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An Extended Dynamic Model for Kinematic Positioning

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The main problems faced by a dynamic model within a Kalman filter occur when the system experiences unexpected dynamic conditions, a change in data acquisition rate, or when the dynamics of the system are non-linear. To minimize the errors produced from dynamic modelling in unusual conditions, an extended dynamic model is developed in this paper, and its usefulness demonstrated through comparison of the performance of a Kalman filter's response to simulated data with a standard dynamic model and the extended dynamic model. The results show that, in use, the proposed extended dynamic model is superior to a standard dynamic model, due mainly to its ability to adapt to a wider range of dynamic conditions, which in turn ensures the optimization of the Kalman filter and the consequent generation of reliable positioning results.

KEY WORDS

1. Positioning. 2. Modelling. 3. Simulation.

1. INTRODUCTION. In many applications, a primary sensor has to be positioned and oriented in space at some instant in time. This instant in time does not often coincide with the same point in time at which a discrete trajectory point has been measured. Therefore extrapolation becomes necessary. In the Kalman filtering algorithm, the dynamic model predicts the state of the system based upon a best-filtered estimate of the current state. In other words, part of the dynamic model's role is to predict the path of the object in question, based upon the best estimate of the object's current trajectory characteristics determined from the filter. In order to ensure that an accurate extrapolation is made, the dynamic model must be as accurate as possible.

A number of different approaches to dynamic modelling have appeared in literature, especially in the radar tracking community, which is mainly concerned with the problem of tracking a manoeuvring target. One approach to dynamically modelling such an object is to model the manoeuvre as a stochastic departure from a nominated base trajectory. This has been done in numerous ways, for example by increasing the driving noise of the Kalman filter during periods of manoeuvring (Thorp, 1973). Another method is to model the noisy component as a random process with a known exponential auto-correlation function (Singer, 1970). The algorithms do work well within the context of their assumptions. However, some *a priori* statistical descriptions must be given to the manoeuvre process, and often this requires more knowledge about the object than is normally available. In addition, if the assumptions do not correspond to the actual characteristics of the manoeuvres, filter performance may be degraded.

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An alternative approach is to consider a manoeuvre as an inherent part of an object's dynamics. That is, while the manoeuvres may only occur at irregular intervals, and last for an undetermined length of time, they will not be modelled as noise. One such method introduces an extra estimable parameter (Chan *et al.*, 1979). At each epoch a manoeuvre parameter is solved for using a least squares process. If the parameter is significantly different to the noise level expected from the filter, then the manoeuvre parameter is used to update the filter states. The detection and adaptation scheme used in this approach also requires a significant amount of computation and memory. Another methodology uses a default trajectory (i.e. constant velocity) in the absence of any manoeuvres (Bar-Shalom, 1982), but once a manoeuvre is determined to have occurred, then a different state model is used by the filter to account for the manoeuvre. While this technique holds a lot of promise, the models used are unable to deal with time-correlated acceleration, or time-correlated velocity motion, and is therefore still susceptible to running the filter sub-optimally.

In this paper, an Extended Dynamic model is presented, which is able to be adapted to a large range of dynamic conditions. The use of this extended dynamic model ensures that the filter is run optimally at all times, during periods of manoeuvring, as well as during periods of relative stability. There are a number of factors that need to be addressed to ensure that a general dynamic model is able to adapt and run smoothly. While the emphasis of this paper is on the formulation and application of a general dynamic model, other important issues of adaptive dynamic modelling, such as the control of the dynamic noise matrix, determination of the correlation parameters and detection algorithms will be briefly discussed as they are subjects of ongoing investigations.

The models described in Sections 2 and 3 are restricted to the one-dimensional case, and do not consider an order higher than constant acceleration. This has been done for the sake of readability so that the principles of the extended dynamic model are demonstrated. The model can be easily extended to more dimensions or to a higher order.

