J. Plasma Physics (2000), *vol.* 64, *part* 4, *pp.* 371–377. Printed in the United Kingdom

© 2000 Cambridge University Press

Pressure determinations for incompressible fluids and magnetofluids

BRIAN T. KRESS and DAVID C. MONTGOMERY

Department of Physics and Astronomy, Dartmouth College, Hanover, NH 03755-3528, USA

(Received 24 April 2000)

Abstract. Certain unresolved ambiguities surround pressure determinations for incompressible flows: both Navier–Stokes and magnetohydrodynamic (MHD). For uniform-density fluids with standard Newtonian viscous terms, taking the divergence of the equation of motion leaves a Poisson equation for the pressure to be solved. But Poisson equations require boundary conditions. For the case of rectangular periodic boundary conditions, pressures determined in this way are unambiguous. But in the presence of 'no-slip' rigid walls, the equation of motion can be used to infer both Dirichlet and Neumann boundary conditions on the pressure P, and thus amounts to an over-determination. This has occasionally been recognized as a problem, and numerical treatments of wall-bounded shear flows usually have built in some relatively *ad hoc* dynamical recipe for dealing with it – often one that appears to 'work' satisfactorily. Here we consider a class of solenoidal velocity fields that vanish at no-slip walls, have all spatial derivatives, but are simple enough that explicit analytical solutions for P can be given. Satisfying the two boundary conditions separately gives two pressures, a 'Neumann pressure' and a 'Dirichlet pressure', which differ non-trivially at the initial instant, even before any dynamics are implemented. We compare the two pressures, and find that, in particular, they lead to different volume forces near the walls. This suggests a reconsideration of no-slip boundary conditions, in which the vanishing of the tangential velocity at a no-slip wall is replaced by a local wall-friction term in the equation of motion.

Dedication

This paper is dedicated to Dr J. P. Dougherty in his honour on the occasion of his 65th birthday.

1. Introduction

It has long been the case that pressure determinations for incompressible flows, both Navier–Stokes and MHD, are known to be highly non-local. Taking the divergence of the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = \frac{\mathbf{j}\times\mathbf{B}}{\rho c} - \boldsymbol{\nabla}P + \nu\nabla^2\mathbf{v}, \tag{1.1}$$

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press

and using $\nabla \cdot \mathbf{v} = 0$ leaves us with a Poisson equation for the pressure P, which is said to function as an equation of state:

$$\nabla^2 P = -\nabla \cdot \left(\mathbf{v} \cdot \nabla \mathbf{v} - \frac{\mathbf{j} \times \mathbf{B}}{\rho c} \right). \tag{1.2}$$

Here $\mathbf{v} = \mathbf{v}(x, t)$ is the fluid velocity field as a function of position and time, $\mathbf{B}$ is the magnetic field, $\mathbf{j} = c\nabla \times \mathbf{B}/4\pi$ is the electric current density, c is the speed of light, ν is the kinematic viscosity, assumed spatially uniform and constant, and P is the pressure normalized to the mass density ρ, also spatially uniform. Equations (1.1) and (1.2) are written for MHD. Their Navier–Stokes equivalents can be obtained simply by dropping the terms containing $\mathbf{B}$ and $\mathbf{j}$.

If we are to solve (1.2) for P, boundary conditions are required. In the immediate neighbourhood of a stationary 'no-slip' boundary, both the terms on the left-hand side of (1.1) vanish, and we are left with the following equation for ∇P as a boundary condition:

$$\nabla P = \nu \nabla^2 \mathbf{v} + \frac{\mathbf{j} \times \mathbf{B}}{\rho c}. \tag{1.3}$$

