector of (pseudo-)differential operators $R_\alpha := (\partial^\alpha/\partial x_1, \partial^\alpha/\partial x_2, T_\alpha)$. For a vector field $F \in H^{1/2}_\mathrm{p}(\Gamma_h)^3$,

$$R_\alpha \times F = (\partial^\alpha/\partial x_1, \partial^\alpha/\partial x_2, T_\alpha) \times F, \quad R_\alpha \cdot F = (\partial^\alpha/\partial x_1, \partial^\alpha/\partial x_2, T_\alpha) \cdot F.$$

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Since all components of $R_\alpha$ are bounded operators from $H_{\mathrm{p}}^{1/2}(\Gamma_h)$ into $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$, the operator $F \mapsto R_\alpha \times F$ is bounded from $H_{\mathrm{p}}^{1/2}(\Gamma_h)^3$ into $H_{\mathrm{p}}^{-1/2}(\Gamma_h)^3$, and $F \mapsto R_\alpha \cdot F$ is bounded from $H_{\mathrm{p}}^{1/2}(\Gamma_h)^3$ into $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$. If a biperiodic function $H \in H^1_{\mathrm{loc}}(\mathbb{R}^3_+)$ satisfies the Rayleigh expansion condition, then $T_\alpha H_3 = \partial H_3/\partial x_3$ on $\Gamma_h$. This implies that $e_3 \times (\mathrm{curl}_\alpha H) = e_3 \times (R_\alpha \times H)$ on $\Gamma_h$ (see, *e.g.* [16]).

Assume that $H^s_\alpha$ is a distributional periodic solution to the boundary value problem (2.10) such that $H^s_\alpha$, $\mathrm{curl}_\alpha H^s_\alpha$, and $\mathrm{div}_\alpha H^s_\alpha$ are locally square-integrable, such that the radiation condition (2.13) is satisfied, and such that $\nu \cdot (H^s_\alpha + H^{ir}_\alpha)$ and $\nu \times (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}(H^s_\alpha + H^{ir}_\alpha))$ are continuous over interfaces with normal vector $\nu$ where $\varepsilon_{\mathrm{r}}$ jumps. As noted in [16], this implies that, following the above notation, $H^s_\alpha \in H^1_{\mathrm{p,T}}(\Omega)$. Then the Stokes formula [1,16] implies that

$$\begin{aligned}
&\int_\Omega (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}_\alpha H^s_\alpha \cdot \overline{\mathrm{curl}_\alpha F} - k^2 H^s_\alpha \cdot \overline{F})\,\mathrm{d}x \\
&- \int_{\Gamma_0} e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}_\alpha H^s_\alpha) \cdot \overline{F}\,\mathrm{d}x + \int_{\Gamma_h} e_3 \times (R_\alpha \times H^s_\alpha) \cdot \overline{F}\,\mathrm{d}s \\
&= \int_\Omega (1 - \varepsilon_{\mathrm{r}}^{-1}) \mathrm{curl}_\alpha H^{ir}_\alpha \cdot \overline{\mathrm{curl}_\alpha F}\,\mathrm{d}x - \int_{\Gamma_0} (e_3 \times (1 - \varepsilon_{\mathrm{r}}^{-1}) \mathrm{curl}_\alpha H^{ir}_\alpha) \cdot \overline{F}\,\mathrm{d}x
\end{aligned}$$

for all test functions $F \in H^1_{\mathrm{p,T}}(\Omega)^3$. Since we assumed that

$$0 = e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}_\alpha H_\alpha) = e_3 \times (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}_\alpha (H^s_\alpha + H^{ir}_\alpha)) \quad \text{on } \Gamma_0,$$

the above identity simplifies to

$$\begin{aligned}
\int_\Omega (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}_\alpha H^s_\alpha \cdot \overline{\mathrm{curl}_\alpha F} - k^2 H^s_\alpha \cdot \overline{F})\,\mathrm{d}x + \int_{\Gamma_h} e_3 \times (R_\alpha \times H^s_\alpha) \cdot \overline{F}\,\mathrm{d}s \\
= \int_\Omega (1 - \varepsilon_{\mathrm{r}}^{-1}) \mathrm{curl}_\alpha H^{ir}_\alpha \cdot \overline{\mathrm{curl}_\alpha F}\,\mathrm{d}x - \int_{\Gamma_0} (e_3 \times \mathrm{curl}_\alpha H^{ir}_\alpha) \cdot \overline{F}\,\mathrm{d}x\,.
\end{aligned}$$

By construction, $e_3 \times \mathrm{curl}_\alpha H^{ir}_\alpha$ vanishes on $\Gamma_0$, that is, we can neglect the last term in the last equation. The divergence constraint $\mathrm{div}_\alpha H^s_\alpha = 0$ that follows from (2.10) shows that

$$\begin{aligned}
\mathcal{B}(H^s_\alpha, F) &:= \int_\Omega (\varepsilon_{\mathrm{r}}^{-1} \mathrm{curl}_\alpha H^s_\alpha \cdot \overline{\mathrm{curl}_\alpha F} - k^2 H^s_\alpha \cdot \overline{F})\,\mathrm{d}x + \rho \int_\Omega (\mathrm{div}_\alpha H^s_\alpha)(\overline{\mathrm{div}_\alpha F})\,\mathrm{d}x \\
&\quad + \int_{\Gamma_h} e_3 \times (R_\alpha \times H^s_\alpha) \cdot \overline{F}\,\mathrm{d}s - \int_{\Gamma_h} (R_\alpha \cdot H^s_\alpha)(e_3 \cdot \overline{F})\,\mathrm{d}s \\
&= \int_\Omega (1 - \varepsilon_{\mathrm{r}}^{-1}) \mathrm{curl}_\alpha H^{ir}_\alpha \cdot \overline{\mathrm{curl}_\alpha F}\,\mathrm{d}x\,, \qquad (3.1)
\end{aligned}$$

where $\rho$ is some complex constant with $\mathrm{Re}\,(\rho) \geq c > 0$ and $\mathrm{Im}\,(\rho) < 0$.

We next prove that the bounded sesquilinear form $\mathcal{B} : H^1_{\mathrm{p,T}}(\Omega)^3 \times H^1_{\mathrm{p,T}}(\Omega)^3 \to \mathbb{C}$ satisfies a Gårding inequality (this goes back to [1]), *i.e.* there exist strictly positive constants $c_1$ and $c_2$ such that

$$\mathrm{Re}\,(\mathcal{B}(H, H)) \geq c_1 \int_\Omega |\nabla_\alpha H|^2\,\mathrm{d}x - c_2 \int_\Omega |H|^2\,\mathrm{d}x\,. \qquad (3.2)$$

for all $H \in H^1_{\mathrm{p,T}}(\Omega)^3$.

**Theorem 3.1.** *Assume that $\varepsilon_{\mathrm{r}}^{-1} \in L^\infty(\Omega)$ is positive and bounded away from zero. Set $\mathrm{Re}\,\rho = \inf_\Omega \varepsilon_{\mathrm{r}}^{-1} > 0$ and choose $\mathrm{Im}\,\rho < 0$. Then $\mathcal{B}$ satisfies (3.2).*

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*Proof.* As in [16], proof of Theorem 1 one shows that

$$\mathrm{Re}\,(\mathcal{B}(H,H)) \geq \mathrm{Re}\,(\rho)\int_\Omega (|\operatorname{curl}_\alpha H|^2 + |\mathrm{div}_\alpha H|^2)\,\mathrm{d}x - k^2\int_\Omega |H|^2\,\mathrm{d}x \\ - \mathrm{Re}\int_{\Gamma_h} T_\alpha H\cdot\overline{H}\,\mathrm{d}s - 2\mathrm{Re}\int_{\Gamma_h}\left(\overline{H_3}\frac{\partial^\alpha H_1}{\partial x_1} + \overline{H_3}\frac{\partial^\alpha H_2}{\partial x_2}\right)\mathrm{d}s\,.$$