2. DYNAMIC MODELLING.

2.1. *Kalman Filtering*. A number of filtering/processing options exist for the determination of an object's trajectory. The Kalman filter algorithm is the most popular choice as it provides the best performance in terms of minimizing the mean square estimation error, and has the advantage that it can be easily implemented. The Kalman filter state equations are given by:

(a) Functional Models:

(i) the dynamic model:

$$x_k = \Phi_{k.k-1} x_{k-1} + \tau_k. \tag{1}$$

(ii) the measurement model:

$$z_k = H_k x_k + \varepsilon_k. \tag{2}$$

(b) Prediction phase:

$$\bar{x}_k = \Phi_{k,k-1} \hat{x}_{k-1}, \tag{3}$$

$$Q_{\bar{x}_k} = \Phi_{k,k-1} Q_{\hat{x}_{k-1}} \Phi^T_{k,k-1} + Q_k. \tag{4}$$

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(c) Filtering Phase:

$$\hat{x}_k = \bar{x}_k + G_k v_k, \tag{5}$$

$$Q_{\hat{x}_k} = Q_{\bar{x}_k} - G_k Q_{d_k} G_k^T, \tag{6}$$

$$G_k = Q_{\bar{x}_k} A_k^T Q_{d_k}^{-1}, \tag{7}$$

$$v_k = z_k - A_k \bar{x}_k, \tag{8}$$

$$Q_{d_k} = R_k + A_k Q_{\bar{x}_k} A_k^T. \tag{9}$$

where: $\bar{x}_k$ is a minimum mean square estimate of x_{k+1} given sensor data up to and including time k (the one sample ahead prediction); $\hat{x}_k$ is the minimum mean square estimate of x_k given sensor data up to and including time k (filtered estimate); z_k is the observation vector at time k; $Q_{\hat{x}_k}$ and $Q_{\bar{x}_k}$ are the covariance matrices of the filtered and one-sample-ahead prediction errors respectively; A_k is the design matrix relating the observations to the parameters; R_k is the matrix of measurement noise matrix; G_k is the gain matrix; v_k is the innovation vector and Q_{dk} is its covariance matrix; $\Phi_{k,k-1}$ is the transition matrix; Q_k is the dynamic noise matrix and τ_k is the random error vector.

It can be seen that the dynamic model plays a critical role in Kalman filtering, especially in the determination of $\bar{x}_k$, and $Q_{\bar{x}_k}$.

2.2. *Dynamic Modelling*. In the context of Kalman filtering a dynamic system is typically seen as a set of ordinary differential equations describing the evolution over time of the state of a physical system. For practical problems, one is mainly interested in knowing the state of a system at discrete time intervals, which usually correspond to the sampling period of the system.

The role of the dynamic model is to describe the behaviour of an object with as few parameters as possible. However the dynamic model must also describe all of the dynamics of the system. In the special case when the measurement data is either determined to be unreliable, or is not available, then the dynamic model may also take the role of the filter. Using the dynamic model in such circumstances to provide the current state estimate will degrade the estimation accuracy, but at least the filter will be able to continue to operate without modification. In fact, one can use the prediction algorithm continually to estimate the state from the last available estimate until the measurements become available or have been determined to be reliable again.

In the following equations only one dimension is shown for the sake of illustration, however it is easy to expand them to account for higher dimensions. When modelling for kinematic positioning, the positions at time t_1 and t_{i+1} may be related by:

$$p_{i+1} = p_i + \dot{p}_i \Delta t + \ddot{p}_i \frac{\Delta t^2}{2} + \dddot{p}_i \frac{\Delta t^3}{6} + \cdots, \tag{10}$$

where: the number of dots above the parameter indicates the order of differentiation.

Equation (10) shows that for correct position interpolation all of the derivatives of the position vector p_i are needed. In reality these are not available, and the model

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is approximated by a truncated series, either using a constant velocity model or a constant acceleration model:

$$p_{i+1} = p_i + \dot{p}_i \Delta t, \tag{11}$$

$$p_{i+1} = p_i + \dot{p}_i \Delta t + \ddot{p}_i \frac{\Delta t^2}{2}, \tag{12}$$

where: p_i and $\dot{p}_i$ of equation (11) and (12) are considered constant for the time interval Δt, and $\ddot{p}_i$ is considered constant in equation (12) for the same time interval. The validity of these assumptions is dependent upon the length of Δt and the degree of linearity of the object's motion.

