109.09 Equable triangles on the Eisenstein lattice

Introduction

This Note concerns triangles whose vertices lie on the *Eisenstein lattice*, which is the lattice in the complex plane generated by the elements 1 and $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. We investigate triangles that are *equable*, that is, they have equal perimeter and area. Our study is inspired by the classification of equable Heron triangles, which have integer sides and area and are known to be realisable on the integer lattice [1]. There are precisely five triangles of the preceding type, up to Euclidean motions; see [2] and [3]. By comparison, on the Eisenstein lattice, we find the following result.

Theorem 1: There are only two equable triangles having vertices on the Eisenstein lattice, up to Euclidean motions. They are realized by the following vertices.

(a) $A = 8 + 4\omega, B = 4 + 8\omega, C = 0,$

(b) $A = 6 + 3\omega, B = 8 + 16\omega, C = 0.$

Figure 1 shows the two triangles, with the first (equilateral) triangle translated by 5 to the right.

Proof of the Theorem

Consider an equable triangle T with vertices $A = a_1 + a_2\omega$, $B = b_1 + b_2\omega$ and $C = 0$, where $a_1, a_2, b_1, b_2 \in \mathbb{N} \cup \{0\}$. Let a, b, c denote the lengths of the sides AC, BC, AB respectively. Notice that the squares of the side lengths are integers; for example, $a^2 = a_1^2 - a_1a_2 + a_2^2$. Moreover, the signed area of T is $\frac{\sqrt{3}}{4}(a_1b_2 - a_2b_1)$.

Lemma 1: The side lengths a, b, c are each of the form $\sqrt{3}n$, for some $n \in \mathbb{N}$.

Proof: From above, as T is equable, $a + b + c = \frac{\sqrt{3}}{4}(a_1b_2 - a_2b_1)$, so

$$\sqrt{3a^2} + \sqrt{3b^2} + \sqrt{3c^2} + \sqrt{3}(a + b + c) = \tfrac{3}{4}(a_1b_2 - a_2b_1),$$

which is rational. But it is well known that if $\sum_{i=1}^{n} \sqrt{m_i}$ is rational for integers $m_1, \ldots, m_n$, then $\sqrt{m_i}$ is rational for each i; see for example [4] or [5]. So $\sqrt{3a^2}$, $\sqrt{3b^2}$, $\sqrt{3c^2}$ are each rational. Hence, as $3a^2$, $3b^2$, $3c^2$ are integers, it follows that $\sqrt{3a^2}$, $\sqrt{3b^2}$, $\sqrt{3c^2}$ are also integers. So the side lengths a, b, c are each of the form $\frac{n}{\sqrt{3}}$, for some $n \in \mathbb{N}$. Thus, since the squares of the side lengths are integers, the required result follows.

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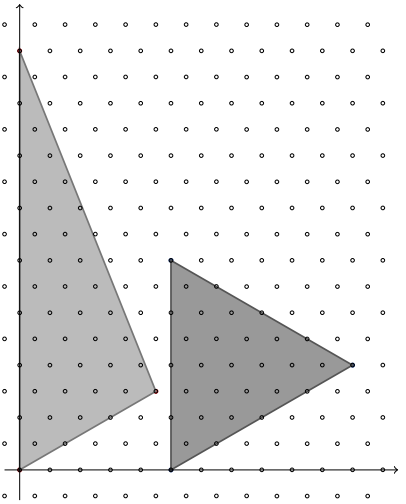


FIGURE 1: Two equable triangles on the Eisenstein lattice

We follow the reasoning used in the proof of the equable Heron triangle theorem given in the Appendix in [6]. For equable triangles with sides a, b, c, Heron's formula gives

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 16(a+b+c)^2. \qquad (1)$$

Let $u = \frac{1}{\sqrt{3}}(-a+b+c)$, $v = \frac{1}{\sqrt{3}}(a-b+c)$, $w = \frac{1}{\sqrt{3}}(a+b-c)$, so that $u, v, w \in \mathbb{N}$ by the above lemma, and $a = \frac{\sqrt{3}}{2}(v+w)$, $b = \frac{\sqrt{3}}{2}(u+w)$, $c = \frac{\sqrt{3}}{2}(u+v)$. Then (1) gives

$$3uvw = 16(u+v+w), \qquad (2)$$

and we may assume without loss of generality that $u < v < w$. Note that by construction, u, v, w have the same parity, and from (2), u, v, w are necessarily even. Let $u = 2x$, $v = 2y$, $w = 2z$, so $a = 3(y+z)$, $b = 3(x+z)$, $c = 3(x+y)$. Then $3xyz = 4(x+y+z)$. Thus

$$y \leqslant z = \frac{4(x+y)}{3xy-4},$$

so $3xy^2 - 8y - 4x \leqslant 0$. Hence

$$x \leqslant y = \frac{4+\sqrt{16+12x^2}}{3x}$$

so $3xy^2 \leqslant 4+\sqrt{16+12x^2}$. Hence $(3x^2-4)^2 \leqslant (16+12x^2)$. Thus $9x^4 - 36x^2 \leqslant 0$, which gives $x \leqslant 2$. Then

$$y \leqslant \frac{4+\sqrt{16+12x^2}}{3x} \leqslant \frac{4+\sqrt{16+12}}{3},$$

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since the function $\dfrac{4 + \sqrt{16 + 12x^2}}{3x}$ is decreasing for positive x. So, as y is an integer, $y \leqslant 3$. Then, considering the values $x \leqslant 2$, $y \leqslant 3$ and $z = \dfrac{4(x+y)}{3xy}$, we find there are only two solutions, which have the following integer values for x, y, x:

$$1, 2, 6 \quad \text{and} \quad 2, 2, 2,$$

which imply finally the values for a, b, c:

$$8\sqrt{3},\ 7\sqrt{3},\ 3\sqrt{3} \qquad \text{and} \qquad 4\sqrt{3},\ 4\sqrt{3},\ 4\sqrt{3}.$$

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CHRISTIAN AEBI
Collège Calvin, Geneva, Switzerland 1211
e-mail: *christian.aebi@edu.ge.ch*
GRANT CAIRNS
Department of Mathematical and Physical Sciences, La Trobe University, Melbourne, Australia 3086
e-mail: *G.Cairns@latrobe.edu.au*

109.10 On a geometric theorem of Al–Kūhī

Introduction

In [1] the authors consider the following result which was proved by the Islamic mathematician Al–Kūhī in the tenth century.

Theorem 1: Let AB be a diameter of a circle with centre O, let X be a point on a semi-circle based on AB, and let X be the orthogonal projection of X onto AB; see Figure 1. Then the area of the triangle $\Delta AXX'$ is maximised

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