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Rationality of dynamical canonical height

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Abstract. We present a dynamical proof of the well-known fact that the Néron–Tate canonical height (and its local counterpart) takes rational values at points of an elliptic curve over a function field $k = \mathbb{C}(X)$, where X is a curve. More generally, we investigate the mechanism by which the local canonical height for a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over a function field k can take irrational values (at points in a local completion of k), providing examples in all degrees $\deg f \geq 2$. Building on Kiwi's classification of non-archimedean Julia sets for quadratic maps [Puiseux series dynamics of quadratic rational maps. *Israel J. Math.* **201** (2014), 631–700], we give a complete answer in degree 2 characterizing the existence of points with irrational local canonical heights. As an application we prove that if the heights $\widehat{h}_f(a)$, $\widehat{h}_g(b)$ are rational and positive, for maps f and g of multiplicatively independent degrees and points $a, b \in \mathbb{P}^1(\bar{k})$, then the orbits $\{f^n(a)\}_{n\geq 0}$ and $\{g^m(b)\}_{m\geq 0}$ intersect in at most finitely many points, complementing the results of Ghioca *et al* [Intersections of polynomials orbits, and a dynamical Mordell–Lang conjecture. *Invent. Math.* **171** (2) (2008), 463–483].

1. *Introduction*

Throughout this article, we let K be an algebraically closed field of characteristic 0, X a smooth projective curve defined over K, and $k = K(X)$ its function field. We examine the values of the canonical height function associated to dynamical systems

$$f : \mathbb{P}^1 \to \mathbb{P}^1$$

defined over k, of degree $\deg(f) \geq 2$. The canonical height $\widehat{\mathrm{h}}_f$ was introduced in [**CS**] for general polarized dynamical systems f on projective varieties. It measures arithmetic complexity in orbits, and it has a decomposition as a sum of local canonical heights $\widehat{\lambda}_{f,v}$, indexed by the places v of the function field k. In the special case where f is a Lattès map

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on $\mathbb{P}^1$, i.e. the quotient of an endomorphism of an elliptic curve, the canonical height of f coincides with the projection of the Néron–Tate canonical height on the elliptic curve. In general, Call and Silverman showed that, in the presence of a 'weak Néron model', the values of the local canonical heights $\widehat{\lambda}_{f,v}$ are equal to certain intersection numbers, exactly as for abelian varieties [**CS**, §6]. Unfortunately, these weak Néron models often fail to exist in the dynamical setting [**Hs**, Theorem 3.1]; in fact, as observed in [**In**, §1], *every* non-isotrivial and non-Lattès example f will have some place v where the weak Néron model fails to exist. This constitutes a fundamental obstruction to using weak Néron models to study the canonical height in general.

In our paper, we study the local heights, building on the methods of [**CS**, §5], by analyzing the action of f on the Berkovich projective line $\mathbf{P}^1_v$ over a local field $\mathbb{C}_v$. We use the dynamics of a Lattès map to give a new proof that the local and global canonical heights are aways rationally valued for elliptic curves over function fields. Our main results show, by contrast, that the local canonical heights for general maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ can take irrational values. Finally, we relate these observations to the question of orbit intersections and the dynamical Mordell–Lang conjecture.

1.1. *Elliptic rationality.* Suppose that E is an elliptic curve over $k = K(X)$. The *Néron–Tate canonical height* $\widehat{\mathrm{h}}_E : E(\bar{k}) \to \mathbb{R}$ is defined as follows (see, for example, [**Si1**]). We let $\pi : E \longrightarrow \mathbb{P}^1$ be a non-constant even function (such as the identification of a point P with $-P$) and define

$$\widehat{\mathrm{h}}_E(P) := \frac{1}{\deg(\pi)} \cdot \lim_{m\to\infty} \frac{h(\pi([m]P))}{m^2}, \tag{1.1}$$

where $[m]$ is the multiplication-by-m map on the elliptic curve E and h is a logarithmic Weil height on $\mathbb{P}^1(\bar{k})$. As shown in [**Si1**, Ch. III], $\widehat{\mathrm{h}}_E(P) \geq 0$ for each $P \in E(\bar{k})$, with equality if and only if P is a torsion point for E or the pair $(E,\ P)$ is *isotrivial* in the sense that E is isomorphic (through some isomorphism ι defined over a finite extension of k) to an elliptic curve E_0 defined over K, while $\iota(P) \in E_0(K)$. The canonical height has a decomposition into Néron local heights,

$$\widehat{\mathrm{h}}_E = \sum_{v\in M_k} \widehat{\lambda}_{E,v},$$

where each $\widehat{\lambda}_{E,v}$ is well defined on the non-zero points of E and M_k denotes the set of all places of the function field k/K. We note that the construction of $\widehat{\lambda}_{E,v}$ depends on a choice, but this choice does not affect the question about its rationality; essentially, the local canonical height is unambiguous modulo $\mathbb{Q}$ (for more details, see §3).

For each place $v \in M_k$, we let $\mathbb{C}_v$ be a minimal complete and algebraically closed extension of k. Note that $v(\mathbb{C}_v^*) = \mathbb{Q}$. We begin this article with a dynamical proof of a well-known 'local rationality' statement.

THEOREM 1.1. *Suppose E is an elliptic curve over k. Fix any place $v \in M_k$. Then, for every point $P \in E(\mathbb{C}_v) \setminus \{O\}$, the local canonical height $\widehat{\lambda}_{E,v}(P)$ is a rational number. Consequently, for all $P \in E(\bar{k})$, the canonical height $\widehat{\mathrm{h}}_E(P)$ is a rational number.*

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See [**Si1**, Ch. III, Theorem 9.3] for a proof that $\widehat{\mathrm{h}}_E(P) \in \mathbb{Q}$ for any $P \in E(\bar{k})$; see [**CS**, §6 and p. 203] and also [**La2**, Ch. 11 Theorem 5.1] for the proofs that each local function $\widehat{\lambda}_{E,v}$ also takes values in $\mathbb{Q}$. Both of these facts can be explained by means of intersection theory since the local heights correspond to intersection numbers on a Néron local model.

1.2. *Local irrationality.* As for elliptic curves, the canonical height of a morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ has a local decomposition,

$$\widehat{\mathrm{h}}_f(x) = \sum_{v \in M_k} \widehat{\lambda}_{f,v}(x)$$

(see §3.2). The local canonical heights depend on a choice, but their rationality or irrationality is independent of the choice, as $v(\mathbb{C}_v^*) = \mathbb{Q}$. Also, as for elliptic curves, it is easy to show that local rationality holds for polynomial maps [**In**] (see also the discussion immediately following [**GT1**, Fact 3.6]); i.e. for any polynomial map $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over k and every $a \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$, the value of $\widehat{\lambda}_{f,v}(a)$ is a rational number because of the strongly attracting behavior of ∞. Consequently, the global canonical height $\widehat{\mathrm{h}}_f(P)$ is rational for all $P \in \mathbb{P}^1(\bar{k})$ and any polynomial f.

One ingredient in our proof of Theorem 1.1—in studying the dynamics of an associated map on $\mathbb{P}^1$—is that all Type II points in the Julia set within the Berkovich projective line $\mathbf{P}^1_v$ are preperiodic. This is true for all rational maps f defined over k [**DF**, Proposition 3.9], but nevertheless, the local heights are not always rational.

THEOREM 1.2. *For every degree $d \geq 2$, there exist a rational function $f \in k(z)$ of degree d, a place $v \in M_k$, and a point $x \in \mathbb{P}^1(\mathbb{C}_v)$ for which the local canonical height $\widehat{\lambda}_{f,v}(x)$ lies in $\mathbb{R} \setminus \mathbb{Q}$.*

The examples are inspired by the classical dynamical construction of a Markov partition, encoding orbits by their itineraries; see Proposition 6.1. In fact, there is nothing special about the irrational values that arise as local heights, as exhibited by the following example.

THEOREM 1.3. *Let $k = \mathbb{C}(t)$ and $v = \mathrm{ord}_{t=0}$ so that $v(t) = 1$. Let*

$$f(z) = \frac{(z+1)(z-t)}{z+t}.$$

Then, for all real numbers $-1 \leq \alpha \leq 0$, there exists a point $x \in \mathbb{P}^1(\mathbb{C}_v)$ so that

$$\widehat{\lambda}_{f,v}(x) = \alpha.$$

Remark 1.4. By choosing other normalizations for $\widehat{\lambda}_{f,v}$ in Theorem 1.3, the interval $[-1, 0]$ of values can be shifted to any interval of the form $[n, n+1]$ with $n \in \mathbb{Z}$; see §2.4.

Recall that polynomials only have rational values for their local heights. For maps of degree 2, building on the work of Kiwi [**Ki1**], we give a complete description of which maps can produce irrational local heights.

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THEOREM 1.5. *Fix a place $v \in M_k$. For each $f \in k(z)$ of degree $d = 2$, exactly one of the following holds:*

(1) *f has potential good reduction at v;*
(2) *f is strongly polynomial-like at v; or*
(3) *there are points in $\mathbb{P}^1(\mathbb{C}_v)$ with irrational local canonical heights.*

The map f is said to have *potential good reduction* at v if, after a change of coordinates by a Möbius transformation μ defined over a finite extension k' of k, the reduction $\mu \circ f \circ \mu^{-1} \mod v'$ has the same degree as f for a place v' of k' over v. For a map f of degree greater than 1, this will hold at all but finitely many places v of k, and it holds at all places v if and only if f is isotrivial [**Ba**] (see also [**PST**] when f is an endomorphism of $\mathbb{P}^N$). We say that a quadratic map f is *strongly polynomial-like* at v if its three fixed-point multipliers $\{\alpha, \beta, \gamma\}$ satisfy $|\alpha|_v = |\beta|_v > 1$ and $|\gamma|_v < 1$. Note that the quadratic polynomials are characterized, up to conjugacy by an automorphism of $\mathbb{P}^1$, by the condition that one fixed-point multiplier is $\gamma = 0$; furthermore, if the polynomial does not have potential good reduction, then the fixed-point formula guarantees that the other two multipliers satisfy $|\alpha|_v = |\beta|_v > 1$ (see §7.2).

From [**Ki1**, Theorems 2 and 3], our strong polynomial-like condition implies that the Julia set of f is a Cantor set contained in $\mathbb{P}^1(\mathbb{C}_v)$, with $f|J(f)$ conjugate to the full one-sided 2-shift, but the conditions are not equivalent. There exist maps with Cantor Julia set that do admit points with irrational local height; these are precisely the quadratic maps with fixed-point multipliers satisfying $|\alpha|_v > |\beta|_v > 1 > |\gamma|_v = |1/\beta|_v > 0$. Our proof of Theorem 1.5 shows that, in this polynomial-like but not strongly polynomial-like case, the points with irrational local height are necessarily in the Cantor Julia set.

1.3. Global rationality? The constructions we exploit in Theorem 1.2 suggest that the points $x \in \mathbb{P}^1(\mathbb{C}_v)$ with irrational local height will be transcendental over k. We expect an affirmative answer to the following question.