An alternative method is to model the velocity and/or acceleration components as a process that is correlated in time. If we assume that the acceleration component can be modelled by an exponential correlation function, then (Singer, 1970):

$$r_a(\tau) = E\{v(t)v(t+\tau)\} = \sigma_a^2 e^{(-\alpha_2|\tau|)}, \tag{13}$$

where: σ_a^2 is the variance of the acceleration, and α_2 is the reciprocal of the acceleration time constant.

The correlation parameter α will permit the modelling of different dynamics. For instance, a vehicle travelling with a sustained level of acceleration would be best modelled by a small α_2 value. On the other hand, an object with a rapid fluctuating acceleration would be better modelled with a large α_2 value. The value of α_2 is a way of adjusting the dynamic model. Hence in this case, a very low α_2 would correspond to a second-order model (constant acceleration), and a very high α_2 corresponds to a first-order model (constant velocity). The white noise component of the acceleration model permits the model to superimpose random acceleration components over its sustained acceleration capabilities.

The state equation for the time correlated acceleration model can be written as:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha_2 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t). \tag{14}$$

Note that the above equation is of the standard form:

$$\dot{x} = Ax + Bw(t). \tag{15}$$

The discrete version is:

$$X(k+1) = F(k+1,k)X(k) + u(k), \tag{16}$$

where:

$$F(k+1,k) = e^{A(t_{k+1}-t_k)}. \tag{17}$$

Equation (17) is the state transition matrix of the linear time invariant system between the instant k and $k+1$. The state transition matrix is obtained through a series expansion of the right-hand side of Equation (17) with the substitution of the matrix A, giving:

$$F(t) = \begin{bmatrix} 1 & T & c \\ 0 & 1 & d \\ 0 & 0 & e \end{bmatrix}, \tag{18}$$

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$$c = \frac{T^2}{2!} + \frac{-\alpha_2 T^3}{3!} + \frac{\alpha_2^2 T^4}{4!} + \cdots = \frac{(e^{-\alpha_2} - 1 + \alpha_2 T)}{\alpha_2^2}, \tag{19}$$

$$d = T + \frac{-\alpha_2 T^2}{2!} + \frac{\alpha_2^2 T^3}{3!} + \frac{-\alpha_2^3 T^4}{4!} + \cdots = \frac{(1 - e^{-\alpha_2 T})}{\alpha_2}, \tag{20}$$

$$e = 1 - \alpha_2 T + \frac{\alpha_2^2 T^2}{2!} - \frac{\alpha_2^3 T^3}{3!} + \frac{\alpha_2^4 T^4}{4!} + \cdots = e^{-\alpha_2 T}. \tag{21}$$

3. THE EXTENDED DYNAMIC MODEL. For the purposes of demonstrating the performance, the dynamic model will be restricted so that acceleration is the highest order. There are, however, no reasons why the same principles could not be applied to a higher order model. The extended dynamic model permits both the velocity and acceleration components to be correlated in time. Again each of these components can be represented by an exponential correlation function:

$$\begin{aligned} r_v(\tau) &= E\{v(t)v(t+\tau)\} = \sigma_v^2 e^{(-\alpha_1|\tau|)} \\ r_a(\tau) &= E\{v(t)v(t+\tau)\} = \sigma_a^2 e^{(-\alpha_2|\tau|)} \end{aligned} \tag{22}$$

where: σ_v^2 is the variance of the velocity, σ_a^2 is the variance of the acceleration, α_1 is the reciprocal of the velocity time constant, and α_2 is the reciprocal of the acceleration time constant.