We now focus on the Navier–Stokes case, where the magnetic terms disappear from (1.3), for simplicity. All the complications of MHD are illustrated by this simpler case. It is apparent that (1.3) must apply to all components of ∇P, and that while the normal component of ∇P is enough to determine P through Neumann boundary conditions, the tangential components of (1.3) at the wall equally well determine P through Dirichlet boundary conditions. This is a problem for whose resolution some inventive procedures have been proposed to resolve, usually by some degree of 'preprocessing' or various dynamical recipes that seem to lead to approximately no-slip velocity fields after a few time steps (see e.g. Gresho 1991; Roache 1982; Canuto et al. 1988). It is not our purpose to review or critique these recipes, but rather to focus on a set of velocity fields, related to Chandrasekhar–Reid functions (Chandrasekhar 1961), for which (1.2) is explicitly soluble at a level where the Neumann or Dirichlet conditions can be exactly implemented. In Sec. 2, we explore the difference between the two pressures so arrived at. Then, in Sec. 3, we propose a replacement for the long-standing practice of demanding that all components of a solenoidal $\mathbf{v}$ vanish at material walls, in favour of a replacement by a wall-friction term for which the above mathematical difficulty is no longer present. Of course, similar statements and options will apply to all comparable incompressible MHD problems.

2. Pressure determinations

Restricting attention at present to the Navier–Stokes case, we consider two-dimensional, solenoidal, velocity fields obtained from the following stream function:

$$\psi(x, y) = C_{k\lambda} \cos(kx) \left[\cos(\lambda y) + A_{k\lambda} \cosh(ky) \right]. \tag{2.1}$$

The hyperbolic-cosine term in (2.1) contributes a potential-flow velocity component to $\mathbf{v}$, which makes it possible to demand that $\mathbf{v}$ obey two boundary

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press

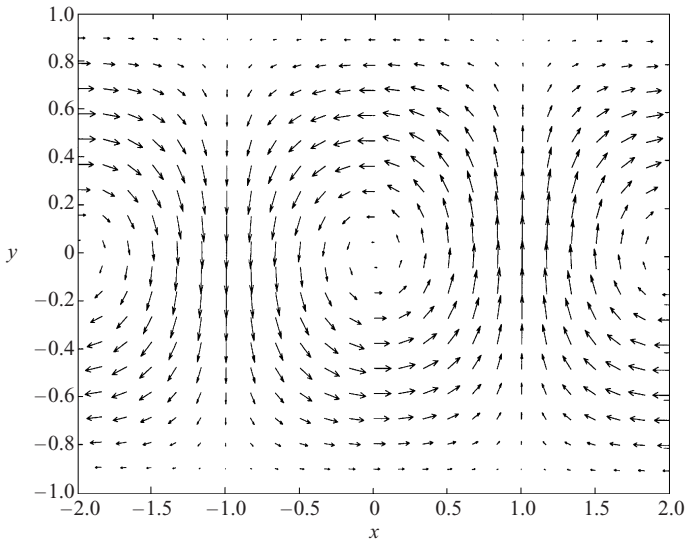


Figure 1. Velocity field: $\mathbf{v} = \boldsymbol{\nabla}\psi \times \hat{\mathbf{e}}_z$ using ψ from (2.1) with $k = \frac{1}{2}\pi$, $\lambda = 2.6424$ and $A_{k\lambda} = 0.3499$.

conditions: the vanishing of both components at rigid material walls (Chandrasekhar 1961). The function in (2.1) is even in x and y, but can obviously be converted into an odd or mixed one by the appropriate trigonometric substitutions.

The velocity field $\mathbf{v} = \boldsymbol{\nabla}\psi \times \hat{\mathbf{e}}_z$ has only x and y components, and is periodic in x, with an arbitrary wavenumber k. $C_{k\lambda}$ is a normalizing constant, and the constants λ and $A_{k\lambda}$ can systematically be found numerically to any desired accuracy so that both components of $\mathbf{v}$ vanish at symmetrically placed no-slip walls at $y = a$ and $y = -a$. In fact, for given k, an infinite sequence of such pairs of λ and $A_{k\lambda}$ can be determined straightforwardly. Thus any such $\mathbf{v}$, or superposition thereof, not only is solenoidal, but also has both components zero at $y = \pm a$, and all spatial derivatives exist. Moreover, the 'source' term, or $\boldsymbol{\nabla}\cdot(\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v})$, from the right-hand side of (1.2), is of a relatively simple nature for such a $\mathbf{v}$, since every term in it can be written as a product of exponentials of kx, λy and ky. It is straightforward to find an inhomogeneous solution for P, which then is the same for all boundary conditions for a given $\mathbf{v}$ of the form stated. To this inhomogeneous part of P must be added a solution of Laplace's equation. This can be chosen so that the total P may satisfy either the normal component of (1.3) at the walls, or the tangential component of it, but not both. The determination involves only simple but tedious algebra.