Question 1.6. Given any rational function f of degree greater than or equal to 2 defined over k, is $\widehat{\mathrm{h}}_f(P)$ a rational number for every $P \in \mathbb{P}^1(\bar{k})$?

1.4. Orbit intersections. We conclude with an application of rationality, partly motivating our study. For any rational function f of degree greater than 1 defined over a field F of characteristic 0, and for any $a \in \mathbb{P}^1(F)$, we denote by $\mathcal{O}_f(a)$ the orbit of a under the iterates of f, i.e. the set of all $f^n(a)$ for $n \geq 0$. It is natural to ask under what hypotheses the intersection $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ can be infinite. For example, if f and g share an iterate, and if a and b lie in the same grand orbit for this iterate, then the intersection is infinite. It is expected that infinite intersection can only arise from such special arrangements. It is known that if f and g are polynomials, then an f-orbit can intersect a g-orbit in infinitely many points only when f and g share a common iterate (see [**GTZ2**] and [**GTZ1**]). On the other hand, almost nothing is known for general rational functions.

The question was originally motivated by the dynamical Mordell–Lang conjecture (see [**GT2**]). A special case of the dynamical Mordell–Lang conjecture predicts that if C is an

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irreducible curve inside an arbitrary variety X defined over a field K of characteristic 0 and endowed with an endomorphism $\Phi : X \longrightarrow X$, if there exists $\alpha \in X(K)$ such that C contains infinitely many points in common with the orbit of α under Φ, then C must be periodic.

We work over function fields k/K as above, and we prove the result for rational functions under additional simplifying hypotheses.

THEOREM 1.7. *Fix $a, b \in \mathbb{P}^1(k)$ and $f, g \in k(z)$ with $d = \deg(f)$ and $e = \deg(g)$ both greater than* 1. *Assume the following hypotheses are met:*

(1) *d and e are multiplicatively independent;*

(2) *the pair (f, a) is not isotrivial over k;*

(3) $\widehat{\mathrm{h}}_f(a), \widehat{\mathrm{h}}_g(b) \in \mathbb{Q}$.

Then $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ is finite.

Furthermore, if hypotheses (2)–(3) hold, while condition (1) is replaced by the weaker hypothesis that

(1′) *f and g do not share a common iterate,*

then the dynamical Mordell–Lang conjecture yields that $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ is finite.

Hypothesis (1) ensures that f and g do not share a common iterate. Obviously, if f and g share a common iterate then $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ may be infinite (e.g. when $a = b$). Hypothesis (2) means there does not exist any invertible $\mu \in k(z)$ for which both $\mu \circ f \circ \mu^{-1} \in K(z)$ and $\mu(a) \in \mathbb{P}^1(K)$. We believe Theorem 1.7 should hold without hypothesis (2); we use it to guarantee that if a is not preperiodic for f, then $\widehat{\mathrm{h}}_f(a) > 0$ ([**Ba**], [**De2**]). As mentioned above, we suspect that hypothesis (3) is always satisfied; the rationality of heights provides the decisive Diophantine ingredient needed in our proof.

Finally, we note that essentially the same proof as the one we employ for Theorem 1.7 would work under the more general hypotheses that f and g are polarizable endomorphisms of degrees d and e with respect to the same ample line bundle on a projective variety Z. Condition (2) would be replaced with the assumption that f does not descend to an endomorphism defined over the constant field K. Then one would use [**CH1**] and [**CH2**] instead of [**De2**] or [**Ba**] to infer that $\widehat{\mathrm{h}}_f(a) > 0$ if a is not preperiodic under the action of f. However, since checking whether hypothesis (3) is always met for rational functions appears to be difficult, one might expect this same condition for the general case of polarizable endomorphisms to be very challenging.

2. *Notation*

We fix some notation and terminology that we use throughout the article.

2.1. Places of function fields. Let K be an algebraically closed field of characteristic 0, and let k be the function field of a projective, smooth curve X defined over K. We let M_k be the set of all *places* of k which are trivial on K; each place $v \in M_k$ corresponds to a point $P(v) \in X(K)$. By abuse of notation, we identify each place $v \in M_k$ with both its valuation function $v(\cdot)$ and also with its corresponding non-archimedean absolute value $|\cdot|_v$ (note that $|x|_v := e^{-v(x)}$ for each $x \in k$). We normalize each valuation v so that $v(k) = \mathbb{Z} \cup \{\infty\}$ (note that $v(0) = \infty$). In other words, for a non-zero $x \in k$, the valuation $v(x)$ represents

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the order of vanishing at $P(v)$ of the function x (seen as a map $X \mapsto \mathbf{P}^1$). We have the following *product formula* (see [**BG**]) for each non-zero $x \in k$:

$$\prod_{v \in M_k} |x|_v = 1. \tag{2.1}$$

Note that in (2.1), each local degree (usually denoted by N_v) equals 1 since the residue field at each place v is canonically isomorphic to K.

For each place $v \in M_k$ we define $r_v : \mathbf{P}^1(k) \longrightarrow \mathbf{P}^1(K)$ to be the reduction map modulo v. So, for each $P \in \mathbf{P}^1(k)$ we let $a, b \in k$ such that $\min\{v(a), v(b)\} = 0$ and $P = [a : b]$; then $r_v(P)$ is obtained by reducing modulo v both a and b. Hence, for $x \in k$ such that $|x|_v \leq 1$, $r_v(x)$ is simply the image of x in the residue field at the place v, while if $|x|_v > 1$, then $r_v(x) = \infty$.

For each $v \in M_k$ we let k_v be the completion of k at the place v; we let $\mathbb{C}_v$ be the completion of a given algebraic closure of k_v. Then $\mathbb{C}_v$ is a complete, algebraically closed field, and we fix an embedding of $\bar{k}$ into $\mathbb{C}_v$. We extend naturally the reduction map $r_v : \mathbf{P}^1(\mathbb{C}_v) \longrightarrow \mathbf{P}^1(K)$.

We let $\mathcal{O}_v = \{x \in \mathbb{C}_v : |x|_v \leq 1\} = \{x : v(x) \geq 0\}$. For each $a \in \mathbb{C}_v$ and each $r \geq 0$, we let

$$D(a, r) = \{x \in \mathbb{C}_v : |x - a|_v \leq r\}$$

and we refer to a set of the form $\{x : |x - a| < r\}$ as an *open disk*.

2.2. Weil height. We define the Weil logarithmic height function $h(\cdot)$ on $\mathbb{P}^1(\bar{k})$ as follows: for each $P := [x_0 : x_1] \in \mathbb{P}^1(\bar{k})$ we let

$$h(P) := \frac{1}{[k(x_0, x_1) : k]} \sum_{v \in M_k} \sum_{\substack{\sigma : k(x_0, x_1) \hookrightarrow \bar{k} \\ \sigma|_k = \mathrm{id}|_k}} \log \max\{|\sigma(x_0)|_v, |\sigma(x_1)|_v\}.$$

We refer the reader to [**BG**] for more details about the Weil height for function fields.

2.3. Rational functions on function fields. Let $f \in k(z)$ be a rational function of degree $d \geq 2$. We consider homogeneous presentations

$$F(z, w) = (P(z, w), Q(z, w))$$

of the rational function $f(z) = P(z, 1)/Q(z, 1)$, where P and Q are homogeneous polynomials of degree d. The presentation F is said to be *normalized (at the place v)* if the coefficients of P and Q lie in $\mathcal{O}_v$, and if at least one coefficient of P and of Q has valuation 0. The rational function determined by $F \mod v$ is called the *reduction* of f; it has degree between 0 and $d = \deg f$. The map f is said to have *good reduction* if the degree is equal to d. The map f is said to have *potential good reduction* if, after a change of coordinates by conjugation by a map defined over a finite extension k_1 of k, the resulting map has good reduction at a place of k_1 lying over v. See also [**Si2**] for more information on these fundamental concepts.

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2.4. Canonical height. The canonical height of the dynamical system $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over k of degree $d \geq 2$ is defined by

$$\widehat{\mathrm{h}}_f(P) = \lim_{n\to\infty} \frac{1}{d^n} h(f^n(P)).$$

There exists a local decomposition associated to a choice of divisor on $\mathbb{P}^1$; the standard choice is to take ∞ as a divisor. That is, if we write $f(z) = P(z)/Q(z)$, where P and Q have no common roots, then there is a unique function $\widehat{\lambda}_{f,v}$ satisfying [**Si2**, Theorem 3.27]:

(1) for all $a \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$ with $Q(a) \neq 0$,

$$\widehat{\lambda}_{f,v}(f(a)) = d\ \widehat{\lambda}_{f,v}(a) - \log |Q(a)|_v,$$

and

(2) $a \mapsto \widehat{\lambda}_{f,v}(a) - \log \max\{|a|_v, 1\}$ extends to a bounded continuous function on $\mathbb{P}^1(\mathbb{C}_v)$.

Note that if we replace the pair (P, Q) with (sP, sQ) for $s \in k^*$, the local height is transformed by adding

$$\frac{1}{d-1} \log |s|_v = -\frac{1}{d-1}\, v(s) \ \in \mathbb{Q}$$

[**Si2**, Remark 3.28].

For a polynomial $f \in k[z]$ of degree $d \geq 2$ and for any $v \in M_k$, the local canonical height $\widehat{\lambda}_{f,v}$ is given by

$$\widehat{\lambda}_{f,v}(x) = -\lim_{n\to\infty} \frac{\min\{0, v(f^n(x))\}}{d^n}$$

upon taking denominator $Q = 1$.

As shown in [**CS**], if $x = (x_0 : x_1) \in \mathbf{P}^1(\bar{k})$ then

$$\widehat{\mathrm{h}}_f(x) = \frac{1}{[k(x_0, x_1) : k]} \sum_{v\in M_k} \sum_{\substack{\sigma : k(x_0,x_1)\hookrightarrow \bar{k} \\ \sigma|_k = \mathrm{id}|_k}} \widehat{\lambda}_{f,v}(x).$$

Also, as proved in [**CS**], the difference $|\widehat{\mathrm{h}}_f(x) - h(x)|$ is uniformly bounded over $x \in \mathbb{P}^1(\bar{k})$ (this property will be used in §8). Finally, we note the result of Baker [**Ba**], who proved that unless f is *isotrivial* (i.e. there exists a linear map $\mu \in \bar{k}(z)$ such that $\mu^{-1} \circ f \circ \mu \in K(z)$), we have that $\widehat{\mathrm{h}}_f(x) = 0$ if and only if x is preperiodic (this property will be used in §8).

3. *Computing the local canonical height*

We let $f \in k(z)$ be a rational function of degree $d \geq 2$. In this section, we compute the local canonical height $\widehat{\lambda}_{f,v}$ following the approach of [**CS**, §5] and [**CG**], and we explain how to determine its rationality. We also show that rationality does not depend on choices of coordinates, so in particular, if f has potential good reduction, then all points have rational canonical height.