The state equation for the time correlated velocity and acceleration model can be written as:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha_1 & 1 \\ 0 & 0 & -\alpha_2 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} w(t). \tag{23}$$

Note again that the above equation is of the standard form:

$$\dot{x} = Ax + Bw(t). \tag{24}$$

The discrete version is:

$$X(k+1) = F(k+1, k)X(k) + u(k), \tag{25}$$

where:

$$F(k+1, k) = e^{A(t_{k+1} - t_k)}. \tag{26}$$

Equation (26) is the state transition matrix of the linear time invariant system between the instant k and $k+1$. The state transition matrix is obtained through a series expansion of the right-hand side of Equation (26) with the substitution of the matrix A, giving:

$$F(t) = \begin{bmatrix} 1 & a & c \\ 0 & b & d \\ 0 & 0 & e \end{bmatrix}, \tag{27}$$

$$a = T - \frac{\alpha_1 T^2}{2!} + \frac{\alpha_1^2 T^3}{3!} - \frac{\alpha_1^3 T^4}{4!} + \cdots = \frac{(1 - e^{-\alpha_1 T})}{\alpha_1}, \tag{28}$$

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$$b = 1 - \alpha_1 T + \frac{\alpha_1^2 T^2}{2!} - \frac{\alpha_1^3 T^3}{3!} + \frac{\alpha_1^4 T^4}{4!} + \cdots = e^{-\alpha_1 T}, \tag{29}$$

$$c = \frac{T^2}{2!} + \frac{(-\alpha_1 - \alpha_2)T^3}{3!} + \frac{(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2)T^4}{4!} + \cdots, \tag{30}$$

$$d = T + \frac{(-\alpha_1 - \alpha_2)T^2}{2!} + \frac{(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2)T^3}{3!} + \frac{(-\alpha_1^3 - \alpha_1^2\alpha_2 - \alpha_1\alpha_2^2 - \alpha_2^3)T^4}{4!} + \cdots, \tag{31}$$

$$e = 1 - \alpha_2 T + \frac{\alpha_2^2 T^2}{2!} - \frac{\alpha_2^3 T^3}{3!} + \frac{\alpha_2^4 T^4}{4!} + \cdots = e^{-\alpha_2 T}. \tag{32}$$

In reality, systems tend to have a degree of unpredictability in their behaviour, which may be due to unknown or neglected inputs. These effects are not always negligible, and where possible should be accounted for. As will be shown later, the extended dynamic model has the ability to adapt to a wide range of dynamic conditions through the manipulation of the time correlation parameters for velocity and acceleration. By adjusting the values of α_1 and α_2 the extended dynamic model is able to model dynamics ranging from constant position, constant velocity, time correlated velocity, constant acceleration, time correlated acceleration, and time correlated velocity and acceleration.

As can be seen in Table 1, by varying the values of the correlation time for the velocity and acceleration components it is possible to adapt the dynamic model to a wide variety of other states and standard dynamic models. This allows the filter to adapt to the dynamics being experienced.

4. SIMULATION, RESULTS AND DISCUSSION.

In the following set of simulations, the performance of the filter is analysed through the application of different dynamic models to a variety of simulated conditions. First a scenario for the simulation is described, which is then analysed using a standard dynamic model, and its performance is compared to that of the adaptive dynamic model. A sample rate of 10 Hz has been used, with the noise level for the position components assumed to be normally distributed with a standard deviation of 1 cm.

4.1. *Adapting to a Constant Position Model.* In this scenario, the platform was stationary. As can be seen from Figure 1 the extended dynamic model performs in nearly an identical fashion to the constant position model. Thus there appears to be no loss in performance from using the extended dynamic model during periods of relative stability when compared to the standard dynamic model.

4.2. *Adapting to a Constant Velocity Model.* In this simulation, the platform was assumed to move at a constant velocity of 3 m/s in the east direction, and 2 m/s in the north direction. From Figure 2, it can be seen that the extended dynamic model performs almost identically to the constant velocity model. Thus there is no loss in performance from using the extended dynamic model during periods of no manoeuvres, or relative inactivity.

4.3. *Adapting to a Time Correlated Velocity Model.* A platform was simulated to have an initial velocity of 40 m/s in the east direction and 50 m/s in the north direction. The velocity components were also simulated to have a time correlation period of 20 s.

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Table 1. Summarising Model type vs. Correlation time values and the resulting dynamic model.