The following identity follows from integrations by parts, the periodicity, and the vanishing normal component of $H$ on $\Gamma_0$,

$$\int_\Omega (|\operatorname{curl}_\alpha H|^2 + |\mathrm{div}_\alpha H|^2)\,\mathrm{d}x = \int_\Omega |\nabla_\alpha H|^2\,\mathrm{d}x + 2\mathrm{Re}\int_{\Gamma_h}\left(\overline{H_3}\frac{\partial^\alpha H_1}{\partial x_1} + \overline{H_3}\frac{\partial^\alpha H_2}{\partial x_2}\right)\mathrm{d}s\,.$$

In consequence,

$$\mathrm{Re}\,(\mathcal{B}(H,H)) \geq \mathrm{Re}\,(\rho)\int_\Omega |\nabla_\alpha H|^2\,\mathrm{d}x - k^2\int_\Omega |H|^2\,\mathrm{d}x \\ - \mathrm{Re}\int_{\Gamma_h} T_\alpha H\cdot\overline{H}\,\mathrm{d}s - 2(1-\mathrm{Re}\,(\rho))\,\mathrm{Re}\int_{\Gamma_h}\left(\frac{\partial^\alpha H_1}{\partial x_1} + \frac{\partial^\alpha H_2}{\partial x_2}\right)\overline{H_3}\,\mathrm{d}s\,.$$

Precisely as in [16] one shows now by a Fourier series argument that

$$-\mathrm{Re}\int_{\Gamma_h} T_\alpha H\cdot\overline{H}\,\mathrm{d}s - 2(1-\mathrm{Re}\,(\rho))\mathrm{Re}\int_{\Gamma_h}\left(\frac{\partial^\alpha H_1}{\partial x_1} + \frac{\partial^\alpha H_2}{\partial x_2}\right)\overline{H_3}\,\mathrm{d}s \geq \mathrm{Re}\int_{\Gamma_h} K(H)\cdot\overline{H}\,\mathrm{d}s \\ \geq -C\int_\Omega |H|^2\,\mathrm{d}x$$

for a finite-dimensional operator $K$ on $H_{\mathrm{p}}^{1/2}(\Gamma_h)^3$. Note that the last inequality follows from $|\int_{\Gamma_h} K(H)\cdot\overline{H}\,\mathrm{d}s| \leq C\int_\Omega |H|^2\,\mathrm{d}x$ due to the finite-dimensional range of $K$ and the fact that on finite-dimensional spaces all norms are equivalent. The last inequality implies a Gårding inequality for $\mathcal{B}$. □

For simplicity we write from now on $H$ for the searched-for scattered field $H_\alpha^s$ in (3.1) and replace the source function $\operatorname{curl} H_\alpha^{ir}$ by a $G \in H_{\mathrm{p}}^1(\Omega)^3$. The last theorem implies the following corollary.

**Corollary 3.2.** *The variational problem to find $H \in H_{\mathrm{p,T}}^1(\Omega)^3$ such that*

$$\mathcal{B}(H,F) = \int_\Omega (1-\varepsilon_{\mathrm{r}}^{-1})G\cdot\overline{\operatorname{curl}_\alpha F}\,\mathrm{d}x \qquad \text{for all} \quad F \in H_{\mathrm{p,T}}^1(\Omega)^3 \tag{3.3}$$

*satisfies the Fredholm alternative,* i.e.*, uniqueness of solution implies existence of solution.*

Note that this formulation corresponds to the usual variational formulation of the Maxwell equations with perfectly conducting magnetic boundary conditions in smooth bounded domains, see, *e.g.* [9], Section 4.5(b). For special material parameters $\varepsilon_{\mathrm{r}}^{-1}$ in

$$W_{\mathrm{p}}^{1,\infty}(\Omega) := \{f \in L^\infty(\Omega) :\ f = \tilde{f}|_\Omega \text{ for some } 2\pi\text{-biperiodic } \tilde{f} \in W^{1,\infty}(\mathbb{R}^3)\}$$

we will in the sequel of the paper establish a uniqueness result via a Rellich identity. The next lemma will be useful when proving this identity.

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**Lemma 3.3.** *Assume that* $\varepsilon_{\mathrm{r}}^{-1} \in W_{\mathrm{p}}^{1,\infty}(\Omega)$ *is positive and bounded away from zero, and that* $G \in H_{\mathrm{p}}^{1}(\Omega)^3$. *Then a solution* $H \in H_{\mathrm{p,T}}^{1}(\Omega)^3$ *to problem* (3.3) *satisfies*

$$\operatorname{curl}_\alpha(\varepsilon_{\mathrm{r}}^{-1}\operatorname{curl}_\alpha H) - k^2 H = \operatorname{curl}_\alpha((1-\varepsilon_{\mathrm{r}}^{-1})G) \qquad \text{in } L^2(\Omega)^3, \tag{3.4}$$

$$\operatorname{div}_\alpha H = 0 \qquad \text{in } L^2(\Omega), \tag{3.5}$$

$$e_3 \times (\varepsilon_{\mathrm{r}}^{-1}\operatorname{curl}_\alpha H) = e_3 \times ((1-\varepsilon_{\mathrm{r}}^{-1})G) \qquad \text{in } H_{\mathrm{p}}^{-1/2}(\Gamma_0)^3, \tag{3.6}$$

$$e_3 \cdot H = 0 \qquad \text{in } H_{\mathrm{p}}^{1/2}(\Gamma_0). \tag{3.7}$$

*Moreover,*

$$e_3 \times R_\alpha \times H = e_3 \times \operatorname{curl}_\alpha H \quad \text{in } H_{\mathrm{p}}^{-1/2}(\Gamma_h)^3 \quad \text{and} \quad R_\alpha \cdot H = 0 \quad \text{in } H_{\mathrm{p}}^{-1/2}(\Gamma_h), \tag{3.8}$$

*and* $\partial H/\partial x_3 = T_\alpha(H)$ *holds in* $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$.

*Proof.* The proof that $\operatorname{div}_\alpha H = 0$ is analogous to the proof of [16], Theorem 2. In consequence, using a test function $F \in C_0^\infty(\Omega)^3$ in the variational problem (3.3) shows that the solution $H$ satisfies the differential equation (3.4) in the distributional sense. Since $H \in H_{\mathrm{p,T}}^{1}(\Omega)^3$, (3.4) holds in the $L^2$-sense if the right-hand side belongs to $L^2(\Omega)^3$, which holds if $\varepsilon_{\mathrm{r}}^{-1} \in W_{\mathrm{p}}^{1,\infty}(\Omega)$ and $G \in H_{\mathrm{p}}^{1}(\Omega)^3$.

Multiplying (3.4) by $F \in H_{\mathrm{p,T}}^{1}(\Omega)^3$, using the Stokes formula, and subtracting the resulting expression from the variational formulation (3.3), we find that

$$\int_{\Gamma_h} e_3 \times (R_\alpha \times H)\cdot \overline{F}\,\mathrm{d}s - \int_{\Gamma_h} (R_\alpha \cdot H)(e_3 \cdot \overline{F})\,\mathrm{d}s - \int_{\Gamma_h} e_3 \times \operatorname{curl}_\alpha H \cdot \overline{F}\,\mathrm{d}s + \int_{\Gamma_0} e_3 \times (\varepsilon_{\mathrm{r}}^{-1}\operatorname{curl}_\alpha H)\cdot \overline{F}\,\mathrm{d}s - \int_{\Gamma_0} e_3 \times ((1-\varepsilon_{\mathrm{r}}^{-1})G)\cdot \overline{F}\,\mathrm{d}s = 0.$$

If we choose $F$ such that $F|_{\Gamma_h} = 0$, then we see that $e_3 \times (\varepsilon_{\mathrm{r}}^{-1}\operatorname{curl}_\alpha H - (1-\varepsilon_{\mathrm{r}}^{-1})G) = 0$ in $H_{\mathrm{p}}^{-1/2}(\Gamma_0)$. If $e_3 \cdot F|_{\Gamma_h} = 0$, it follows that $e_3 \times (R_\alpha \times H) = e_3 \times \operatorname{curl}_\alpha H$ in $H_{\mathrm{p}}^{-1/2}(\Gamma_h)^3$. Hence, $R_\alpha \cdot H = 0$ in $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$. These identities imply that $\partial H/\partial x_3 = T_\alpha(H)$ in $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$ due to [16], Lemma 1. □