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3.1. Orders of vanishing. Take a homogeneous presentation of f, writing

$$F(z, w) = (P(z, w), Q(z, w))$$

as in §2.3. Fix $v \in M_k$. Given any point $a \in \mathbb{P}^1(\mathbb{C}_v)$, we express a in homogeneous coordinates $A = (A_1, A_2) \in (\mathbb{C}_v)^2$ so that $\min\{v(A_1), v(A_2)\} = 0$. We define *orders*

$$\sigma(F, a) = \min\{v(P(A)), v(Q(A))\}. \tag{3.1}$$

Note that this is well defined, as it does not depend on our choice of representative of A, subject to the chosen normalization condition. It does depend on the choice of F, since

$$\sigma(\alpha F, a) = \sigma(F, a) + v(\alpha) \tag{3.2}$$

for any $\alpha \in k^*$.

The function $\sigma(F, \cdot)$ is uniformly bounded on $\mathbb{P}^1(\mathbb{C}_v)$, as first observed in [**CS**]; in fact, for normalized F, we have

$$0 \le \sigma(F, a) \le v(\mathrm{Res}(F)) \tag{3.3}$$

for all $a \in \mathbb{P}^1(\mathbb{C}_v)$ [**BR**, Lemma 10.1], where $\mathrm{Res}(F)$ is the homogeneous resultant of the pair (P, Q). In particular, if f has good reduction at v, which is equivalent to $v(\mathrm{Res}(F)) = 0$, then $\sigma(F, a) = 0$ for all a in $\mathbb{P}^1(\mathbb{C}_v)$. Also, restricting to k or its completion k_v (or any finite extension), the function $\sigma(F, \cdot)$ takes only finitely many rational values. Also, note that $v(\mathbb{C}_v) = \mathbb{Q}$.

From the definitions, we compute that

$$\sigma(F^2, a) = d\,\sigma(F, a) + \sigma(F, f(a)),$$

and

$$\begin{aligned}\sigma(F^n, a) &= d\,\sigma(F^{n-1}, a) + \sigma(F, f^{n-1}(a)) \\ &= d^{n-1}\,\sigma(F, a) + d^{n-2}\,\sigma(F, f(a)) + \cdots + \sigma(F, f^{n-1}(a))\end{aligned} \tag{3.4}$$

for all $n \ge 1$. The uniform bound on $\sigma(F, \cdot)$ shows that the limit

$$\eta(F, a) := \lim_{n\to\infty} \frac{\sigma(F^n, a)}{d^n} = \frac{1}{d}\sum_{n=0}^{\infty} \frac{\sigma(F, f^n(a))}{d^n} \tag{3.5}$$

exists.

LEMMA 3.1. ([**CS**, Theorem 5.3], [**CG**, §3]) *The local canonical height (relative to ∞) for $F = (P, Q)$ is computed as*

$$\widehat{\lambda}_{f,v}(a) = -\eta(F, a) - \min\{0, v(a)\}$$

for all $a \ne \infty$.

Proof. Recall that $\lambda_{f,v}$ is characterized by conditions (1) and (2) in §2.4. If we write $f(z) = P(z, 1)/Q(z, 1)$, and if we fix any point $a \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$ for which $f(a) \ne \infty$, we obtain

$$\begin{aligned}-\eta(F, f(a)) - \min\{0, v(f(a))\} &= d(-\eta(F, a) - \min\{0, v(a)\}) \\ &\quad + d\,\min\{0, v(a)\} + \sigma(F, a) - \min\{0, v(f(a))\}.\end{aligned}$$

The lemma is proved by a straightforward computation showing that

$$d\,\min\{0, v(a)\} + \sigma(F, a) - \min\{0, v(f(a))\} = -\log|Q(a, 1)|_v.$$

Also, in order to check condition (2) in §2.4, one uses (3.3) coupled with (3.5). □

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The rationality of $\widehat{\lambda}_{f,v}(a)$ thus hinges on the values of the sequence

$$\sigma_n := \sigma(F,\ f^n(a)) \ \in \ \mathbb{Q} \cap [0,\ v(\mathrm{Res}\ F)]. \tag{3.6}$$

We note the following basic fact about d-ary expansions for the proofs of our main theorems.

PROPOSITION 3.2. *Fix an integer $d \geq 2$. Suppose that $\sigma_n \in \{0, 1, \ldots, d-1\}$ for all n. Then*

$$\eta = \frac{1}{d} \sum_{n=0}^{\infty} \frac{\sigma_n}{d^n}$$

is rational if and only if the sequence $\{\sigma_n\}$ is eventually periodic.

3.2. Transforming the local canonical height. Recall from §2.4 that $\widehat{\lambda}_{f,v}$ depends on the choice of polynomials P and Q representing $f = P/Q$. However, regardless of the choice, the property whether $\widehat{\lambda}_{f,v}(\cdot)$ takes rational values or not is unchanged.

Next we show that replacing a rational function $\Phi(z) = P(z)/Q(z)$ by conjugate rational function Ψ, i.e. for which there exists a Möbius transformation $\eta \in k(z)$ such that $\Phi = \eta^{-1} \circ \Psi \circ \eta$, the property of taking only rational values by the local canonical height is unaffected. Since each linear transformation η is a composition of finitely many linear transformations of the form

(1) $z \mapsto cz$ for some constant $c \in k^*$,
(2) $z \mapsto z + c$ for some constant $c \in k^*$, and
(3) $z \mapsto 1/z$,

it suffices to consider separately each of the cases (1)–(3).

Case 1. The change of coordinates given by multiplicative translation. For each place v, we have that the local canonical heights are related by

$$\widehat{\lambda}_{\Phi,v}(x) = \widehat{\lambda}_{\Psi,v}(cx) + v(c). \tag{3.7}$$

The proof goes through checking the properties (1) and (2) of §2.4 for the local canonical height. Since $\Psi(z) = P(z)/Q(z)$, we have $\Phi(z) = P(cz)/(c \cdot Q(cz))$. We take the denominator of $\Phi(z)$ to be $c^{1-d}Q(cz)$. Using (3.7) as a definition for $\widehat{\lambda}_{\Phi,v}(\cdot)$ in terms of $\widehat{\lambda}_{\Psi,v}(\cdot)$, we see that

$$\begin{aligned}\widehat{\lambda}_{\Phi,v}(x) + \min\{v(x), 0\} &= \widehat{\lambda}_{\Psi,v}(cx) + v(c) + \min\{v(x), 0\} \\ &= -\min\{v(cx), 0\} + O(1) + \min\{v(cx),\ v(c)\} \\ &= O(1),\end{aligned}$$

where in the penultimate equality we used property (2) for $\widehat{\lambda}_{\Psi,v}$. This shows that $\widehat{\lambda}_{\Phi,v}$ satisfies property (2) as well. Now, for property (1) for $\widehat{\lambda}_{\Phi,v}$, we have

$$\begin{aligned}\widehat{\lambda}_{\Phi,v}(\Phi(x)) &= \widehat{\lambda}_{\Psi,v}(c\Phi(x)) + v(c) \\ &= \widehat{\lambda}_{\Psi,v}(\Psi(cx)) + v(c) \\ &= d\,\widehat{\lambda}_{\Psi,v}(cx) - \log|Q(cx)|_v + v(c) \\ &= d\,\widehat{\lambda}_{\Phi,v}(x) - \log|c \cdot Q(cx)|_v,\end{aligned}$$

which yields formula (ii) for $\lambda_{\Phi,v}$ since $-\log|c|_v = v(c)$ and the denominator for $\Phi(z)$ is $c^{1-d}Q(cz)$.

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Case 2. The change of coordinates given by additive translation. In this case,

$$\Phi(z) = \Psi(z+c) - c = \frac{P(z+c)}{Q(z+c)} - c = \frac{P(z+c) - cQ(z+c)}{Q(z+c)}.$$

We observe that the denominator $Q(z+c)$ remains coprime with the numerator $P(z+c) - cQ(z+c)$; also, $\max\{\deg(P(z+c) - cQ(z+c)), \deg(Q(z+c))\} = d$. We claim that for any place v, we have

$$\widehat{\lambda}_{\Phi,v}(x) = \widehat{\lambda}_{\Psi,v}(x+c). \tag{3.8}$$

Again, we only need to check that $\widehat{\lambda}_{\Phi,v}$ satisfies conditions (1)–(2) above. Condition (2) follows because

$$\begin{aligned}\widehat{\lambda}_{\Phi,v}(x) + \min\{v(x), 0\} &= \widehat{\lambda}_{\Psi,v}(x+c) + \min\{v(x), 0\} \\ &= (\widehat{\lambda}_{\Psi,v}(x+c) + \min\{v(x+c), 0\}) \\ &\quad + (\min\{v(x), 0\} - \min\{v(x+c), 0\}) \\ &= O(1).\end{aligned}$$

The first parenthesis above is $O(1)$ using condition (2) applied to the local canonical height $\widehat{\lambda}_{\Psi,v}$. Now, for condition (2), we check that

$$\begin{aligned}\widehat{\lambda}_{\Phi,v}(\Phi(x)) &= \widehat{\lambda}_{\Psi,v}(\Phi(x) + c) \\ &= \widehat{\lambda}_{\Psi,v}(\Psi(x+c)) \\ &= d \cdot \widehat{\lambda}_{\Psi,v}(x+c) - \log|Q(x+c)|_v,\end{aligned}$$

which proves condition (1) for $\widehat{\lambda}_{\Phi,v}$ because the denominator of $\Phi(z)$ is indeed $Q(z+c)$.

Case 3. The change of coordinates given by inversion. Since $\Phi(z) = 1/\Psi(1/z)$,

$$\Phi(z) = \frac{Q\left(\frac{1}{z}\right)}{P\left(\frac{1}{z}\right)} = \frac{z^d \cdot Q\left(\frac{1}{z}\right)}{z^d \cdot P\left(\frac{1}{z}\right)}.$$

So the denominator of $\Phi(z)$ is $z^d \cdot P(1/z)$, which is a polynomial coprime with the polynomial $z^d \cdot Q(1/z)$, which is the numerator of $\Phi(z)$. We claim that for any place v,

$$\widehat{\lambda}_{\Phi,v}(x) = \widehat{\lambda}_{\Psi,v}\left(\frac{1}{x}\right) + v\left(\frac{1}{x}\right). \tag{3.9}$$

Now, to check condition (2) for the function $\widehat{\lambda}_{\Phi,v}(\cdot)$ defined above, we compute

$$\begin{aligned}\widehat{\lambda}_{\Phi,v}(x) + \min\{v(x), 0\} &= \widehat{\lambda}_{\Psi,v}\left(\frac{1}{x}\right) + v\left(\frac{1}{x}\right) + \min\{v(x), 0\} \\ &= \widehat{\lambda}_{\Psi,v}\left(\frac{1}{x}\right) + \min\left\{0, v\left(\frac{1}{x}\right)\right\} \\ &= O(1),\end{aligned}$$

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because $\widehat{\lambda}_{\Psi,v}$ satisfies condition (2). For condition (1), we have that

$$\begin{aligned}
\widehat{\lambda}_{\Phi,v}(\Phi(x)) &= \widehat{\lambda}_{\Psi,v}\left(\frac{1}{\Phi(x)}\right) + v\left(\frac{1}{\Phi(x)}\right) \\
&= \widehat{\lambda}_{\Psi,v}\left(\Psi\left(\frac{1}{x}\right)\right) + v\left(\Psi\left(\frac{1}{x}\right)\right) \\
&= d \cdot \widehat{\lambda}_{\Psi,v}\left(\frac{1}{x}\right) - \log\left|Q\left(\frac{1}{x}\right)\right|_v + v\left(P\left(\frac{1}{x}\right)\right) - v\left(Q\left(\frac{1}{x}\right)\right) \\
&= d \cdot \left(\widehat{\lambda}_{\Psi,v}\left(\frac{1}{x}\right) + v\left(\frac{1}{x}\right)\right) + v(x^d) + v\left(P\left(\frac{1}{x}\right)\right) \\
&= d \cdot \widehat{\lambda}_{\Phi,v}(x) + v\left(x^d \cdot P\left(\frac{1}{x}\right)\right) \\
&= d \cdot \widehat{\lambda}_{\Phi,v}(x) - \log\left|x^d \cdot P\left(\frac{1}{x}\right)\right|_v,
\end{aligned}$$

as desired.