Model Type	α_1	α_2	Resulting Dynamic Model
Constant Position	$\to\infty$	$\to\infty$	$F(T)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$
Time Correlated Velocity	>0	$\to\infty$	$F(T)=\begin{bmatrix}1&T&0\\0&1&0\\0&0&0\end{bmatrix}$
Constant Velocity	$\to 0$	$\to\infty$	$F(T)=\begin{bmatrix}1&a&0\\0&b&0\\0&0&0\end{bmatrix}$
Time Correlated Acceleration	$\to 0$	>0	$F(T)=\begin{bmatrix}1&T&c\\0&1&d\\0&0&e\end{bmatrix}$
Constant Acceleration	$\to 0$	$\to 0$	$F(T)=\begin{bmatrix}1&T&T^2\\0&1&T\\0&0&1\end{bmatrix}$
Time Correlated Acceleration and Velocity	>0	>0	$F(T)=\begin{bmatrix}1&a&c\\0&b&d\\0&0&e\end{bmatrix}$

The innovation sequence plots in Figure 3 show that the constant velocity model introduces an initial large bias into the system, which is slow to converge to a small offset bias from the zero mean. The time correlated velocity model does, however, model the dynamics well, and maintains the expected zero mean. As can be seen in Figure 3, the extended dynamic model is better able to model a manoeuvre from the nominal trajectory than a constant velocity model.

4.4. *Adapting to a Constant Acceleration Model.* In this simulation, a platform was simulated to be moving at a constant velocity (40 m/s in the east direction and 50 m/s in the north direction) for the first 100 epochs. At epoch 101 the object was simulated to experience a constant acceleration of 3 m/s/s in the east direction and 2 m/s/s in the north direction. Figure 4 shows that the constant velocity model has a large initial divergence from zero in the innovation sequence mean, which is then sustained for the remainder of the simulation. The adapted extended dynamic model is able to adjust to the change in conditions, and after a short initialisation period the filter returns to optimal performance with no bias. Hence the adaptive extended dynamic model is effective in modelling manoeuvres that transition from a constant velocity to a constant acceleration.

4.5. *Adapting to a Time Correlated Acceleration Model.* A platform was simulated to be moving at a constant velocity (40 m/s in the east direction and 50 m/s in the north direction) with an initial jump in acceleration of 30 m/s/s in the east direction and 20 m/s/s in the north direction. The acceleration component was modelled to have a time correlation period of 20 s. Figure 5 shows the responses of a constant acceleration model compared to that of the adapted extended dynamic model. The constant acceleration model maintains a bias in the filter, as shown by the innovation sequence plot (Figure 5, left). The extended dynamic model is able to maintain an optimal filter without any bias being introduced. Therefore the adaptive extended dynamic model is

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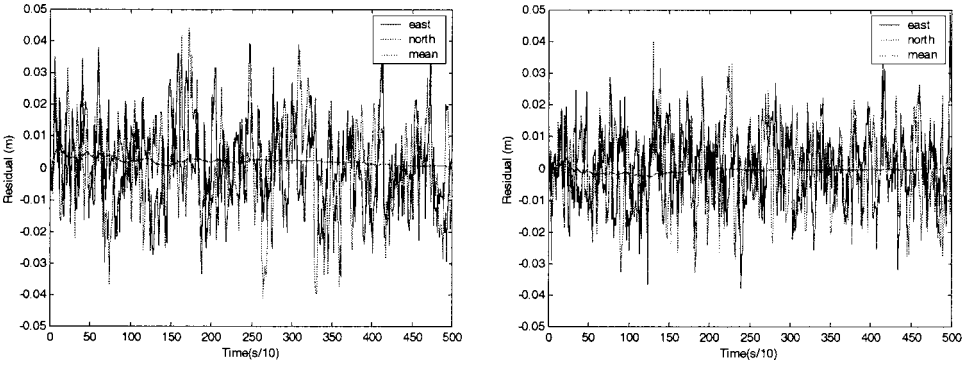


Figure 1. Innovation sequence plots for a constant position model (left), and the adapted extended dynamic model (right).