**Remark 3.4.** Instead of the above variational formulation in $H_{\mathrm{p,T}}^{1}(\Omega)$, one can also consider formulations in $H_{\mathrm{p}}(\operatorname{curl}_\alpha, \Omega)$, the natural energy space for the second-order Maxwell equations (2.10), see, *e.g.* [1]. In $H_{\mathrm{p}}(\operatorname{curl}_\alpha, \Omega)^3$ there is no bounded trace operator for the normal component of the field, and in consequence, the formulation (3.3) needs to be adapted. Usually, one replaces $F \mapsto e_3 \times (R_\alpha \times F) \times e_3$ by $Q(e_3 \times H)$, where $Q$ is a bounded operator between the natural trace spaces $H_{\mathrm{p,div}}^{-1/2}(\Gamma_h)$ and $H_{\mathrm{p,curl}}^{-1/2}(\Gamma_h)$, defined by

$$(QF)(x) = -\sum_{n\in\Lambda} \frac{1}{i\beta_n}\{k^2 \hat{F}_{T,n} - [(n+\alpha)\cdot \hat{F}_n](n+\alpha)\}\mathrm{e}^{in\cdot x}, \quad \text{for } F(x) = \sum_{n\in\Lambda} \hat{F}_n \mathrm{e}^{in\cdot x}, \tag{3.9}$$

see, *e.g.* [1]. Obviously this definition only makes sense if all $\beta_n$ are non-zero. If this is the case, then the variational formulation (3.3) is equivalent to the formulation in $H_{\mathrm{p}}(\operatorname{curl}_\alpha, \Omega)^3$ obtained using $Q$. Under the assumption that $\beta_n \neq 0$, all subsequent results could also be obtained *via* the formulation in $H_{\mathrm{p}}(\operatorname{curl}_\alpha, \Omega)^3$.

## 4. Integral identities

This section is concerned with technical lemmas that will be used to derive the Rellich identity and solution bounds subsequently. Roughly speaking, for deriving the Rellich identity, we will multiply the Maxwell equations (3.4) by $x_3 \partial H/\partial x_3$ and integrate by parts. Therefore, it is the aim of the technical lemmas in this section

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to analyze the term $\mathrm{Re}\int_\Omega x_3 \partial H/\partial x_3 \cdot \overline{\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H)}\,\mathrm{d}x$ for a solution $H \in H^1_{\mathrm{p,T}}(\Omega)^3$ to the problem (3.3). Note that the first two lemmas need the function $H$ to be in $H^2_\mathrm{p}(\Omega)^3$. These lemmas for the magnetic field formulation actually correspond to the ones for the electric field formulation in [12], Section 3.

We need to introduce some notation. For a vector field $F = (F_1, F_2, F_3)^\top$ we denote by $F_T = (F_1, F_2, 0)^\top$ its transverse part. Recall that $\partial^\alpha f/\partial x_j = \partial f/\partial x_j + i\alpha_j f$ for a scalar function $f$ and $j = 1, 2, 3$. Further, we introduce

$$\nabla_T f := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, 0\right)^\top, \quad \nabla_{\alpha,T} f := \left(\frac{\partial^\alpha f}{\partial x_1}, \frac{\partial^\alpha f}{\partial x_2}, 0\right)^\top, \quad \overrightarrow{\mathrm{curl}}_{\alpha,T} f := \left(\frac{\partial^\alpha f}{\partial x_2}, -\frac{\partial^\alpha f}{\partial x_1}, 0\right)^\top,$$

and, for a vector field $F = (F_1, F_2, F_3)^\top$,

$$\mathrm{div}_{\alpha,T} F := \frac{\partial^\alpha F_1}{\partial x_1} + \frac{\partial^\alpha F_2}{\partial x_2} \quad \text{and} \quad \mathrm{curl}_{\alpha,T}\, F := \frac{\partial^\alpha F_2}{\partial x_1} - \frac{\partial^\alpha F_1}{\partial x_2}.$$

It is straightforward to show that $\mathrm{div}_{\alpha,T}\overrightarrow{\mathrm{curl}}_{\alpha,T} = 0$ as well as $\mathrm{curl}_{\alpha,T}\,\nabla_{\alpha,T} = 0$. Moreover, a tedious computation shows that

$$\mathrm{curl}_\alpha F = (\mathrm{curl}_{\alpha,T}\, F_T) e_3 + \overrightarrow{\mathrm{curl}}_{\alpha,T} F_3 - \frac{\partial (F \times e_3)}{\partial x_3},$$

and further

$$|\,\mathrm{curl}_\alpha F|^2 = |\,\mathrm{curl}_{\alpha,T}\, F_T|^2 + |\overrightarrow{\mathrm{curl}}_{\alpha,T} F_3|^2 + \left|\frac{\partial F_T}{\partial x_3}\right|^2 - 2\mathrm{Re}\left(\overline{\nabla_{\alpha,T} F_3} \cdot \frac{\partial F_T}{\partial x_3}\right). \tag{4.1}$$

**Lemma 4.1.** *Assume that $\varepsilon_\mathrm{r}^{-1} \in W^{1,\infty}_\mathrm{p}(\Omega)$ is positive and bounded away from zero and that $H \in H^2_\mathrm{p}(\Omega)^3$. Then*

$$\begin{aligned} 2\mathrm{Re}\int_\Omega x_3 \frac{\partial H}{\partial x_3} \cdot \overline{\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H)}\,\mathrm{d}x &= -\int_\Omega \frac{\partial(x_3 \varepsilon_\mathrm{r}^{-1})}{\partial x_3} |\,\mathrm{curl}_\alpha H|^2\,\mathrm{d}x + h\int_{\Gamma_h} |\,\mathrm{curl}_\alpha H|^2\,\mathrm{d}s \\ &\quad + 2\mathrm{Re}\int_\Omega \varepsilon_\mathrm{r}^{-1}\left(e_3 \times \frac{\partial H}{\partial x_3}\right) \cdot \overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x + 2h\mathrm{Re}\int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot (e_3 \times \overline{\mathrm{curl}_\alpha H})\,\mathrm{d}s\,. \end{aligned} \tag{4.2}$$

*Proof.* Denote by $\nu$ the outward unit normal to $\Omega$. Using integration by parts and noting that $\nu = e_3$ on $\Gamma_h$, and that the boundary term on $\Gamma_0$ vanishes since $x_3 = 0$ on $\Gamma_0$, we find that

$$\begin{aligned} &2\mathrm{Re}\int_\Omega x_3 \frac{\partial H}{\partial x_3} \cdot \overline{\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H)}\,\mathrm{d}x \\ &\quad = 2\mathrm{Re}\int_\Omega \varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha\left(x_3 \frac{\partial H}{\partial x_3}\right) \cdot \overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x + 2\mathrm{Re}\int_{\partial\Omega} x_3 \frac{\partial H}{\partial x_3} \cdot (\nu \times \overline{\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H})\,\mathrm{d}s \\ &\quad = \int_\Omega \varepsilon_\mathrm{r}^{-1} x_3 \frac{\partial |\,\mathrm{curl}_\alpha H|^2}{\partial x_3}\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \varepsilon_\mathrm{r}^{-1}\left(e_3 \times \frac{\partial H}{\partial x_3}\right) \cdot \overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x \\ &\qquad + 2h\mathrm{Re}\int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot (e_3 \times \overline{\mathrm{curl}_\alpha H})\,\mathrm{d}s \\ &\quad = -\int_\Omega \frac{\partial(x_3 \varepsilon_\mathrm{r}^{-1})}{\partial x_3} |\,\mathrm{curl}_\alpha H|^2\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \varepsilon_\mathrm{r}^{-1}\left(e_3 \times \frac{\partial H}{\partial x_3}\right) \cdot \overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x \\ &\qquad + h\int_{\Gamma_h} |\,\mathrm{curl}_\alpha H|^2\,\mathrm{d}s + 2h\mathrm{Re}\int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot (e_3 \times \overline{\mathrm{curl}_\alpha H})\,\mathrm{d}s\,. \end{aligned}$$

□

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The next lemma continues the analysis of the term $\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}(e_3\times\partial H/\partial x_3)\cdot\overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x$ in the right hand side of (4.2).