PROPOSITION 3.3. *If rational function $f \in k(z)$ of degree greater than or equal to 2 has potential good reduction at the place v, then $\widehat{\lambda}_{f,v}(a) \in \mathbb{Q}$ for all $a \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$.*

Proof. Let F be a normalized homogeneous presentation of f. Suppose that f has good reduction. As observed immediately after (3.3), good reduction implies that $\eta(F, a) \equiv 0$. Therefore, from Lemma 3.1, we have $\widehat{\lambda}_{f,v}(a) = -\min\{0, v(a)\} \in \mathbb{Q}$ for all $a \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$. On the other hand, if f has potential good reduction, then equations (3.7), (3.8), and (3.9) show that $\widehat{\lambda}_{f,v}$ will again take only rational values. □

4. *The spine in Berkovich space*

We begin this section with the background we need and the terminology we use about the Berkovich projective line $\mathbf{P}^1_v$ over $\mathbb{C}_v$. We introduce the *spine* of a rational function f inside $\mathbf{P}^1_v$, a finite tree that captures the geometry of its zeros and poles. It is our tool for computing the order function (3.1) for arbitrary rational functions f.

4.1. Berkovich space and definition of the Julia set. Useful introductions to the Berkovich projective line $\mathbf{P}^1_v$ are given in [**BR**] and [**Fa1**]. See [**Ki1**, §§1 and 2] for a presentation of the relevant dynamical notions. We summarize the most important things here.

In the weak topology, the projective line $\mathbb{P}^1(\mathbb{C}_v)$ is a dense subset of $\mathbf{P}^1_v$. Its points are called Type I or classical. The closure inside $\mathbf{P}^1_v$ of a disk $D(a, r) = \{z : |z - a|_v \le r\}$ in $\mathbb{C}_v$ is called a Berkovich closed disk. It has a unique boundary point inside $\mathbf{P}^1_v$. If the radius r lies in the value group $|\mathbb{C}_v^*| = \mathbb{Q}$, this boundary point is called a Type II point, and it is a branching point of the tree structure on $\mathbf{P}^1_v \setminus \mathbb{P}^1(\mathbb{C}_v)$. In this way, we identify Type II points with disks of rational radius; we sometimes write $\zeta = D(a, r)$ where ζ is the Type II point bounding the disk $D(a, r)$. Given a Type II point $\zeta \in \mathbf{P}^1_v$, the connected components of $\mathbf{P}^1_v \setminus \{\zeta\}$ are called *directions from* ζ. The directions are open sets in the weak topology, and their intersections with the set of Type I points are disks in $\mathbb{P}^1(\mathbb{C}_v)$ of

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the form $\{z : |z-a|_v < r\}$ or $\{z : |z-a|_v > r\}$. There is a distinguished Type II point ζ_0 called the *Gauss point* corresponding to the unit disk $D(0, 1)$. The set of directions from ζ_0 is isomorphic to $\mathbb{P}^1(\mathcal{O}_v/\mathfrak{m}_v) = \mathbb{P}^1(K)$.

The Type II points may also be identified with Möbius changes of coordinates on $\mathbb{P}^1(\mathbb{C}_v)$. More precisely, the group $\mathrm{PSL}_2(\mathbb{C}_v)$ acts on $\mathbf{P}^1_v$, and the stabilizer of the Gauss point ζ_0 is equal to $\mathrm{PSL}_2(\mathcal{O}_v)$. For any Type II point ζ of $\mathbf{P}^1_v$, there is a unique $M \in \mathrm{PSL}_2(\mathbb{C}_v)$, modulo postcomposition by an element of the stabilizer, that sends ζ to ζ_0. See the discussion in [**Ki2**, §3].

Now suppose that f is a rational function of degree $d \geq 2$ defined over k. As a map, f takes (sufficiently small) disks in $\mathbb{P}^1(\mathbb{C}_v)$ to disks; the image of an arbitrary disk will be either another disk or all of $\mathbb{P}^1(\mathbb{C}_v)$. In this way f extends naturally and continuously to $\mathbf{P}^1_v$. Topologically, it behaves much like a holomorphic branched cover of the Riemann sphere; see [**Fa1**, **Fa2**] for a study of the branching structure. See also [**DF**, Proposition 2.1] for a description of its local topological degrees, exactly analogous to the behavior of proper analytic maps on regions in the complex plane, built on a non-archimedean version of the argument principle.

Given any Type II point ζ, the action of f on its set of directions is easily computed. Indeed, under a pair of Möbius changes of coordinates A and B, we can move ζ and its image $f(\zeta)$ to the Gauss point, so that $B \circ f \circ A^{-1}$ fixes the Gauss point ζ_0. The reduction modulo v of $B \circ f \circ A^{-1}$ (expressed in a homogeneous, normalized form) determines a rational function of degree $d(\zeta) \geq 1$ which determines the action from the set of directions at ζ mapping to those of $f(\zeta)$. In particular, the set of directions from ζ always maps surjectively over the set of directions from $f(\zeta)$. Moreover, a direction will always map surjectively over its image direction; but there are particular directions from ζ that will map over all of $\mathbf{P}^1_v$. These are called *bad* directions, and they arise when the degree of the map defined by $B \circ f \circ A^{-1}$ mod v satisfies $d(\zeta) < d$; a bad direction at ζ is sent by A to a direction from ζ_0 containing both a pole and a zero of $B \circ f \circ A^{-1}$.

A point $x \in \mathbf{P}^1_v$ belongs to the *Julia set* of f if, for every open neighborhood V containing x, the union $\bigcup_{n\geq 0} f^n(V)$ omits at most two points of $\mathbf{P}^1_v$. Otherwise, x is said to be in the *Fatou set* of f. For example, a map f has good reduction if and only if its Julia set is equal to the Gauss point $\{\zeta_0\}$. It has potential good reduction if and only if the Julia set consists of a single Type II point.

4.2. The spine of a rational map. Given a rational function f defined over k, write f in homogeneous coordinates as

$$F(z, w) = (P(z, w), Q(z, w)).$$

Recall that F is normalized if all coefficients lie in $\mathcal{O}_v$ and if P and Q each have at least one coefficient of valuation 0. Recall the order function $\sigma(F, \cdot)$ on $\mathbb{P}^1(\mathbb{C}_v)$, defined above in (3.1):

$$\sigma(F, a) = \min\{v(P(A)), v(Q(A))\}$$

where $A = (A_1, A_2)$ is any expression for $a \in \mathbb{P}^1(\mathbb{C}_v)$ in homogeneous coordinates with $\min\{v(A_1), v(A_2)\} = 0$. The following result, proved in §4.3, allows us to define the spine of a rational function.

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PROPOSITION 4.1. *There is a well-defined and continuous function*

$$\zeta \mapsto \sigma_F(\zeta)$$

on Type II points ζ in $\mathbf{P}^1_v$, satisfying:

(i) *$\sigma_F(\zeta_0) = 0$ for the Gauss point ζ_0 if and only if F is normalized;*

(ii) *the function σ_F is monotone (non-strictly) increasing along any path from ζ_0 to a zero or pole of f and locally constant elsewhere; and*

(iii) *for all but finitely many directions U from ζ, and every point $a \in U \cap \mathbb{P}^1(\mathbb{C}_v)$, the order $a \mapsto \sigma(F, a)$ is constant equal to $\sigma_F(\zeta)$.*

The *spine* S_f is the smallest, closed, connected subset of $\mathbf{P}^1_v$, containing the Gauss point ζ_0, so that the order function σ_F is constant on connected components of $\mathbf{P}^1_v \setminus S_f$. From Proposition 4.1, the spine is contained in the finitely branched tree spanning the zeros and poles of f and containing the Gauss point. It is not a dynamical object, and it is not a conjugacy invariant. We will show the following theorem.

THEOREM 4.2. *The spine S_f is the connected hull of the Gauss point ζ_0 and its set of preimages $f^{-1}(\zeta_0)$ in $\mathbf{P}^1_v$.*

In particular, the spine S_f is equal to the Gauss point $\{\zeta_0\}$ if and only if f has good reduction.

4.3. Computing the order function. Fix a normalized homogeneous presentation of F as

$$F(z, w) = (P(z, w), Q(z, w)) = \left(c \prod_{i=1}^{d} (\beta_i z - \alpha_i w), u \prod_{j=1}^{d} (\delta_j z - \gamma_j w)\right) \tag{4.1}$$

with $\min\{v(c), v(u)\} = \min\{v(\alpha_i), v(\beta_i)\} = \min\{v(\delta_j), v(\gamma_j)\} = 0$ for all i and j. Observe that $\{z_i = \alpha_i/\beta_i\}$ is the set of zeros of f and $\{p_j = \gamma_j/\delta_j\}$ is the set of poles of f.

Each point a in the disk $D(0, 1)$ in $\mathbb{P}^1(\mathbb{C}_v)$ will be represented in homogeneous coordinates as $A = (a, 1)$. Then

$$\sigma(F, a) := \min\{v(P(A)), v(Q(A))\}$$

can be computed by observing that

$$v(\beta_i x - \alpha_i) = \begin{cases} 0 & \text{if } |z_i|_v > 1, \\ v(a - z_i) & \text{if } |z_i|_v \leq 1, \end{cases}$$

and

$$v(\delta_i x - \gamma_i) = \begin{cases} 0 & \text{if } |p_i|_v > 1, \\ v(a - p_i) & \text{if } |p_i|_v \leq 1. \end{cases}$$

Therefore, still assuming that $|a|_v \leq 1$,

$$\sigma(F, a) = \min\left\{v(c) + \sum_{\{i:|z_i|_v \leq 1\}} v(a - z_i)\,,\ v(u) + \sum_{\{i:|p_i|_v \leq 1\}} v(a - p_i)\right\}. \tag{4.2}$$

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Similarly, a point a with $|a|_v > 1$, allowing also $a = \infty$, may be represented as $A = (1, 1/a)$. Then $\sigma(F, a)$ can be computed by observing that

$$v(\beta_i - \alpha_i y) = \begin{cases} 0 & \text{if } |z_i|_v \le 1, \\ v(1/a - 1/z_i) & \text{if } |z_i|_v > 1, \end{cases}$$

and

$$v(\delta_i - \gamma_i y) = \begin{cases} 0 & \text{if } |p_i|_v \le 1, \\ v(1/a - 1/p_i) & \text{if } |p_i|_v > 1. \end{cases}$$

Consequently, for all $|a|_v > 1$, we compute that

$$\sigma(F, a) = \min\left\{ v(c) + \sum_{\{i:|z_i|_v>1\}} v(1/a - 1/z_i)\,,\ v(u) + \sum_{\{i:|p_i|>1\}} v(1/a - 1/p_i) \right\}. \tag{4.3}$$

Proof of Proposition 4.1. We will use condition (iii) of the proposition to define σ_F for the F given by formula (4.1); it is a normalized homogeneous presentation for f. Because $\min\{v(c), v(u)\} = 0$, the equations (4.2) and (4.3) show that $\sigma(F, a) = 0$ unless a lies in one of the finitely many directions from the Gauss point ζ_0 which contains a pole or a zero. Thus $\sigma_F(\zeta_0) = 0$. Recalling that $\sigma(sF, a) = \sigma(F, a) + v(s)$ for any $s \in k^*$ from (3.2), we have the proof of (i).