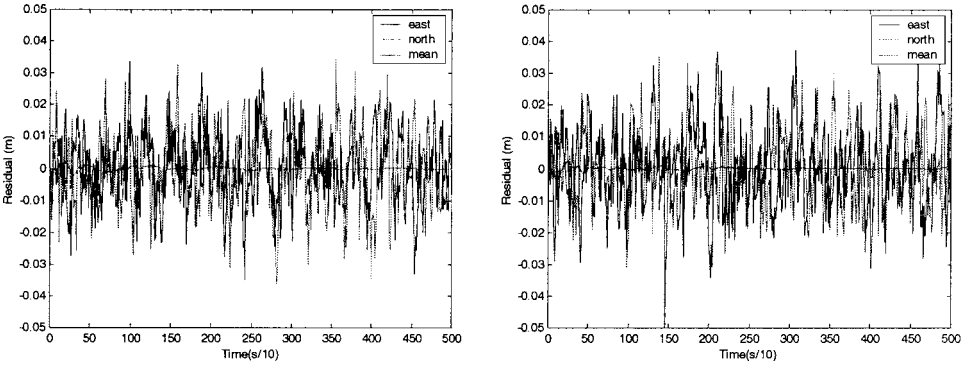


Figure 2. Innovation sequence plots for a constant velocity model (left), and the adapted constant velocity model (right).

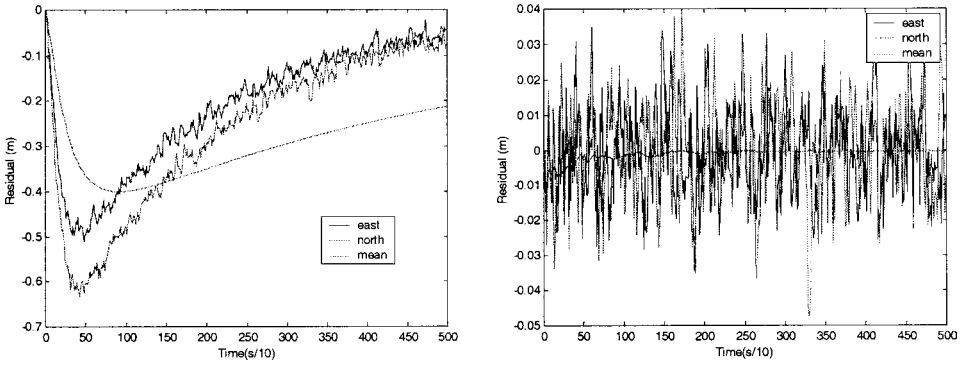


Figure 3. Innovation sequence plots of a constant velocity model (left), and the adapted time-correlated velocity model (right).

superior in performance to the standard acceleration model, as it is capable of modelling manoeuvres that a constant acceleration model cannot.

4.6. *Time-Correlated Velocity and Time-Correlated Acceleration Model.* A platform was simulated to be moving at a constant velocity (40 m/s in the east direction and

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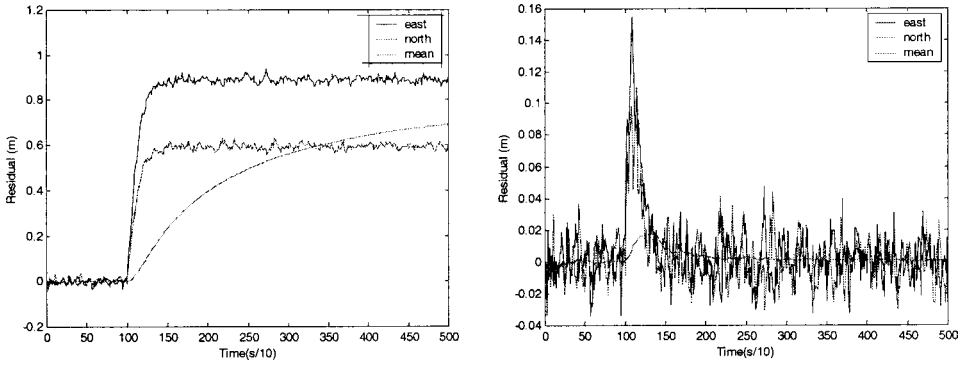


Figure 4. Innovation sequence plots of a constant acceleration model (left), and the adapted constant acceleration model (right).