We illustrate, in Fig. 1, an arrow plot of the velocity field given by choosing $k = \frac{1}{2}\pi$, $\lambda = 2.6424$ and $A_{k\lambda} = 0.3499$, in units of $a = 1$. The two pressures resulting from the satisfaction of the normal and tangential components of (1.3) can best be compared by comparing their respective values of $\boldsymbol{\nabla}P$, since P itself is indeterminate up to an additive constant in both cases. In Fig. 2, we display, as an arrow plot, the difference between the pressure gradients associated with the velocity field shown in Fig. 1. We have rewritten (1.1)–(1.3) in dimensionless

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press

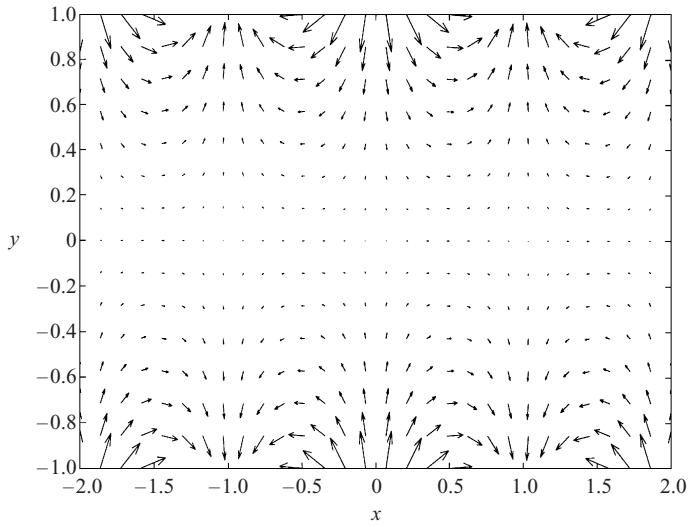


Figure 2. The difference between pressure gradients: $\nabla P_N - \nabla P_D$, with $Re = 2293$.

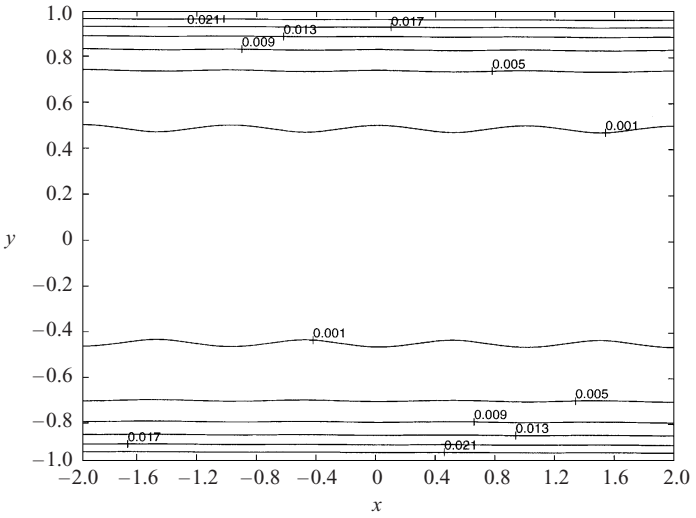


Figure 3. Normalized mean square pressure-gradient difference: $(\nabla P_D - \nabla P_N)^2 / \langle (\nabla P_N)^2 \rangle$, with $Re = 2293$. Note that the fractional difference between the two values of ∇P is significant only near the wall.

units for this purpose, with the kinematic viscosity being replaced by the reciprocal of a Reynolds number, which may be defined as

$$Re = \left(\frac{\langle \mathbf{v}^2 \rangle}{k^2 + \lambda^2} \right)^{1/2} \Big/ \nu$$

Here the angle brackets refer to the mean of $\mathbf{v}^2$ taken over the two-dimensional box, containing one period in the x direction and from $y = -a$ to $y = a$. The value of Re used to construct Fig. 2 is $Re = 2293$, with the dimensionless version of $C_{k\lambda} = 5000$ in (2.1). The two pressures are similar but not identical.