Equations (4.2) and (4.3) also show that the dependence of $\sigma(F, a)$ upon a only incorporates its distance to the zeros and poles of f. In particular, points lying in all but finitely many directions from any Type II point ζ will have the same order. This shows that σ_F is well defined, satisfying (iii) of the proposition, and it is continuous.

Now fix any two points $x \neq y$ in the disk $D(0, 1) = \{z : |z|_v \le 1\}$. The function

$$a \mapsto v(a - y)$$

is (non-strictly) increasing, for a in $D(0, 1) \setminus \{z : |z - y|_v < |x - y|_v\}$, as $v(a - x)$ increases from 0 to ∞. Its maximum value over this domain is $v(x - y)$. Therefore, for any Type II point of the form $\zeta = D(x, r)$ with $r \le 1$, equation (4.2) shows that the order σ_F can only increase (or remain constant) along the path from ζ_0 towards ζ. Similarly for Type II points lying in the direction of ∞ from ζ_0, using (4.3). This proves (ii). □

4.4. Proof of Theorem 4.2. We begin with a lemma about the preimages of the Gauss point. Let $F = (P, Q)$ be a normalized presentation of f. Recall that a point $a \in \mathbb{P}^1(\mathbb{C}_v)$ will be expressed in homogeneous coordinates as $A = (a, 1)$ for $|a|_v \le 1$ and $A = (1, 1/a)$ for $|a|_v > 1$.

LEMMA 4.3. *Let ζ_0 be the Gauss point, and suppose $f(\zeta) = \zeta_0$. Then for a in all but finitely many directions from ζ, we have $v(P(A)) = v(Q(A)) = \sigma_F(\zeta)$.*

Proof. Recall that the space of directions from any Type II point $\zeta \in \mathbf{P}^1_v$ is identified with $\mathbb{P}^1(K)$. The action of f, from the space of directions from ζ to the space of directions from ζ_0, is by a rational function of degree greater than or equal to 1. Furthermore, the image of all but finitely many directions from ζ is equal to a direction from ζ_0. In particular, all but finitely many directions from ζ are sent to directions from ζ_0 that do *not* contain 0 or ∞.

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This implies that $|f(a)|_v = 1$ for any point a in those directions, so for these a, we must have $v(P(A)) = v(Q(A))$. □

Suppose that $v(c) = v(u) = 0$ in the expression (4.1) for F. Then, from formulas (4.2) and (4.3), we find that $\sigma(F, a) > 0$ if and only if a lies in a direction from ζ_0 containing both a zero and a pole. We now show that this condition is equivalent to there being a preimage of ζ_0 in that direction. Let C denote the path joining a zero and pole lying in the same direction from ζ_0. By continuity, the image $f(C)$ must contain the path joining 0 and ∞, so in particular it contains ζ_0. In other words, there is a preimage of ζ_0 inside U. Conversely, if a direction U from ζ_0 contains a preimage ζ of ζ_0, then we examine the action of f on directions from ζ. Each direction from ζ maps surjectively over its image direction (and possibly also over all of $\mathbf{P}^1_v$), and the action on the space of directions is surjective. Therefore, restricting to the directions from ζ that are contained in U (which is all but one direction), we conclude that U must contain at least one zero or pole of f. So let us assume that the direction U contains a zero z_i. This implies that for all $a \in U$, we have $v(a - z_i) > 0$ (or that $v(1/a - 1/z_i) > 0$ in the case where U is the direction of ∞), so that $v(P(A)) > 0$. But by Lemma 4.3, we know that $\sigma_F(\zeta) = v(Q(A)) = v(P(A)) = \sigma(F, a)$ for a in all but finitely many directions from ζ, and therefore $\sigma_F(\zeta) > 0$. Consequently, there must also be a pole lying in the direction U. If U contains a pole of f rather than a zero, the argument is similar, so we see that U contains both a pole and a zero.

Let T_f be the finite tree spanned by the Gauss point ζ_0 and its finitely many preimages, with its vertices consisting of the elements of $\{\zeta_0\} \cup f^{-1}(\{\zeta_0\})$ together with any additional branching points in T_f. We call a preimage ζ of ζ_0 an *end* of T_f if $\zeta \neq \zeta_0$ and if the valence of ζ in T_f is equal to 1. The preceding paragraph shows that, under the assumption that $v(c) = v(u) = 0$, the tree T_f is a subset of the span of the zeros and poles and ζ_0 inside $\mathbf{P}^1_v$; also that $\sigma_F(\zeta) > 0$ for each preimage ζ of ζ_0. It remains to show that σ_F achieves its maximum at ends of T_f and not before, when moving along a path from ζ_0 towards an end.

Note that there cannot be a zero and a pole in the same connected component of $\mathbf{P}^1 \setminus T_f$; if there were, then the continuity argument above shows that the path between them would contain another preimage of ζ_0. Let ζ be an end of T_f. Recall that, for a in all but finitely many directions from ζ, we have that $v(P(A)) = v(Q(A))$ by Lemma 4.3. If we take a in any direction disjoint from T_f, then at most one of $v(P(A))$ and $v(Q(A))$ can be larger than the value of $\sigma_F(\zeta)$. Therefore, the order function σ_F must remain constant in that direction. This shows that σ_F achieves its maximum on T_f and that the spine S_f is a subset of T_f.

On the other hand, suppose that ζ' is a Type II point lying on the path between ζ_0 and an endpoint ζ of T_f. We will see that $\sigma_F(\zeta') < \sigma_F(\zeta)$. For ζ' sufficiently close to ζ, the point $f(\zeta')$ lies in the direction from ζ_0 determined by the reduction map from ζ to ζ_0. We consider three cases. If this image direction does not point to either 0 or ∞, then necessarily the number of zeros and poles beyond ζ (outside T_f) must coincide—because the action on the space of directions by f has a well-defined degree. Thus, moving from ζ to ζ' will reduce the valuations $v(P(A))$ and $v(Q(A))$ simultaneously; this shows that the order function decreases. If $f(\zeta')$ lies in the direction of 0 from ζ_0, this means exactly

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that $v(P(A)) > v(Q(A))$ for points a in the annulus between ζ and ζ'. On the other hand, there must be at least one pole in the directions from ζ disjoint from T_f, and therefore $v(Q(A))$ will have decreased from the value $\sigma_F(\zeta)$ for all a in the annulus between ζ and ζ'. This shows that $\sigma_F(\zeta') < \sigma_F(\zeta)$. Similarly if $f(\zeta')$ lies in the direction of ∞ from ζ_0.

Now suppose $v(c) > v(u) = 0$ in the expression (4.1) for F. This means that $f(\zeta_0)$ lies in the direction of 0 from ζ_0. Then $\sigma(F, a) > 0$ if and only if a lies in a direction containing a pole, from the formulas (4.2) and (4.3) for $\sigma(F, \cdot)$. But any direction containing a pole p must also contain a preimage of ζ_0, because the image of the path $[\zeta_0, p]$ must contain ζ_0 by continuity. Conversely, if a direction U from ζ_0 contains a preimage ζ of ζ_0, it must also contain a pole, by the surjectivity of f on directions. Indeed, each direction from ζ maps surjectively over its image direction from ζ_0. The only concern is if the direction from ζ back to ζ_0 is the unique direction sent towards ∞. But since ζ_0 itself is moving towards 0, it would follow that the direction is a bad direction (otherwise its image would be exactly equal to the direction from ζ_0 to infinity) – so that direction maps surjectively over all of $\mathbf{P}^1_v$ and by surjectivity of directions at all Type II points between ζ and ζ_0, it follows that there must be a pole between ζ and ζ_0.

Similarly, if $0 = v(c) < v(u)$, then $f(\zeta_0)$ lies in the direction of ∞ from ζ_0. It follows that $\sigma(F, a) > 0$ if and only if a lies in a direction containing a zero, and again this is equivalent to the statement that the direction contains a preimage of ζ_0. □

4.5. Spine examples. We conclude this section by computing the spine in a few examples. We do so directly, with formulas (4.2) and (4.3), using the definition of the spine. We also do the computation indirectly, appealing to Theorem 4.2. The first example has potential good reduction. The second is a quadratic polynomial that does not have potential good reduction.

Example 4.4. Let $k = K(t)$ and $v = \mathrm{ord}_{t=0}$. Consider

$$g(z) = \frac{z^2 - 1}{z}.$$

This function g has good reduction, so its spine S_g is the singleton $\{\zeta_0\}$ and the order function is $\equiv 0$. Conjugating by $A(z) = tz$, we define

$$f(z) = A \circ g \circ A^{-1}(z) = \frac{z^2 - t^2}{z}.$$

The zeros of f lie at $z_1 = t$ and $z_2 = -t$. The poles of f lie at 0 and ∞. In the form (4.1), a homogeneous presentation of f is given by

$$F(z, w) = (z^2 - t^2 w^2, zw) = ((z - tw)(z + tw), zw)$$

with $c = u = 1$. It is immediate from (4.3) that $\sigma(F, a) = 0$ for all $|a|_v > 1$. Using (4.2), we compute that

$$\sigma(F, a) = \min\{v(a - t) + v(a + t),\ v(a)\}$$

for each $a \in \mathbb{P}^1(\mathbb{C}_v)$ with $|a|_v \leq 1$. Consequently,

$$\sigma(F, a) = \begin{cases} 0 & \text{for } v(a) \leq 0, \\ v(a) & \text{for } 0 \leq v(a) \leq 2, \\ 2 & \text{for } 2 \leq v(a), \end{cases}$$

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and the spine S_f is the interval $[\zeta_0, \zeta_2]$ in $\mathbf{P}^1_v$, from the Gauss point ζ_0 and the point ζ_2 representing $D(0, |t^2|_v)$.

To compute the spine of f with Theorem 4.2, we need only determine the preimages of the Gauss point ζ_0 under f. Observing first that

$$f(z) = z + O(t^2)$$

we see that f must fix the Gauss point; the reduction map has degree 1. We also compute that

$$f(t^2 z) = -1/z + O(t^2),$$

which shows that the point $\zeta_2 = D(0, |t^2|_v)$ is another preimage of ζ_0. Because f has degree 2, there are the only preimages of ζ_0. Thus, the spine S_f is the convex hull of ζ_0 and ζ_2.