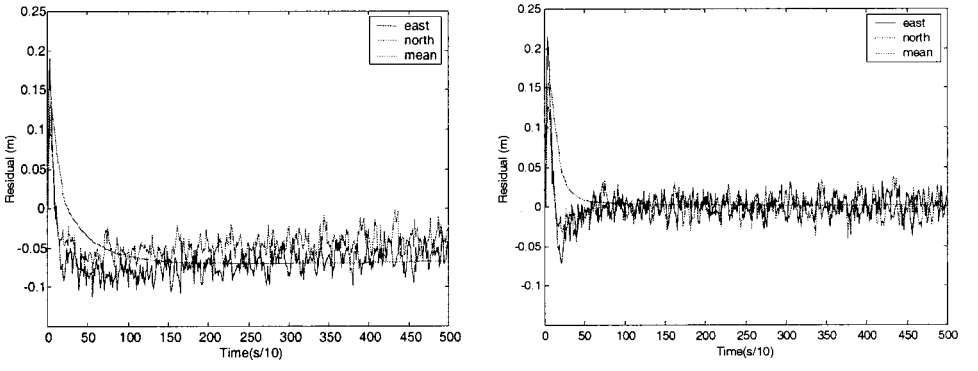


Figure 5. Innovation sequence plots of a constant acceleration model (left), and the adapted time-correlated acceleration model (right).

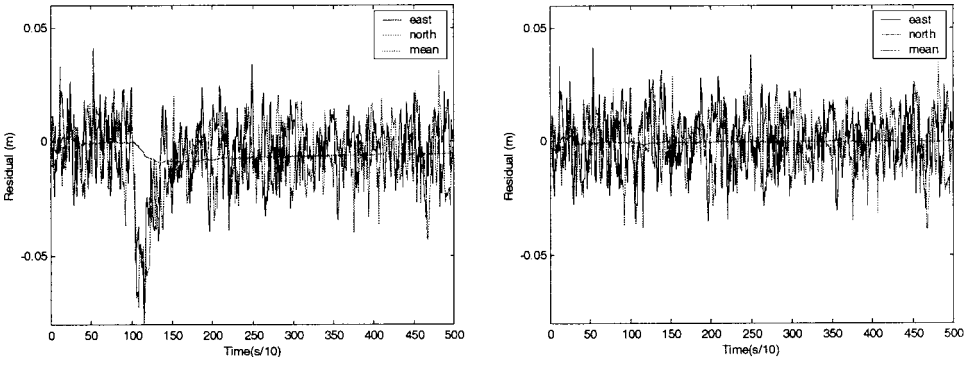


Figure 6. Innovation sequence plots of a time-correlated acceleration model (left), and adapted time-correlated velocity and acceleration (right).

50 m/s in the north direction) for the first 100 epochs. After epoch 101 the platform experienced a time-correlated velocity period of 100 s and a time-correlated acceleration period of 20 s, with an initial jump in acceleration of 30 m/s/s in the east direction and 20 m/s/s in the north direction. Figure 6 shows the response of the two filters with

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different dynamic models. In the left hand plot, it can be seen that the time-correlated acceleration model performs very well for the first 100 epochs. However, even after adapting the time-correlated acceleration model to account for the simulated acceleration component, there still remains a noticeable bias in the innovation sequence. The extended dynamic model (right hand plot) is able to accurately model the simulated dynamic conditions quickly, removing the modelling bias from the filter. Hence the adaptive extended dynamic model is able to model these conditions very well, unlike the time-correlated acceleration model.

5. CONCLUSION. The extended dynamic model has been described as comprising time-correlated velocity and time-correlated acceleration components. The main advantage of this dynamic model is its ability to be adapted easily to model fractional as well as standard dynamic occurrences through the simple manipulation of the time-correlation parameters.

The simulations have shown that the extended dynamic model is superior to the standard dynamic models whenever a manoeuvre takes place. The extended dynamic model is best suited to situations in which a wide range of dynamics is likely to be experienced.

Further investigations are being conducted into the formulation of an appropriate dynamic noise matrix that can also be easily adapted to suit a large range of dynamic conditions. Detection algorithms are also being investigated so that they are sensitive to detecting modelling errors, with the minimum number of false alarms. The issue of determining the most appropriate correlation parameters in real time will also be addressed.

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