**Lemma 4.2.** *Assume that $\varepsilon_{\mathrm{r}}^{-1}\in W^{1,\infty}_{\mathrm{p}}(\Omega)$ is positive and bounded away from zero. Then for all $H\in H^2_{\mathrm{p}}(\Omega)^3$ the following identity holds,*

$$\begin{aligned}2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\left(e_3\times\frac{\partial H}{\partial x_3}\right)\cdot\overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x &= 2\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\left|\frac{\partial H}{\partial x_3}\right|^2\mathrm{d}x + 2\mathrm{Re}\int_\Omega \nabla\varepsilon_{\mathrm{r}}^{-1}\cdot\frac{\partial H}{\partial x_3}\overline{H_3}\,\mathrm{d}x\\ &\quad -2\mathrm{Re}\int_\Omega \frac{\partial(\varepsilon_{\mathrm{r}}^{-1}\overline{H_3})}{\partial x_3}\mathrm{div}_\alpha H\,\mathrm{d}x - 2\mathrm{Re}\int_{\Gamma_h}\left(\frac{\partial H_3}{\partial x_3}-\mathrm{div}_\alpha H\right)\overline{H_3}\,\mathrm{d}s\\ &\quad -2\mathrm{Re}\int_{\Gamma_0}\varepsilon_{\mathrm{r}}^{-1}\overline{H_3}\mathrm{div}_{\alpha,T}H_T\,\mathrm{d}s\,. \end{aligned}\tag{4.3}$$

*Proof.* First, we have

$$\begin{aligned}2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\left(e_3\times\frac{\partial H}{\partial x_3}\right)\cdot\overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x &= 2\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\left|\frac{\partial H_T}{\partial x_3}\right|^2\mathrm{d}x\\ &\quad -2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\cdot\nabla_T\overline{H_3}\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\cdot i\alpha\overline{H_3}\,\mathrm{d}x\,. \end{aligned}\tag{4.4}$$

Second, we compute that

$$\begin{aligned}-2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\cdot\nabla_T\overline{H_3}\,\mathrm{d}x &= 2\mathrm{Re}\int_\Omega \mathrm{div}_T\left(\varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\right)\overline{H_3}\,\mathrm{d}x\\ &= 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\mathrm{div}_T\left(\frac{\partial H_T}{\partial x_3}\right)\overline{H_3}\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \nabla_T\varepsilon_{\mathrm{r}}^{-1}\cdot\frac{\partial H_T}{\partial x_3}\overline{H_3}\,\mathrm{d}x\\ &= -2\mathrm{Re}\int_\Omega \frac{\partial\varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\overline{H_3}\mathrm{div}_T H_T\,\mathrm{d}x - 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial\overline{H_3}}{\partial x_3}\mathrm{div}_T H_T\,\mathrm{d}x\\ &\quad + 2\mathrm{Re}\int_\Omega \nabla_T\varepsilon_{\mathrm{r}}^{-1}\cdot\frac{\partial H_T}{\partial x_3}\overline{H_3}\,\mathrm{d}x + 2\mathrm{Re}\int_{\Gamma_h}\overline{H_3}\mathrm{div}_T H_T\,\mathrm{d}s - 2\mathrm{Re}\int_{\Gamma_0}\varepsilon_{\mathrm{r}}^{-1}\overline{H_3}\mathrm{div}_T H_T\,\mathrm{d}s \end{aligned}$$

Now, using the identity $\mathrm{div}_T H_T = -\partial H_3/\partial x_3 + \mathrm{div}_\alpha H - i\alpha\cdot H$, we obtain that

$$\begin{aligned}-2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\cdot\nabla_T\overline{H_3}\,\mathrm{d}x &= 2\mathrm{Re}\int_\Omega \frac{\partial\varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\overline{H_3}(i\alpha\cdot H)\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \frac{\partial\varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\overline{H_3}\frac{\partial H_3}{\partial x_3}\,\mathrm{d}x\\ &\quad - 2\mathrm{Re}\int_\Omega \frac{\partial\varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\overline{H_3}\mathrm{div}_\alpha H\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial\overline{H_3}}{\partial x_3}(i\alpha\cdot H)\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\left|\frac{\partial H_3}{\partial x_3}\right|^2\mathrm{d}x\\ &\quad - 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial\overline{H_3}}{\partial x_3}\mathrm{div}_\alpha H\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \nabla_T\varepsilon_{\mathrm{r}}^{-1}\cdot\frac{\partial H_T}{\partial x_3}\overline{H_3}\,\mathrm{d}x + 2\mathrm{Re}\int_{\Gamma_h}\overline{H_3}\mathrm{div}_T H_T\,\mathrm{d}x\\ &\quad - 2\mathrm{Re}\int_{\Gamma_0}\varepsilon_{\mathrm{r}}^{-1}\overline{H_3}\mathrm{div}_T H_T\,\mathrm{d}s \end{aligned}$$

Applying Green formula to the term $2\mathrm{Re}\int_\Omega(\partial\varepsilon_{\mathrm{r}}^{-1}/\partial x_3)\overline{H_3}(i\alpha\cdot H)\,\mathrm{d}x$, we have

$$\begin{aligned}-2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\cdot\nabla_T\overline{H_3}\,\mathrm{d}x &= -2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial H_T}{\partial x_3}\cdot i\alpha\overline{H_3}\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\left|\frac{\partial H_3}{\partial x_3}\right|^2\mathrm{d}x\\ &\quad - 2\mathrm{Re}\int_\Omega \frac{\partial\varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\overline{H_3}\mathrm{div}_\alpha H\,\mathrm{d}x - 2\mathrm{Re}\int_\Omega \varepsilon_{\mathrm{r}}^{-1}\frac{\partial\overline{H_3}}{\partial x_3}\mathrm{div}_\alpha H\,\mathrm{d}x + 2\mathrm{Re}\int_\Omega \nabla\varepsilon_{\mathrm{r}}^{-1}\cdot\frac{\partial H}{\partial x_3}\overline{H_3}\,\mathrm{d}x\\ &\quad - 2\mathrm{Re}\int_{\Gamma_h}\left(\frac{\partial H_3}{\partial x_3}-\mathrm{div}_\alpha H\right)\overline{H_3}\,\mathrm{d}s - 2\mathrm{Re}\int_{\Gamma_0}\varepsilon_{\mathrm{r}}^{-1}\overline{H_3}\mathrm{div}_{\alpha,T}H_T\,\mathrm{d}s \end{aligned}$$

Now the claim follows from substituting this identity into equation (4.4). □

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In the following final lemma of this section we will reformulate the term $\mathrm{Re}\int_\Omega x_3 \partial H/\partial x_3 \cdot \overline{\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H)}\,\mathrm{d}x$ for a solution $H \in H^1_{\mathrm{p,T}}(\Omega)^3$ to the problem (3.3) using the last two lemmas.

**Lemma 4.3.** *Assume that $\varepsilon_\mathrm{r}^{-1} \in W^{1,\infty}_\mathrm{p}(\Omega)$ is positive and bounded away from zero. Then any solution $H \in H^1_{\mathrm{p,T}}(\Omega)^3$ to the problem (3.3) satisfies*

$$\begin{aligned}2\mathrm{Re}\int_\Omega x_3\frac{\partial H}{\partial x_3}\cdot\overline{\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H)}\,\mathrm{d}x &= -\int_\Omega \frac{\partial(x_3\varepsilon_\mathrm{r}^{-1})}{\partial x_3}|\,\mathrm{curl}_\alpha H|^2\,\mathrm{d}x + h\int_{\Gamma_h}|\,\mathrm{curl}_\alpha H|^2\,\mathrm{d}s\\ &\quad +2\int_\Omega \varepsilon_\mathrm{r}^{-1}\left|\frac{\partial H}{\partial x_3}\right|^2\mathrm{d}x + 2\mathrm{Re}\int_\Omega \nabla\varepsilon_\mathrm{r}^{-1}\cdot\frac{\partial H}{\partial x_3}\overline{H_3}\,\mathrm{d}x - 2\mathrm{Re}\int_{\Gamma_h}\overline{H_3}\frac{\partial H_3}{\partial x_3}\,\mathrm{d}s\\ &\quad +2h\mathrm{Re}\int_{\Gamma_h}\frac{\partial H_T}{\partial x_3}\cdot(e_3\times\overline{\mathrm{curl}_\alpha H})\,\mathrm{d}s\,.\end{aligned}$$