Example 4.5. Let $k = K(t)$ and $v = \mathrm{ord}_{t=0}$. Consider

$$f(z) = t^{-3} z(z - t).$$

In the form (4.1), a homogeneous presentation of f is given by

$$F(z, w) = (z(z - tw), t^3 w^2)$$

so that $c = 1$ and $u = t^3$ has $v(u) = 3$. For $|a|_v > 1$, we compute from (4.3) that $\sigma(F, a) = 0$, as the zeros of f lie at 0 and t. For $|a|_v \leq 1$, we have

$$\sigma(F, a) = \min\{v(a) + v(a - t),\ 3\}.$$

We deduce that the spine S_f has the shape of a Y. The three tips of the Y lie at the Gauss point ζ_0, at the point $\zeta_{0,2}$ representing the disk $D(0, |t^2|_v)$, and at the point $\zeta_{t,2}$ representing the disk $D(t, |t^2|_v)$, with the center of the Y at $\zeta_{0,1}$ representing the disk $D(0, |t|_v)$. The function σ_F of Proposition 4.1 satisfies $\sigma_F(\zeta_0) = 0$, $\sigma_F(\zeta_{0,1}) = 2$, $\sigma_F(\zeta_{0,2}) = 3$, and $\sigma_F(\zeta_{t,2}) = 3$. The function σ_F is linear along the three branches joining the end points to the center $\zeta_{0,1}$ of the Y.

To compute the spine S_f using Theorem 4.2, we observe that

$$f(t^2 z) = -z + O(t)$$

and

$$f(t + t^2 z) = z + O(t).$$

These computations show that the points $\zeta_{0,2} = D(0, |t^2|_v)$ and $\zeta_{t,2} = D(t, |t|^2)$ are both sent by f to ζ_0. As f has degree 2, these are the only preimages of ζ_0. Therefore, the spine is the convex hull of the set $\{\zeta_0, \zeta_{0,2}, \zeta_{t,2}\}$.

5. *Elliptic curves, dynamically*

In this section we present our dynamical proof of Theorem 1.1.

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5.1. The Lattès tent. Let E be an elliptic curve over function field $k = K(X)$. Via the quotient identifying a point P with its inverse $-P$, the multiplication-by-2 endomorphism on E induces a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ defined over k of degree 4. The Néron–Tate canonical height on E is equal to $1/2$ times the dynamical canonical height of f (see (1.1) and §2.4).

For a given place v, either E has potential good reduction, or the j-invariant satisfies $|j|_v > 1$ and we may present E locally as $\mathbb{C}_v^*/q^{\mathbb{Z}}$ for some $|q|_v < 1$ [**Si1**, Ch. V, Theorem 5.3(a), Remark 3.1.2]. In the latter case, from [**FRL**, Proposition 5.1] and its proof, we know that the action of f on the Berkovich projective line $\mathbf{P}_v^1$ has a very simple form.

Indeed, after a change of coordinates (putting our E into Legendre form), we may assume that our f is

$$f(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)}$$

for some $|t|_v < 1$. Observe that ∞ is a fixed point for f, and the points $a = 0, 1, t$ are sent to ∞ under one iteration of f. In homogeneous coordinates, we have

$$F(z, w) = ((z^2 - tw^2)^2, 4zw(z-w)(z-tw)).$$

For simplicity, we will assume that $v(t) = 1$. The general case has an identical proof. Also, it is convenient to set $v(0) = \infty$ and $v(\infty) = -\infty$.

PROPOSITION 5.1. *For any $r \in \mathbb{Q}$ and point $a \in \mathbb{C}_v$ with $v(a) = r$, we have*

$$\sigma(F, a) = \begin{cases} 0 & \text{if } r \le 0, \\ 2r & \text{if } 0 \le r \le 1, \\ 2 & \text{if } r \ge 1, \end{cases}$$

and

$$v(f(a)) \begin{cases} < 0 & \text{if } r < 0, \\ < 0 & \text{if } r = 0 \text{ and } r_v(a) = 1, \\ = 0 & \text{if } r = 0 \text{ and } r_v(a) \ne 1, \\ = 2r & \text{if } 0 < r < 1/2, \\ = 1 & \text{if } r = 1/2 \text{ and } r_v(a/t^{1/2}) \ne \pm 1, \\ > 1 & \text{if } r = 1/2 \text{ and } r_v(a/t^{1/2}) = \pm 1, \\ = 2 - 2r & \text{if } 1/2 < r < 1, \\ = 0 & \text{if } r = 1 \text{ and } r_v(a/t) \ne 1, \\ < 0 & \text{if } r = 1 \text{ and } r_v(a/t) = 1, \\ < 0 & \text{if } r > 1, \end{cases}$$

where $r_v(\alpha)$ represents (as always) the residue class of $\alpha \in \mathcal{O}_v$.

The first piece of Proposition 5.1 implies that the spine of f is the closed interval

$$S_f = [\zeta_0, \zeta_1] \subset \mathbf{P}_v^1,$$

where ζ_r is the Type II point representing the disk $D(0, |t^r|)$. The second piece of the proposition describes the action of f on the spine. It is exactly the tent map T_2 of slope ± 2 on the unit interval; see Figure 5.1. This is also included in the statement of [**FRL**, Proposition 5.1]. For this example, the Julia set of f coincides with the spine, because it is

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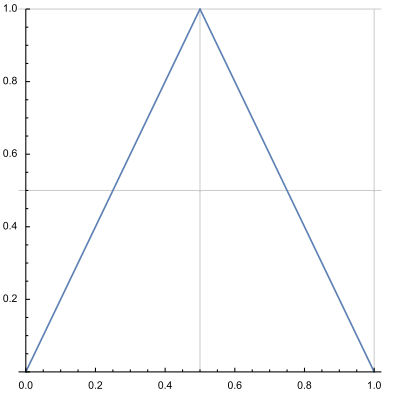


FIGURE 5.1. The graph of the tent map T_2 of slope ± 2 on the interval $[0, 1]$.

totally invariant for f. The two pieces of the proposition together allow us to iterate points and compute the sequence $c_n = \sigma(F, f^n(a))$ for any starting point a. The key observation is that all rational points in the interval $[0, 1]$ are preperiodic for the tent map.

Proof of Proposition 5.1. Recall that we are assuming $v(t) = 1$. Fix $a \neq 0, 1, t$ with $|a|_v \leq 1$, and write $A = (a, 1)$. Let $r = v(a)$. Then

$$F(A) = (a^4 - 2ta^2 + t^2, -4a^2 + 4a^3 + 4ta - 4ta^2).$$

For $0 < r < 1$, the minimum valuation occurs only for the term $-4a^2$, so $\sigma(F, a) = 2r$. For $r = 0$, the first coordinate has valuation 0 while the second coordinate has valuation greater than or equal to 0, so again $\sigma(F, a) = 2r = 0$. For $r = 1$, the first coordinate has valuation 2, while the second coordinate has valuation greater than or equal to 2, and again $\sigma(F, a) = 2r = 2$. For $r > 1$, we see that the minimal valuation occurs only for t^2, so we have $\sigma(F, a) = 2$. For $r < 0$, we should consider a lift $A = (1, b)$ with $v(b) = -r > 0$. Then

$$F(A) = (1 - 2b^2t + b^4t^2, 4b - 4b^2 - 4tb^2 + 4tb^3),$$

and we see immediately that the minimal valuation occurs only for the term 1, so $\sigma(F, a) = 0$.

Assuming $0 \leq r \leq 1$, upon dividing $F(A)$ by t^{2r}, we see that the first coordinate of $F(A)/t^{2r}$ has order $= \min\{2 - 2r, 2r, 1\}$, unless $r = v(a) = 0, 1/2, 1$ when there can be cancelation. This shows that $v(f(a))$ has the desired value for $0 < r < 1/2$ and $1/2 < r < 1$.

To understand the case $r = v(a) = 1/2$, we can compute directly that

$$f(zt^{1/2})/t = \frac{-(z+1)^2(z-1)^2}{4z^2} + O(t^{1/2}).$$

The leading term in this expansion describes the action of f on the space of directions from the point $\zeta_{1/2}$ to the point ζ_1. It is of degree 4, sending both 0 and ∞ to ∞, and sending ± 1 to 0. This proves the desired statement for $r = 1/2$.

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For $r = 0$, we compute that

$$f(z) = \frac{z^2}{4(z-1)} + O(t),$$

so we know that the Gauss point ζ_0 is fixed. The action on directions to ζ_0 fixes 0 and ∞, and sends 1 to ∞, but all other directions will have $v(f(a)) = v(a) = 1$. This proves the desired statement for $r = 0$. We have also proved the statement for $r < 0$, because the direction of ∞ from ζ_0 is fixed (and the Berkovich disk is forward invariant).

For $r = 1$, we compute that

$$f(zt) = \frac{-1}{4z(z-1)} + O(t),$$

showing that f sends ζ_1 to the Gauss point ζ_0 and describing the action on directions. Indeed, the two directions 0 and 1 are sent to ∞, while ∞ is the unique zero. We conclude that $v(a) = 1 \implies v(f(a)) = 0$ for all $z = a/t$ satisfying $r_v(z) \neq 1$, and $v(f(a)) < 0$ for all $z = a/t$ satisfying $r_v(z) = 1$. The computation also shows that $v(a) > 1$ implies that $v(f(a)) < 0$, completing the proof of the proposition. □

5.2. Proof of Theorem 1.1. Fix $a \in \mathbb{P}^1(\mathbb{C}_v)$ and set $c_n = \sigma(F, f^n(a))$. Recall from Proposition 3.2 that it suffices to show that the sequence c_n of rational numbers is eventually periodic. Look at the sequence of rational numbers $r_n = v(f^n(a))$. If $r_n \in [0, 1]$ for all n, then from Proposition 5.1, the sequence r_n is equal to a rational orbit of the tent map T_2, so it must be preperiodic. Then, since $c_n = 2r_n$, we conclude that c_n also cycles eventually. If there exists an n_0 so that $r_{n_0} < 0$, then Proposition 5.1 implies that $r_n < 0$ for all $n \geq n_0$, and $c_n = 0$ for all $n \geq n_0$. If there exists an n_0 so that $r_{n_0} > 1$, then $r_{n_0+1} < 0$ and $c_n = 0$ for all $n > n_0$. We conclude that $\eta(F, a) = \sum_n c_n/d^n$ is a rational number. Therefore, from Lemma 3.1, the value $\widehat{\lambda}_{f,v}(a)$ is also rational.

6. *Irrationality*

In this section, we prove Theorems 1.2 and 1.3.

Actually, we prove a more technical statement, about the realization of arbitrary sequences of two numbers as the order itinerary of a point in $\mathbb{P}^1(\mathbb{C}_v)$. Let $\mathbb{N}_0$ denote the non-negative integers.