In Fig. 3, a fractional measure of the difference between the 'Neumann

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press

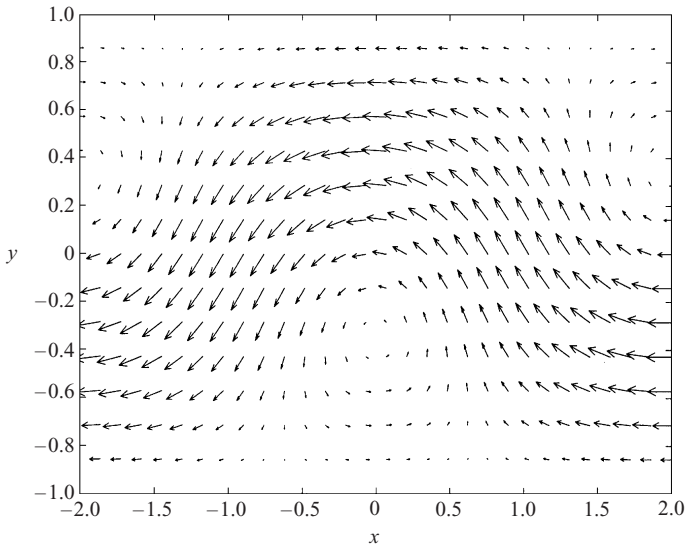


Figure 4. Velocity field shown in Fig. 1 plus parabolic plane Poiseuille flow: $\mathbf{v} = \boldsymbol{\nabla}\psi \times \hat{\mathbf{e}}_z + 2(y^2-1)\,\hat{\mathbf{e}}_x$, with $C_{k\lambda} = 1$.

pressure' P_N and the 'Dirichlet pressure' P_D is exhibited as a contour plot of the scalar ratio

$$\frac{(\boldsymbol{\nabla}P_D - \boldsymbol{\nabla}P_N)^2}{\langle(\boldsymbol{\nabla}P_N)^2\rangle}. \tag{2.2}$$

There is no absolute significance to the numerical value of this ratio. It initially increases with Re, approaching a maximum of about 2% near the wall for $Re \gtrsim 10$. It is considered interesting, however, that the fractional difference is nearly x-independent where it is largest. That occurs formally because the algebra reveals it to be dominated by a term that varies as $\cosh(4ky) - \cos(4kx)$ in a region where $ky \gtrsim 1$.

It is amusing but perhaps not significant to superpose the velocity field from (2.1) with a parabolic plane Poiseuille flow of a larger amplitude. The resulting flow field is shown in Fig. 4, and it bears a striking but perhaps not significant similarity to the flow patterns seen in two-dimensional plane Poiseuille flow (Jones and Montgomery 1994) when linear stability thresholds are approached. The pressure-gradient difference for this case will be fractionally smaller than in Fig. 3, since pure parabolic plane Poiseuille flow is a rare case where the two pressures happen to agree, and it quantitatively dominates the pressures determined from (1.2) in this example.

3. Discussion and a possible modification

An alternative to the no-slip condition is the 'Navier' boundary condition (Lamb 1932): the slip velocity at the wall surface is taken to be proportional to the rate of shear at the wall. This may be expressed $\Delta V = L_s\,\dot{\gamma}$, where ΔV is the slip velocity of the fluid at the wall, $\dot{\gamma}$ is the rate of shear at the wall and L_s is

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press

a constant with the dimension of length. Molecular-dynamic simulations of Newtonian liquids under shear (Thompson and Robbins 1990) have shown this to be the case under some circumstances. In fact, recent work by Thompson and Troian (1997) has shown that, in cases where the shear rate is large, there is a nonlinear relationship between L_s and $\dot{\gamma}$.

We note that the velocity field shown in Fig. 1 does not lead to one that obeys the Navier boundary condition, after an initial time step, where the fluid has been allowed to slip at the wall. If the velocity field determined by (2.1) is advanced in time using (1.1) with the 'Neumann pressure', the proportionality between the slip velocity and the rate of shear at the wall, after the initial time step, varies sinusoidally with x.

It is to be stressed that we are concerned here only with initial conditions, not with circumstances under which initial slip velocities might be coaxed dynamically into vanishing after some time.