*Proof.* It is sufficient to prove that $H$ satisfies (4.2) and

$$\begin{aligned}2\mathrm{Re}\int_\Omega \varepsilon_\mathrm{r}^{-1}\left(e_3\times\frac{\partial H}{\partial x_3}\right)\cdot\overline{\mathrm{curl}_\alpha H}\,\mathrm{d}x &= 2\int_\Omega\varepsilon_\mathrm{r}^{-1}\left|\frac{\partial H}{\partial x_3}\right|^2\mathrm{d}x + 2\mathrm{Re}\int_\Omega\nabla\varepsilon_\mathrm{r}^{-1}\cdot\frac{\partial H}{\partial x_3}\overline{H_3}\,\mathrm{d}x\\ &\quad - 2\mathrm{Re}\int_{\Gamma_h}\overline{H_3}\frac{\partial H_3}{\partial x_3}\,\mathrm{d}s\,. \quad (4.5)\end{aligned}$$

Recall that, for $h > \sup\{x_3 : (x_1,x_2,x_3)^\top \in \mathrm{supp}(\varepsilon_\mathrm{r}-1)\}$, there exists a constant $0<\eta\ll 1$ such that $\varepsilon_\mathrm{r}=1$ in $(0,2\pi)^2\times(h-\eta,h)$. Hence, a solution $H\in H^1_{\mathrm{p,T}}(\Omega)^3$ to the problem (3.3) belongs to $H^1_{\mathrm{p,T}}(\Omega)^3\cap H^2_\mathrm{p}((0,2\pi)^2\times(h-\eta,h))^3$ due to interior elliptic regularity theory. Then one can extend $H$ to a function defined in all of $\mathbb{R}^3$ that is $2\pi$-biperiodic and belongs to $H^1_\mathrm{p}((0,2\pi)^2\times(-\infty,h))^3\cap H^2_\mathrm{p}((0,2\pi)^2\times(h-\eta,\infty))^3$ (This can be seen using [13] combined with suitable cut-off arguments). By abuse of notation, we still denote the extended function by $H$. Let $\phi\in C^\infty(\mathbb{R}^3)$ be a smooth and non-negative function supported in the unit ball and $\int_{\mathbb{R}^3}\phi\,\mathrm{d}x = 1$. For $\delta>0$ and $x\in\mathbb{R}^3$ let $\phi^\delta(x)=\delta^{-3}\phi(x/\delta)$. The convolution $H^\delta := \phi^\delta * H$ belongs to $H^2_\mathrm{p}(\Omega)^3$ and thus satisfies (4.2). Then, from Lemma 3.3 and the fact that $H^\delta\to H$ in $H^1_{\mathrm{p,T}}(\Omega)^3\cap H^2_\mathrm{p}((0,2\pi)^2\times(h-\eta,h))^3$ we obtain that

$$\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H^\delta)\overset{\delta\to0}{\to}\mathrm{curl}_\alpha(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}_\alpha H)\quad\text{in } L^2(\Omega)^3.$$

Moreover, the convergence in $H^2_\mathrm{p}((0,2\pi)^2\times(h-\eta,h))^3$ implies that $\mathrm{curl}_\alpha H^\delta\to\mathrm{curl}_\alpha H$ in $L^2(\Gamma_h)^3$ as $\delta\to0$. Consequently, $H$ satisfies (4.2).

It remains to show that $H$ also satisfies (4.5). The function $H^\delta$ satisfies (4.3) and we consider the limit of this identity as $\delta\to0$. It is easily seen that $\mathrm{div}_\alpha H^\delta\to\mathrm{div}_\alpha H=0$ in $L^2(\Omega)$. Thus, we have

$$e_3\cdot H^\delta\overset{\delta\to0}{\to}e_3\cdot H=0\quad\text{in } H^{1/2}_\mathrm{p}(\Gamma_0),\qquad \mathrm{div}_{\alpha,T}H^\delta_T\overset{\delta\to0}{\to}\mathrm{div}_{\alpha,T}H_T\quad\text{in } H^{-1/2}_\mathrm{p}(\Gamma_0),$$

due to the convergence of $H^\delta$ to $H$ in $H^1_\mathrm{p}(\Omega)^3$. Further, the convergence of $H^\delta$ to $H$ in $H^2_\mathrm{p}((0,2\pi)^2\times(h-\eta,h))^3$ and the fact $\mathrm{div}_\alpha H=0$ on $\Gamma_h$ imply that

$$\frac{\partial H^\delta}{\partial x_3}-\mathrm{div}_\alpha H^\delta\to\frac{\partial H_3}{\partial x_3}-\mathrm{div}_\alpha H=\frac{\partial H_3}{\partial x_3}\quad\text{in } H^{-1/2}_\mathrm{p}(\Gamma_h).$$

Plugging these limits into (4.3) shows that (4.5) holds. □

## 5. Rellich identity and solution estimate

For establishing uniqueness of solution to the variational problem (3.3), we derive in this section the so-called Rellich identity relating $|\,\mathrm{curl}_\alpha H|^2$ and $|\partial H/\partial x_3|^2$ where $H$ is a solution to the homogeneous variational problem

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corresponding to (3.3). Then, under suitable non-trapping and smoothness conditions on the material parameter, integral inequality resulting from this identity allow us to obtain estimate for a solution to the homogeneous problem. As mentioned in the introduction, the Rellich identity and solution estimate obtained in this section are much simpler than the ones in [12], Section 4. It turns out also that the non-trapping assumptions on the parameter material are weaker than the ones in the latter paper.

The proof of the Rellich identity is based on an integration-by-parts technique that goes back to Rellich [15]. Typically, this technique requires more regularity of a solution than just to belong to the energy space. In our case we will roughly speaking multiply the Maxwell equations (3.4), for $G = 0$ in the right hand side, by $x_3\partial H/\partial x_3$ and integrate by parts.

**Lemma 5.1** (Rellich Identity)**.** *Assume that $\varepsilon_{\mathrm{r}}^{-1} \in W_{\mathrm{p}}^{1,\infty}(\Omega)$ is positive and bounded away from zero. Then the following identity holds for all solutions $H \in H_{\mathrm{p,T}}^{1}(\Omega)^3$ to the homogeneous problem corresponding to (3.3),*

$$\int_\Omega \left[ 2\varepsilon_{\mathrm{r}}^{-1} \left| \frac{\partial H}{\partial x_3} \right|^2 - x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2 + 2\mathrm{Re}\left( \nabla \varepsilon_{\mathrm{r}}^{-1} \cdot \frac{\partial H}{\partial x_3} \overline{H_3} \right) \right] \mathrm{d}x + \mathrm{Re} \int_{\Gamma_h} e_3 \times (R_\alpha \times H) \cdot \overline{H}\, \mathrm{d}s - 2\mathrm{Re} \int_{\Gamma_h} T_\alpha(H_3) \overline{H_3}\, \mathrm{d}s = 0. \tag{5.1}$$

*Proof.* Let $H \in H_{\mathrm{p,T}}^{1}(\Omega)^3$ be a solution to the homogeneous problem corresponding to (3.3). First, using integration by parts we have

$$\mathrm{Re} \int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot (e_3 \times \overline{\operatorname{curl}_\alpha H})\, \mathrm{d}s = \int_{\Gamma_h} \left| \frac{\partial H_T}{\partial x_3} \right|^2 \mathrm{d}s + \mathrm{Re} \int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot \overline{\nabla_{\alpha,T} H_3}\, \mathrm{d}s\,.$$