PROPOSITION 6.1. *Suppose f fixes the Gauss point ζ_0 in $\mathbf{P}^1_v$. Let F be a normalized homogeneous presentation of f. Suppose $f(\zeta) = \zeta_0$ where ζ is a Type II point in the direction from ζ_0 containing 0. Assume there exist:*

- *a backward orbit $\{b_n\} \subset \mathbb{P}^1(\mathbb{C}_v)$ of 0 (satisfying $f(b_1) = 0$ and $f(b_{n+1}) = b_n$ for all $n \geq 1$) such that*
$$\sigma(F, b_n) = 0$$
for all n;
- *a point $x \in \mathbb{P}^1(\mathbb{C}_v)$ such that $\sigma(F, f^n(x)) = 0$ for all $n \geq 0$;*
- *preimages of x, 0, and b_n in directions U from ζ where $\sigma(F, \cdot)|U = \sigma_F(\zeta)$.*

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Then $c := \sigma_F(\zeta) > \sigma_F(\zeta_0) = 0$, *and for any sequence* $s = s_0 s_1 s_2 \cdots \in \{0, c\}^{\mathbb{N}_0}$, *there exists a point* $a \in \mathbb{P}^1(\mathbb{C}_v)$ *for which*

$$\sigma(F, f^n(a)) = s_n$$

for all $n \geq 0$.

The idea of the proposition is to exploit a standard dynamical construction. Roughly, we will find disjoint regions D_0 and D_1 in $\mathbb{P}^1(\mathbb{C}_v)$ so that f maps each region over the union $D_0 \cup D_1$. Then, appealing to the completeness of the field $\mathbb{C}_v$, we conclude that for any itinerary $s = (s_n) \in \{0, 1\}^{\mathbb{N}}$, there will be a point $x \in D_{s_0}$ so that $f^n(x) \in D_{s_n}$ for all $n \geq 1$. The hypotheses of Proposition 6.1 allow us to apply Theorem 4.2 to find suitable regions D_0 and D_1 so that $\sigma(F, \cdot)$ is constant on each D_i, with $\sigma(F, D_0) \neq \sigma(F, D_1)$.

6.1. Proof of Theorem 1.2. Assume Proposition 6.1. Fix $d \geq 2$ and $t \in k$ with $v(t) = 1$. Choose a rational function $g \in K(z)$ of degree $d - 1$ for which 0 has infinite orbit and $g(\infty) = \infty$. For example, we could take the polynomial $g(z) = z^{d-1} + 1$. Consider

$$f(z) = g(z)\,\frac{z - t}{z + t} \in k(z).$$

In homogeneous coordinates, let us write

$$F(z, w) = ((z - tw)P_g(z, w), (z + tw)Q_g(z, w))$$

where $g(z) = P_g(z, 1)/Q_g(z, 1)$, with F normalized as in (4.1).

LEMMA 6.2. *The spine of* f *is the interval* $[\zeta_0, \zeta_1]$ *where* ζ_0 *is the Gauss point and* $\zeta_1 = D(0, |t|)$, *and* $\sigma(F, a) = r$ *for all* $v(a) = r$, *with* $0 \leq r \leq 1$.

Proof. The spine must lie in the direction of 0 from ζ_0, because 0 is the only common root of the reduction of F. Observe that

$$F(z, w) = (zP_g(z, w) + O(t), zQ_g(z, w) + O(t)).$$

Thus, for all a with $0 < v(a) < 1$, we can take $A = (a, 1)$, and we find that $\sigma(F, a) = v(a)$. On the other hand, since there are no other zeros and poles of f besides t and $-t$ in the direction of 0 from ζ_0, we use (4.2) to show that $\sigma(F, a) = 1$ for all a with $v(a) \geq 1$. □

The Gauss point ζ_0 is fixed by f, and the induced map on directions is given by g. As the degree of g is $d - 1$, there is a unique other preimage of ζ_0, and this point is $\zeta_1 = D(0, |t|_v)$, mapping with degree 1; in fact

$$f(tz) = g(0)\frac{z - 1}{z + 1} + O(t),$$

describing the induced map on directions from ζ_1 to ζ_0. Since the point t maps to 0 by f, the direction from ζ_1 containing t will map over ζ_1. More generally, the expression for $f(tz)$ implies that the direction from ζ_1 containing zt maps surjectively, and with degree 1, to the direction from ζ_0 containing $g(0)(z - 1/z + 1)$, for all $z \in K$. Moreover, for each b with $|b|_v = 1$, the direction U_b from ζ_0 containing b is sent to the direction $U_{g(b)}$ from ζ_0, surjectively, with multiplicity equal to that of g at b.

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We now apply Proposition 6.1. We let $\{b_n\}$ be any backward orbit of 0 for f; it will satisfy the hypothesis of Proposition 6.1 because the spine is contained in the direction of 0. We let $x = \infty$. Therefore, for any sequence $s \in \{0, 1\}^{\mathbb{N}_0}$, there exists a point $a \in \mathbb{P}^1(\mathbb{C}_v)$ for which

$$\sigma(F, f^n(a)) = s_n.$$

Now choose any aperiodic sequence. It follows from Proposition 3.2 and Lemma 3.1 that the corresponding points will have irrational local canonical height. This completes the proof of Theorem 1.2.

6.2. Proof of Theorem 1.3. Let $k = \mathbb{C}(t)$ and $v = \mathrm{ord}_{t=0}$ so that $v(t) = 1$. Let

$$f(z) = \frac{(z+1)(z-t)}{z+t}.$$

Note that this f is an example of the type given in the beginning of the proof of Theorem 1.2 above, with $d = 2$. Therefore, from the proof of Theorem 1.2, for any sequence $s \in \{0, 1\}^{\mathbb{N}_0}$, there exists a point $a \in \mathbb{P}^1(\mathbb{C}_v)$ for which

$$\sigma(F, f^n(a)) = s_n.$$

In the proof above, we chose the point x (which satisfies $\sigma(F, f^n(x)) = 0$ for all $n \geq 0$) to be ∞; let us take instead $x = 1$. In this way, we continue to have $\sigma(F, f^n(x)) = 0$ for all $n \geq 0$, but in addition we have that $v(x) = 0$.

Now fix any real number $\alpha \in [-1, 0]$. Then $-\alpha$ has a binary expansion of the form

$$-\alpha = \frac{1}{2} \sum_{n=0}^{\infty} \frac{c_n}{2^n}$$

with $c_n \in \{0, 1\}$ for all $n \geq 0$. Taking this sequence $\{c_n\}$ for the itinerary, obtain a point $a \in \mathbb{P}^1(\mathbb{C}_v)$ satisfying $v(a) \geq 0$ for which

$$\eta(F, a) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\sigma(F, f^n(a))}{2^n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{c_n}{2^n} = -\alpha$$

and therefore

$$\widehat{\lambda}_{f,v}(a) = \alpha$$

by Lemma 3.1. This completes the proof of Theorem 1.3.

6.3. Proof of Proposition 6.1. Without loss of generality, we may assume that $\zeta = D(0, r)$ for some $0 < r < 1$. From Proposition 4.1 and Theorem 4.2, we know that $\sigma_F(\zeta) > 0$. Furthermore, for b in all but finitely many directions from ζ_0, we have $\sigma(F, b) = 0$.

Notice that the direction of ∞ from ζ is necessarily a bad direction for ζ, since ζ_0 is fixed. There are at most finitely many other bad directions. Let $z_1, \ldots, z_N$ be points in each of these bad directions from ζ (if they exist).

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We now fix notation for certain subsets of $\mathbb{P}^1(\mathbb{C}_v)$ to be used for the remainder of the proof. For each $b \in \mathbb{C}_v$ with $|b|_v = 1$, we let

$$X_b = \{z \in \mathbb{C}_v : |b - z|_v < 1\},$$

containing all points in the direction of b from ζ_0. Let

$$X_0 = \{z \in \mathbb{C}_v : |z|_v < 1\}$$

and

$$X_\infty = \{z \in \mathbb{C}_v : |z|_v > 1\}.$$

We set

$$X_\zeta = D(0, r) = \{|z|_v \le r\}$$

and

$$X_\zeta^{\text{good}} = X_\zeta \setminus \bigcup_{j=1}^{N} \{z : |z_j - z|_v < r\}.$$

By hypothesis, we have $\sigma(F, X_{b_n}) = 0$ for all $n \ge 1$, and $\sigma(F, X_{f^n(x)}) = 0$ for all $n \ge 0$. It follows from Theorem 4.2 that none of the b_n or $f^n(x)$ lie in bad directions from ζ_0, and therefore $f(X_{b_{n+1}}) = X_{b_n}$ and $f(X_{f^{n-1}(x)}) = X_{f^n(x)}$ for all $n \ge 1$ and $f(X_{b_1}) = X_0$. We also know that $\sigma(F, X_\zeta^{\text{good}}) = \sigma_F(\zeta)$. Furthermore, there exist preimages b_n' of b_n and x' of x in X_ζ^{good}; we set

$$Y_n = \{|z - b_n'|_v < r\}$$

and

$$Y_x = \{|z - x'| < r\}$$

so that $\sigma(F, Y_n) = \sigma(F, Y_x) = \sigma_F(\zeta)$ for all n.

We now begin our construction.

Set $c = \sigma_F(\zeta)$. Fix sequence $s \in \{0, c\}^{\mathbb{N}_0}$. We will construct a nested sequence of regions $D_n \subset \mathbb{P}^1(\mathbb{C}_v)$, for $n \ge 0$, for which

$$\sigma(F, f^j(a)) = s_j$$

for all $j \le n$ and all $a \in D_n$. The conclusion is obtained by choosing any point

$$a \in \bigcap_{n \ge 0} D_n$$

which we will show is non-empty, appealing to the completeness of $\mathbb{C}_v$.

We proceed by induction. For the base case, if $s_0 = c$, we set $D_0 = X_\zeta^{\text{good}}$. Then

$$\sigma(F, a) = c = s_0$$

for all $a \in D_0$. If $s_0 = 0$, we let

$$m(0) = \sup\{n \ge 0 : s_0 = s_1 = \cdots = s_n = 0\}.$$

For $m(0) = \infty$, we take $D_j = X_x$ for all $j \ge 0$. Then

$$\sigma(F, f^j(a)) = 0 = s_j \quad \text{for all } j \ge 0, \text{ for all } a \in X_x,$$

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and we are done. On the other hand, if $m(0)$ is finite, we set $D_0 = D_1 = \cdots = D_{m(0)} = X_{b_{m(0)+1}}$. Note that $f^{m(0)+1}(X_{b_{m(0)+1}}) = X_0 \supset X_\zeta$. Thus, there exists a (closed) disk $U_{m(0)+1} \subset D_{m(0)}$ of radius less than or equal to r sent by $f^{m(0)+1}$ to X_ζ, and a region $D_{m(0)+1} \subset U_{m(0)+1} \subset D_{m(0)}$ sent onto X_ζ^{good} by $f^{m(0)+1}$. By construction, we have

$$\sigma(F, a) = \cdots = \sigma(F, f^{m(0)}(a)) = 0 \quad \text{and} \quad \sigma(F, f^{m(0)+1}(a)) = c$$

for all $a \in D_{m(0)+1}$, so that

$$\sigma(F, f^j(a)) = s_j \quad \text{for all } j \le m(0)+1, \text{ for all } a \in D_{m(0)+1},$$

and $f^{m(0)+1}(D_{m(0)+1}) = X_\zeta^{\text{good}}$.