It is difficult to see in what sense the velocity field obtained from (2.1) might be an unacceptable one from the point of view of the Navier–Stokes or MHD descriptions (Batchelor 1967, p. 149). It seems to have all the properties that are thought to be relevant. The family of functions of the same x-periodicity in (2.1) can be shown to be orthogonal, and is a candidate for a complete set, in which any $\mathbf{v}$ might be expanded, when supplemented by flux-bearing functions of y alone. The mathematical question of which, if any, velocity fields that are both solenoidal and vanish at the wall would lead to Neumann and Dirichlet pressures that were in agreement with each other must remain open. Indeed, the question of whether there are any, without some degree of 'pre-processing', must remain open. This is an unsatisfactory situation for fluid mechanics and MHD, in our opinion, even if it is a not unfamiliar one. The search for alternatives seems mandatory.

One alternative that may be explored is one that seemed some time ago, in a rather different context (Shan and Montgomery 1994a,b), to have worked well enough for MHD. Namely, we may think of replacing the requirement of the vanishing of the tangential velocity at a rigid wall with a wall-friction term, added to the right-hand side of (1.1), of the form

$$-\frac{\mathbf{v}}{\tau(\mathbf{x})}, \tag{3.1}$$

where the coefficient $1/\tau(\mathbf{x})$ vanishes in the interior of the fluid and rises sharply to a large positive value near the wall. The region over which it is allowed to rise should be smaller than the characteristic thickness of any boundary layer that it might be intended to resolve, but seems otherwise not particularly restrictive. Such a term provides a mechanism for momentum loss to the wall, and constrains the tangential velocity to small values, but does not force it to zero. The Dirichlet boundary condition disappears in favour of a relation that permits the time evolution of the tangential components of $\mathbf{v}$, while demanding that P be determined solely by the Neumann condition (the normal component of (1.3) only). In a previous MHD application (Shan and Montgomery 1994a,b) dealing with rotating MHD fluids, the scheme seemed to perform acceptably well, but was not intensively tested or benchmarked sharply against any of the better-understood Navier–Stokes flows. This comparison seems worthy of future attention.

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press

Acknowledgements

The work of one of us (D.C.M.) was supported by hospitality in the Fluid Dynamics Laboratory at the Eindhoven University of Technology in the Netherlands. A preliminary account of this work was presented orally at a meeting of the American Physical Society (Kress and Montgomery 1999).

References

Batchelor, G. K. 1967 *An Introduction to Fluid Mechanics*. Cambridge University Press.

Canuto, C., Hussaini, M. Y., Quarteroni, A. and Zang, T. A. 1988 *Spectral Methods in Fluid Mechanics*. Springer-Verlag, Berlin.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*, p. 634. Oxford University Press.

Gresho, P. M. 1991 Incompressible fluid dynamics: some fundamental formulation issues. *Annu. Rev. Fluid Mech.* **23**, 413–453.

Kress, B. T. and Montgomery, D. C. 1999 No-slip pressures in wall bounded flows. *Bull. Am. Phys. Soc.* **44**(8), 85.

Lamb, H. 1967 *Hydrodynamics*, p. 576. Dover, New York.

Roache, P. J. 1982 *Computational Fluid Dynamics*. Hermosa, Albuquerque, NM.

Jones, W. B. and Montgomery, D. C. 1994 Finite amplitude steady state of high Reynolds number 2-D channel flow. *Physica* **D73**, 227–243.

Shan, X. and Montgomery, D. C. 1994a Magnetohydrodynamic stabilization through rotation. *Phys. Rev. Lett.* **73**, 1624–1627.

Shan, X. and Montgomery, D. C. 1994b Rotating magnetohydrodynamics. *J. Plasma Phys.* **52**, 113–128.

Thompson, P. A. and Robbins, M. O. 1990 Shear flow near solids: epitaxial order and flow boundary conditions. *Phys. Rev.* **A41**, 6830–6837.

Thompson, P. A. and Troian, S. M. 1997 A general boundary condition for liquid flow at solid surfaces. *Nature* **389**, 360–362.

https://doi.org/10.1017/S0022377800008825 Published online by Cambridge University Press