Note that $H$ satisfies the assumptions of Lemma 4.3. Together with the latter equation we obtain

$$\begin{aligned} 2\mathrm{Re} \int_\Omega x_3 \frac{\partial H}{\partial x_3} \cdot \overline{\operatorname{curl}_\alpha(\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H)}\, \mathrm{d}x &= -\int_\Omega \frac{\partial (x_3 \varepsilon_{\mathrm{r}}^{-1})}{\partial x_3} |\operatorname{curl}_\alpha H|^2\, \mathrm{d}x + h \int_{\Gamma_h} |\operatorname{curl}_\alpha H|^2\, \mathrm{d}s \\ &\quad + 2\int_\Omega \varepsilon_{\mathrm{r}}^{-1} \left| \frac{\partial H}{\partial x_3} \right|^2 \mathrm{d}x + 2\mathrm{Re} \int_\Omega \nabla \varepsilon_{\mathrm{r}}^{-1} \cdot \frac{\partial H}{\partial x_3} \overline{H_3}\, \mathrm{d}x - 2\mathrm{Re} \int_{\Gamma_h} \frac{\partial H_3}{\partial x_3} \overline{H_3}\, \mathrm{d}s \\ &\quad - 2h \int_{\Gamma_h} \left| \frac{\partial H_T}{\partial x_3} \right|^2 \mathrm{d}s + 2h\mathrm{Re} \int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot \overline{\nabla_{\alpha,T} H_3}\, \mathrm{d}s\,. \end{aligned}$$

We exploit that $H$ solves (3.4) for $G = 0$,

$$\begin{aligned} 2\mathrm{Re} \int_\Omega x_3 \frac{\partial H}{\partial x_3} \cdot \overline{\operatorname{curl}_\alpha(\varepsilon_{\mathrm{r}}^{-1} \operatorname{curl}_\alpha H)}\, \mathrm{d}x &= k^2 2\mathrm{Re} \int_\Omega x_3 \frac{\partial \overline{H}}{\partial x_3} \cdot H\, \mathrm{d}x = k^2 \int_\Omega x_3 \frac{\partial |H|^2}{\partial x_3}\, \mathrm{d}x \\ &= -k^2 \int_\Omega |H|^2\, \mathrm{d}x + k^2 h \int_{\Gamma_h} |H|^2\, \mathrm{d}s\,. \end{aligned}$$

From the last two equations we conclude that

$$\begin{aligned} &-\int_\Omega \left( \frac{\partial (x_3 \varepsilon_{\mathrm{r}}^{-1})}{\partial x_3} |\operatorname{curl}_\alpha H|^2 - k^2 |H|^2 \right) \mathrm{d}x + 2\int_\Omega \varepsilon_{\mathrm{r}}^{-1} \left| \frac{\partial H}{\partial x_3} \right|^2 \mathrm{d}x + 2\mathrm{Re} \int_\Omega \nabla \varepsilon_{\mathrm{r}}^{-1} \cdot \frac{\partial H}{\partial x_3} \overline{H_3}\, \mathrm{d}x \\ &\quad - 2\mathrm{Re} \int_{\Gamma_h} \overline{H_3} \frac{\partial H_3}{\partial x_3}\, \mathrm{d}s - 2h \int_{\Gamma_h} \left| \frac{\partial H_T}{\partial x_3} \right|^2 \mathrm{d}s + 2h\mathrm{Re} \int_{\Gamma_h} \frac{\partial H_T}{\partial x_3} \cdot \overline{\nabla_{\alpha,T} H_3}\, \mathrm{d}s \\ &\quad + h \int_{\Gamma_h} (|\operatorname{curl}_\alpha H|^2 - k^2 |H|^2)\, \mathrm{d}s = 0. \end{aligned}$$

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Due to the variational formulation (3.3) for $G = 0$,

$$\int_\Omega (\varepsilon_{\mathrm{r}}^{-1} |\operatorname{curl}_\alpha H|^2 - k^2 |H|^2)\,\mathrm{d}x + \mathrm{Re}\int_{\Gamma_h} e_3 \times (R_\alpha \times H)\cdot \overline{H}\,\mathrm{d}s = 0 \tag{5.2}$$

since $\operatorname{div}_\alpha H = 0$ in $\Omega$ and $R_\alpha \cdot H = 0$ in $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$ due to Lemma 3.3. Adding the last two equations yields that the term $\int_\Omega k^2 |H|^2\,\mathrm{d}x$ cancels, and further exploiting $\partial H_3/\partial x_3 = T_\alpha H_3$ on $\Gamma_h$ to yields that

$$\begin{aligned}
&-\int_\Omega x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2\,\mathrm{d}x + 2\int_\Omega \varepsilon_{\mathrm{r}}^{-1} \left|\frac{\partial H}{\partial x_3}\right|^2 \mathrm{d}x + 2\mathrm{Re}\int_\Omega \nabla \varepsilon_{\mathrm{r}}^{-1} \cdot \frac{\partial H}{\partial x_3}\overline{H_3}\,\mathrm{d}x \\
&\quad - 2\mathrm{Re}\int_{\Gamma_h} T_\alpha(H_3)\overline{H_3}\,\mathrm{d}s + \mathrm{Re}\int_{\Gamma_h} e_3 \times (R_\alpha \times H)\cdot \overline{H}\,\mathrm{d}s + 2h\mathrm{Re}\int_{\Gamma_h} \frac{\partial H_T}{\partial x_3}\cdot \overline{\nabla_{\alpha,T} H_3}\,\mathrm{d}s \\
&\quad + h\int_{\Gamma_h} \left(|\operatorname{curl}_\alpha H|^2 - k^2|H|^2 - 2\left|\frac{\partial H_T}{\partial x_3}\right|^2\right)\mathrm{d}s = 0.
\end{aligned}$$

Recall equality (4.1),

$$|\operatorname{curl}_\alpha H|^2 = |\operatorname{curl}_{\alpha,T} H|^2 + |\overrightarrow{\operatorname{curl}}_{\alpha,T} H_3|^2 + \left|\frac{\partial H_T}{\partial x_3}\right|^2 - 2\mathrm{Re}\left(\frac{\partial H_T}{\partial x_3}\cdot \overline{\nabla_{\alpha,T} H_3}\right).$$

Combining the last two equations yields

$$L(H) = h\int_{\Gamma_h} \left(\left|\frac{\partial H_T}{\partial x_3}\right|^2 + k^2|H|^2 - |\operatorname{curl}_{\alpha,T} H|^2 - |\overrightarrow{\operatorname{curl}}_{\alpha,T} H_3|^2\right)\mathrm{d}s$$

where $L(H)$ is the left hand side of (5.1). It remains now to prove that the right hand side of the latter equation vanishes. First, we recall from Lemma 3.3 that $\partial H/\partial x_3 = T_\alpha H$ in $H_{\mathrm{p}}^{-1/2}(\Gamma_h)$ which yields that

$$\int_{\Gamma_h} \left|\frac{\partial H_T}{\partial x_3}\right|^2 = \sum_{n\in\Lambda} |\beta_n \hat{H}_{T,n}|^2, \quad \int_{\Gamma_h} \left|\frac{\partial H_3}{\partial x_3}\right|^2 = \sum_{n\in\Lambda} |\beta_n \hat{H}_{3,n}|^2.$$

Using the latter formulas and replacing $k^2$ by $|n+\alpha|^2 + \beta_n^2$ in the first boundary term in (5.1) yields

$$\begin{aligned}
&\int_{\Gamma_h} \left(\left|\frac{\partial H_T}{\partial x_3}\right|^2 + k^2|H|^2 - |\operatorname{curl}_{\alpha,T} H|^2 - |\overrightarrow{\operatorname{curl}}_{\alpha,T} H_3|^2\right)\mathrm{d}s \\
&= \sum_{n\in\Lambda} \Big[|\beta_n \hat{H}_{T,n}|^2 + (|n+\alpha|^2 + \beta_n^2)(|\hat{H}_{T,n}|^2 + |\hat{H}_{3,n}|^2) - |(n+\alpha)\times \hat{H}_{T,n}|^2 - |n+\alpha|^2|\hat{H}_{3,n}|\Big] \\
&= \sum_{n\in\Lambda} \Big[(\beta_n^2 + |\beta_n|^2)|\hat{H}_{T,n}|^2 + |n+\alpha|^2|\hat{H}_{T,n}|^2 - |(n+\alpha)\times \hat{H}_{T,n}|^2 + \beta_n^2|\hat{H}_{3,n}|\Big].
\end{aligned} \tag{5.3}$$

On the other hand, due to the divergence-free condition, we have

$$\begin{aligned}
\sum_{n\in\Lambda} \Big[|n+\alpha|^2|\hat{H}_{T,n}|^2 - |(n+\alpha)\times \hat{H}_{T,n}|^2\Big] &= \sum_{n\in\Lambda} |(n_1+\alpha_1)\hat{H}_{1,n} + (n_2+\alpha_2)\hat{H}_{2,n}|^2 \\
&= \|\operatorname{div}_{\alpha,T} H_T\|^2_{L^2(\Gamma_h)} = \|\partial H_3/\partial x_3\|^2_{L^2(\Gamma_h)} = \sum_{n\in\Lambda} |\beta_n \hat{H}_{3,n}|^2.
\end{aligned}$$

Now substituting the latter equation into (5.3) leads to

$$\int_{\Gamma_h} \left(\left|\frac{\partial H_T}{\partial x_3}\right|^2 + k^2|H|^2 - |\operatorname{curl}_{\alpha,T} H|^2 - |\overrightarrow{\operatorname{curl}}_{\alpha,T} H_3|^2\right)\mathrm{d}s = 2\sum_{\beta_n \geq 0} \beta_n^2 |\hat{H}_n|^2, \tag{5.4}$$

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where we exploited that $\beta_n$ is either a non-negative or a purely imaginary number. The proof is hence finished if we show that $\sum_{\beta_n \geq 0} \beta_n^2 |\hat{H}_n|^2 = 0$ (since then $L(H) = 0$, which is the claim of the theorem). First, we compute that

$$\begin{aligned}\langle e_3 \times (R_\alpha \times H), H\rangle_{\Gamma_h} &= \sum_{n\in\Lambda} i(n+\alpha)\cdot \overline{\hat{H}_{T,n}}\hat{H}_{3,n} - \sum_{n\in\Lambda} i\beta_n |\hat{H}_{T,n}|^2 \\ &= -\sum_{n\in\Lambda} i\overline{\beta_n}|\hat{H}_{3,n}|^2 - \sum_{n\in\Lambda} i\beta_n|\hat{H}_{T,n}|^2.\end{aligned}$$

Since $\mathrm{Re}\,(\beta_n) \geq 0$ this implies that

$$\mathrm{Im}\,\langle e_3 \times (R_\alpha \times H), H\rangle_{\Gamma_h} = -\sum_{n\in\Lambda} \mathrm{Re}\,(\beta_n)|\hat{H}_n|^2 \leq 0, \text{ and} \tag{5.5}$$

$$\mathrm{Re}\,\langle e_3 \times (R_\alpha \times H), H\rangle_{\Gamma_h} = \sum_{n\in\Lambda} \mathrm{Im}\,(\overline{\beta_n})|\hat{H}_{3,n}|^2 + \sum_{n\in\Lambda} \mathrm{Im}\,(\beta_n)|\hat{H}_{T,n}|^2. \tag{5.6}$$

(The second equation will be exploited later on). Taking the imaginary part of the variational formulation (3.3) with $G = 0$ and $F = H$, and exploiting Lemma 3.3, we obtain that

$$0 = \mathrm{Im}\,\langle e_3 \times (R_\alpha \times H), H\rangle_{\Gamma_h} \overset{(5.5)}{=} -\sum_{n\in\Lambda} \mathrm{Re}\,(\beta_n)|\hat{H}_n|^2.$$

This implies that $|\hat{H}_n|^2 = 0$ for all $n$ such that $\mathrm{Re}\,(\beta_n) > 0$. Since $\beta_n$ is either purely imaginary or non-negative, we conclude that $\sum_{\beta_n \geq 0} \beta_n^2 |\hat{H}_n|^2 = 0$. □

The next Poincaré-like result is classical (see, *e.g.* [8] for a proof).

**Lemma 5.2.** *For $u \in \{v \in H^1_{\mathrm{p}}(\Omega) : v|_{\Gamma_0} = 0\}$ there holds $2\|u\|^2_{L^2(\Omega)} \leq h^2 \|\partial u/\partial x_3\|^2_{L^2(\Omega)}$.*

The following assumptions on $\varepsilon_{\mathrm{r}}^{-1}$ will guarantee a stability estimate and a uniqueness statement for a solution to the variational problem (3.3):

(a) $\varepsilon_{\mathrm{r}}^{-1} \in W^{1,\infty}_{\mathrm{p}}(\Omega)$ satisfies $\dfrac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \leq 0$ in $\Omega$,

(b) It holds that $\dfrac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} < 0$ in a non-empty open ball $B \subset \Omega$,

(c) There exists $\delta > 1/2$ such that $\dfrac{\delta}{2}\|\nabla_T \varepsilon_{\mathrm{r}}^{-1}\|^2_{L^\infty(\Omega)^2} + \dfrac{\sqrt{2}}{h}\left\|\dfrac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3}\right\|_{L^\infty(\Omega)} < \dfrac{2}{h^2}$. (5.7)

**Remark 5.3.** Note that (5.7)(a) implies that $\varepsilon_{\mathrm{r}}^{-1} \geq 1$, since, by construction, $\varepsilon_{\mathrm{r}}^{-1} = 1$ in $\{h - \eta < x_3 < h\}$ for some small $\eta > 0$. For the case of periodic non-absorbing structures, the main difference between these non-trapping conditions and the ones for the scalar case in [7] is the additional condition (5.7)(c). This condition arises from estimating the term $2\mathrm{Re}\int_\Omega (\nabla \varepsilon_{\mathrm{r}}^{-1} \cdot \partial H/\partial x_3 \overline{H_3})\,\mathrm{d}x$ in the Rellich identity (5.1) using the Poincaré-like result above. This is natural since the Rellich identity resulting from a similar technique for the scalar case [7] does not have a corresponding term.

Let us construct a function $\varepsilon_{\mathrm{r}}^{-1}$ that satisfies the above assumptions (5.7). Choose constants $0 < h_1 < h_2 < h$, $\lambda > 0$, and a $C^1$-smooth cut-off function $\chi \in C^1((0,2\pi)^2)$ with compact support in $(0,2\pi)^2$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ in $(\pi/2, 3\pi/2)^2$. For $x = (x_1, x_2, x_3)^\top \in \Omega$, we define

$$\varepsilon_{\mathrm{r}}^{-1}(x_1,x_2,x_3) = \begin{cases} \lambda\chi(x_1,x_2) + 1, & 0 < x_3 < h_1, \\ \lambda\left(\frac{x_3 - h_2}{h_1 - h_2}\right)\chi(x_1,x_2) + 1, & h_1 < x_3 < h_2, \\ 1, & h_2 < x_3 < h. \end{cases}$$

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Then $\varepsilon_{\mathrm{r}}^{-1}$ is a decreasing function that satisfies (5.7)(a), and condition (5.7)(c) is satisfied when $\lambda > 0$ is small enough. Moreover, $\varepsilon_{\mathrm{r}}^{-1}$ also satisfies condition (5.7)(b) in $(\pi/2, 3\pi/2)^2 \times (h_1, h_2)$. However, $\varepsilon_{\mathrm{r}}^{-1}$ does not satisfy the corresponding conditions (7.2)(b,c) in [12], which require, roughly speaking, strict positivity of $\partial\varepsilon_{\mathrm{r}}/\partial x_3$ in $(0, 2\pi)^2 \times (h_1, h_2)$ (an arbitrary ball $B \subset \Omega$ as in (5.7)(b) is not sufficient for the proof in [12]).

**Lemma 5.4.** *Assume that $\varepsilon_{\mathrm{r}}^{-1}$ satisfies the three assumptions in (5.7). Then there exists $C > 0$ (independent of $k > 0$) such that*

$$C \int_\Omega \left| \frac{\partial H}{\partial x_3} \right|^2 \mathrm{d}x \leq \int_\Omega x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2 \,\mathrm{d}x$$

*for all solutions $H \in H^1_{\mathrm{p,T}}(\Omega)^3$ to the homogeneous problem corresponding to (3.3).*

*Proof.* We first estimate the two boundary terms in (5.1). We find that

$$-2\mathrm{Re} \int_{\Gamma_h} T_\alpha(H_3)\overline{H_3} \,\mathrm{d}s = 2 \sum_{n \in \Lambda} \mathrm{Im}\,(\beta_n) |\hat{H}_{3,n}|^2 \geq 0.$$

Together with (5.6) we obtain

$$\mathrm{Re}\,\langle e_3 \times (R_\alpha \times H), H\rangle_{\Gamma_h} - 2\mathrm{Re} \int_{\Gamma_h} T_\alpha(H_3)\overline{H_3} \,\mathrm{d}s = \sum_{n \in \Lambda} \mathrm{Im}\,(\beta_n) |\hat{H}_n|^2 \geq 0.$$

Therefore, from the Rellich identity (5.1) we deduce $V(H) \leq 0$ where $V(H)$ is the volumetric terms in (5.1). We need now to bound $V(H)$ from below,

$$\begin{aligned} V(H) &= \int_\Omega \left[ 2\varepsilon_{\mathrm{r}}^{-1} \left| \frac{\partial H}{\partial x_3} \right|^2 - x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2 + 2\mathrm{Re} \left( \nabla_T \varepsilon_{\mathrm{r}}^{-1} \cdot \frac{\partial H_T}{\partial x_3} \overline{H_3} + \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \frac{\partial H_3}{\partial x_3} \overline{H_3} \right) \right] \mathrm{d}x \\ &\geq \int_\Omega \left[ 2 \left| \frac{\partial H}{\partial x_3} \right|^2 - x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2 \,\mathrm{d}x \right] \mathrm{d}x - \gamma^{-1} \left\| \frac{\partial H_3}{\partial x_3} \right\|^2_{L^2(\Omega)} - \gamma \left\| \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \right\|^2_{L^\infty(\Omega)} \|H_3\|^2_{L^2(\Omega)} \\ &\qquad - \delta \|\nabla_T \varepsilon_{\mathrm{r}}^{-1}\|^2_{L^\infty(\Omega)^2} \|H_3\|^2_{L^2(\Omega)} - \delta^{-1} \left\| \frac{\partial H_T}{\partial x_3} \right\|^2_{L^2(\Omega)^2} \end{aligned}$$

for arbitrary $\delta, \gamma > 0$. Poincaré's inequality from Lemma 5.2 and the binomial formula imply that

$$\begin{aligned} V(H) \geq \int_\Omega \left[ \left(2 - \frac{\delta h^2}{2} \|\nabla_T \varepsilon_{\mathrm{r}}^{-1}\|^2_{L^\infty(\Omega)^2}\right) \left| \frac{\partial H_3}{\partial x_3} \right|^2 + \frac{2\delta - 1}{\delta} \left| \frac{\partial H_T}{\partial x_3} \right|^2 - x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2 \,\mathrm{d}x \right] \mathrm{d}x \\ - \gamma^{-1} \left\| \frac{\partial H_3}{\partial x_3} \right\|^2_{L^2(\Omega)} - \gamma \left\| \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \right\|^2_{L^\infty(\Omega)} \|H_3\|^2_{L^2(\Omega)} \end{aligned}$$

Again, we exploit Poincaré's inequality, to find that

$$\gamma^{-1} \left\| \frac{\partial H_3}{\partial x_3} \right\|^2_{L^2(\Omega)} + \gamma \left\| \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \right\|^2_{L^\infty(\Omega)} \|H_3\|^2_{L^2(\Omega)} \leq \left( \gamma^{-1} + \gamma \frac{h^2}{2} \left\| \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \right\|^2_{L^\infty(\Omega)} \right) \left\| \frac{\partial H_3}{\partial x_3} \right\|^2_{L^2(\Omega)}.$$

The minimum of $\gamma \mapsto \gamma^{-1} + C\gamma$ is $2\sqrt{C}$. In consequence,

$$\begin{aligned} V(H) \geq \left[ 2 - \frac{\delta h^2}{2} \|\nabla_T \varepsilon_{\mathrm{r}}^{-1}\|^2_{L^\infty(\Omega)^2} - \sqrt{2} h \left\| \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} \right\|_{L^\infty(\Omega)} \right] \int_\Omega \left| \frac{\partial H_3}{\partial x_3} \right|^2 \mathrm{d}x \\ + \frac{2\delta - 1}{\delta} \int_\Omega \left| \frac{\partial H_T}{\partial x_3} \right|^2 \mathrm{d}x - \int_\Omega x_3 \frac{\partial \varepsilon_{\mathrm{r}}^{-1}}{\partial x_3} |\operatorname{curl}_\alpha H|^2 \,\mathrm{d}x. \end{aligned}$$

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Finally, assumption (5.7)(c) implies that there exists $\delta > 1/2$ such that the first bracket on the right-hand side is positive. □

## 6. Uniqueness of solution for all wave numbers

In this section, we prove our main uniqueness result for the electromagnetic scattering problem (3.3), under the assumption that $\varepsilon_r$ satisfies (5.7). As mentioned above, corresponding uniqueness results that hold for all wave numbers currently exist, to the best of our knowledge, only for absorbing materials, see [16], or simpler two-dimensional structures, see [7].

**Theorem 6.1.** *Assume that $\varepsilon_r^{-1}$ satisfies the assumptions (5.7). Then problem (3.3) is uniquely solvable for all right-hand sides $G \in H^1_p(\Omega)$ and for all wave numbers $k > 0$.*

*Proof.* Consider a solution $H \in H^1_{p,T}(\Omega)^3$ to the homogeneous problem corresponding to (3.3). Due to Lemma 5.4 and the assumptions on $\varepsilon_r^{-1}$ we obtain that $\partial H/\partial x_3 = 0$ in $\Omega$ and $\operatorname{curl}_\alpha H = 0$ in the ball $B$ (see assumption (5.7)(b)). Equation (3.4) implies that $H$ vanishes in $B$, too.

Since $H$ is independent of $x_3$, it is sufficient to show that $H$ vanishes on $\Gamma_{h-\eta} = \{(x_1, x_2, x_3) \in \Omega : x_3 = h-\eta\}$ for some (small) $\eta > 0$ to conclude that $H$ vanishes entirely in $\Omega$. If $\eta$ is small enough, then all three components $H_j$, $j = 1, 2, 3$, satisfy

$$\Delta_\alpha H_j + k^2 H_j = 0, \qquad \Delta_\alpha H_j := \Delta H_j + 2i\alpha \cdot \nabla H_j - |\alpha|^2 H_j,$$

in some neighborhood of $\Gamma_{h-\eta}$. Let us denote by $\Delta_2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ the two-dimensional Laplacian. Since $\partial^2 H_j/\partial x_3^2$ vanishes, $H_j|_{\Gamma_{h-\eta}} \in H^1_p(\Gamma_{h-\eta})$ is a weak solution to the two-dimensional equation

$$\Delta_2 H_j + 2i\alpha \cdot \nabla_T H_j + (k^2 - |\alpha|^2) H_j = 0 \quad \text{on } \Gamma_{h-\eta}, \quad j = 1, 2, 3.$$

Standard elliptic regularity results imply that $H_j|_{\Gamma_{h-\eta}}$ belongs to $H^2_p(\Gamma_{h-\eta})$. Moreover, since $H$ vanishes in the open ball $B$ and since $H$ is independent of $x_3$, $H_j$ vanish in a non-empty relatively open subset of $\Gamma_{h-\eta}$.

In this situation, the unique continuation principle stated in Theorem 6.2 (see, *e.g.* [14]) implies that $H_j$ vanishes on $\Gamma_{h-\eta}$ for $j = 1, 2, 3$, and hence $H$ vanishes in $\Omega$. □

**Theorem 6.2.** *Let $\mathcal{O}$ be an open and simply connected set in $\mathbb{R}^2$, and let $u_1, ..., u_m \in H^2(\mathcal{O})$ be real-valued such that*

$$|\Delta u_j| \le C \sum_{l=1}^{m} (|u_l| + |\nabla u_l|) \text{ in } \mathcal{O} \text{ for } j = 1, ..., m. \tag{6.1}$$

*If $u_j$ vanishes in some open and non-empty subset of $\mathcal{O}$ for all $j = 1, ..., m$, then $u_j$ vanish identically in $\mathcal{O}$ for all $j = 1, ..., m$.*

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