We now explain the induction step. Assume that we have a region D_k so that

$$\sigma(F, f^j(a)) = s_j$$

for all $j \le k$ and all $a \in D_k$, with $s_k = c$ and $f^k(D_k) = X_\zeta^{\text{good}}$. Assume further that the region D_k is either an open disk of radius $r_k > 0$ or the complement of finitely many open disks in a closed disk of radius r_k. Suppose $s_{k+1} = c$. Let D be a disk inside X_ζ^{good} sent by f to X_ζ. Then we let U_{k+1} be the disk inside D_k sent by f^k to D. Finally, we let D_{k+1} be the subset of U_{k+1} sent by f^{k+1} to X_ζ^{good}. Note that D_{k+1} is itself the complement of finitely many open disks in U_{k+1}, which is a disk of radius less than or equal to rr_k. Moreover, $\sigma(F, f^{k+1}(a)) = c = s_{k+1}$ for all $a \in D_{k+1}$, with $f^{k+1}(D_{k+1}) = X_\zeta^{\text{good}}$, completing the induction step in this case.

On the other hand, for $s_{k+1} = 0$, we let $m(k+1) \in \mathbb{N} \cup \{\infty\}$ be defined as

$$m(k+1) := \sup\{n \ge k+1 : s_{k+1} = s_{k+2} = \cdots = s_n = 0\}.$$

For $m(k+1) = \infty$, we set $D = Y_x$, sent to X_x by f, and let D_{k+1} be an open disk inside D_k which is sent by f^k to D. We put $D_j = D_{k+1}$ for all $j \ge k+1$. Then

$$\sigma(F, f^n(a)) = s_n \quad \text{for all } n,$$

and we are done. But if $m(k+1)$ is finite, we set $D = Y_{m(k+1)-k}$, which is sent by f to $X_{b_{m(k+1)-k}}$. Then we let $D_{k+1} = D_{k+2} = \cdots = D_{m(k+1)}$ be the preimage of D by f^k inside D_k, so that D_{k+1} is a disk of radius r_k with $f^{k+1}(D_{k+1}) = X_{b_{m(k+1)-k}}$ and $f^{m(k+1)+1}(D_{k+1}) = X_0 \supset X_\zeta$. Then, finally, we let $U_{m(k+1)+1} \subset D_{m(k+1)}$ be the preimage of X_ζ, so $U_{m(k+1)+1}$ is a disk of radius less than or equal to rr_k, and we let $D_{m(k+1)+1}$ be the region inside D_{k+1} sent by $f^{m(k+1)+1}$ to X_ζ^{good}. Then $D_{m(k+1)+1}$ is the complement of finitely many open disks in $U_{m(k+1)+1}$, with

$$\sigma(F, f^j(a)) = 0 = s_j \quad \text{for all } k+1 \le j \le m(k+1),$$

and

$$\sigma(F, f^{m(k+1)+1}(a)) = c = s_{m(k+1)+1}$$

for all $a \in D_{m(k+1)+1}$, and $f^{m(k+1)+1}(D_{m(k+1)+1}) = X_\zeta^{\text{good}}$. This completes the induction argument.

It remains to observe that the nested intersection $\bigcap_n D_n$ is non-empty. If the series s ends with an infinite string of 0s, then the nested intersection coincides with one of the

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entries D_n, which is a non-empty open disk by construction. In general, when there are infinitely many n for which $s_n = c$, we appeal to the completeness of the field $\mathbb{C}_v$. By the induction argument, for each $n \geq 0$, there exists a disk U_n such that $D_n \supset U_{n+1} \supset D_{n+1}$ and the radii of the disks U_n shrink by at least a factor of r for each n with $s_n = c$. Then

$$\bigcap_n D_n = \bigcap_n U_n$$

is non-empty by the completeness of $\mathbb{C}_v$.

This completes the proof of Proposition 6.1.

7. *Quadratic maps*

In this section we prove Theorem 1.5. We begin by stating a result of Kiwi in [**Ki1**]. Applying Kiwi's theorem, our proof will break down into four cases.

7.1. Julia sets in degree **2**. Suppose f is a quadratic rational function defined over $k = K(X)$ (where X is a projective, smooth curve defined over K). Fix $v \in M_k$. Kiwi classified the action of f on its Julia set in the Berkovich line $\mathbf{P}^1_v$. We do not use the full strength of his results. Here is a reformulation of his theorems that will suit our particular needs, combining the main results of [**Ki1**] with some preliminary results appearing in [**Mi**, **Ep**, **De1**].

THEOREM 7.1. *Suppose f has degree 2 and does not have potential good reduction at v. Then exactly one of the following holds.*

(1) *f has an attracting fixed point. In this case, the Julia set $J(f)$ is a Cantor set in $\mathbb{P}^1(\mathbb{C}_v)$; all points of $\mathbf{P}^1_v \setminus J(f)$ tend to the attracting fixed point under iteration.*

(2) *f has a fixed Type II point on which, in suitable coordinates, f acts on the space of directions by*

$$z \mapsto \lambda z$$

for $\lambda \in K^*$ not a root of unity, or by

$$z \mapsto z + 1,$$

and the unique bad direction points towards $z = 1$.

(3) *f has a fixed Type II point ζ on which, in suitable coordinates, f acts on directions by*

$$z \mapsto \lambda z$$

for $\lambda \in K^*$ a primitive pth root of unity, for some $p > 1$; and in the direction of $z = 1$ from ζ, f has a periodic Type II point of period p on which, in suitable coordinates, f^p acts on directions by

$$w \mapsto w + \tau + 1/w$$

for some $\tau \in K$, with exactly two bad directions pointing towards $w = 0, \infty$ and with ζ lying in the direction of $w = \infty$.

(4) *f has a fixed Type II point ζ on which, in suitable coordinates, f acts on directions by*

$$z \mapsto \lambda z$$

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for $\lambda \in K^*$ a primitive pth root of unity, for some $p > 1$; and in the direction of $z = 1$ from ζ, f has a periodic Type II point of period p on which, in suitable coordinates, f^p acts by

$$w \mapsto w + 1$$

with exactly two bad directions pointing towards $w = 0, \infty$ and with ζ lying in the direction of $w = \infty$.

The history of this statement begins with Milnor's description of the moduli space M_2 of quadratic maps over $\mathbb{C}$ in [**Mi**, §3]. He observed that $M_2 \simeq \mathbb{C}^2$ via the identification of the conjugacy class of f with (two of) the symmetric functions in the three fixed-point multipliers. Milnor defined a compactification $\overline{M}_2 \simeq \mathbb{P}^2$ where the boundary points are identified with (unordered) triples $\{\lambda, 1/\lambda, \infty\}$ of limiting fixed-point multipliers, with $\lambda \in \mathbb{C}$; for $\lambda \neq 0$, these boundary points correspond to the action of f on directions at the fixed Type II points appearing in cases (2), (3), and (4). Epstein studied the reduction maps of degree 2 exhibited in case (3) to analyze the structure of hyperbolic components in M_2 [**Ep**]. Such non-trivial reduction maps at periodic Type II points are called 'rescaling limits' and arise as boundary points of M_2 in more elaborate compactifications; see [**Ki2**] and [**De1**].

Remark 7.2. The example of Theorem 1.3 satisfies condition (2) of Theorem 7.1, with the reduction map $z \mapsto z + 1$.

7.2. Fixed-point multipliers. The *multiplier* of a fixed point of f is simply the derivative of f at that point. The four cases of Theorem 7.1, and the value of τ in case (3), can be characterized in terms of the three fixed-point multipliers $\{\alpha, \beta, \gamma\}$ of a quadratic map. To see this, first recall that the conjugacy class of a quadratic map f is uniquely determined by its triple of fixed-point multipliers. In addition, the multipliers are subject to exactly one condition, that

$$\alpha\beta\gamma - (\alpha + \beta + \gamma) + 2 = 0 \tag{7.1}$$

[**Mi**, Lemma 3.1]; see also [**Si2**, Theorems 1.14 and 4.56]. For a characterization of potential good reduction in degree 2, we have the following proposition.

PROPOSITION 7.3. *Fix f of degree 2 defined over k, and fix a place v of k. Then f has potential good reduction at v if and only if the three fixed-point multipliers $\{\alpha, \beta, \gamma\}$ satisfy*

$$|\alpha|_v, |\beta|_v, |\gamma|_v \leq 1.$$

Proof. Suppose that f has potential good reduction. Then under a Möbius change of coordinates, f will have good reduction. For any map of good reduction, the multipliers at all periodic points will be less than or equal to 1 in absolute value [**Si2**, Ch. 2]. As the multipliers are invariant under coordinate change, we deduce that $|\alpha|_v, |\beta|_v, |\gamma|_v \leq 1$. Conversely, suppose the three fixed-point multipliers satisfy $|\alpha|_v, |\beta|_v, |\gamma|_v \leq 1$. Recalling that r_v denotes the reduction map modulo v, assume that $r_v(\alpha) \neq 1$, so that $r_v(\alpha\beta) \neq 1$ by (7.1). Then, exactly as in the proof of [**Mi**, Lemma 3.1], we observe that in suitable coordinates on $\mathbb{P}^1(\mathbb{C}_v)$ the map f can be expressed explicitly in terms of its

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multipliers. For example, we may write

$$f(z) = \frac{z^2 + \alpha z}{\beta z + 1},$$

which has a fixed point of multiplier α at $z = 0$ and a fixed point of multiplier β at ∞. The resultant of a normalized homogeneous presentation F of f is computed as $\mathrm{Res}(F) = 1 - \alpha\beta$ so that $|\mathrm{Res}(F)|_v = 1$. That is, in these coordinates, f will have good reduction. See also [**Si2**, p. 193]. If, on the other hand, we have $r_v(\alpha) = r_v(\beta) = r_v(\gamma) = 1$, then we may choose coordinates on $\mathbb{P}^1(\mathbb{C}_v)$ so that the two critical points of f lie at 1 and -1 and the point 0 is fixed with multiplier α. Then we have

$$f(z) = \frac{\alpha z}{z^2 + \delta z + 1}$$

for some $\delta \in \bar{k}$, which will have good reduction if and only if $|\delta|_v \leq 1$. To determine δ in terms of $\{\alpha, \beta, \gamma\}$, we first compute that the other two fixed-point multipliers of f are (up to ordering)

$$\beta = (4 - 2\alpha - \delta^2 + \delta\sqrt{-4 + 4\alpha + \delta^2})/(2\alpha)$$

and

$$\gamma = (4 - 2\alpha - \delta^2 - \delta\sqrt{-4 + 4\alpha + \delta^2})/(2\alpha).$$

Eliminating α from these equations shows that

$$\beta^2 + \gamma^2 + \beta\gamma(\delta^2 - 2) - \delta^2 = 0.$$

Solving for δ, we find

$$\delta = \pm\frac{\beta - \gamma}{\sqrt{1 - \beta\gamma}}$$

as long as $\beta\gamma \neq 1$. For $\beta\gamma = 1$, so that $\beta = \gamma = 1$ by (7.1), we compute directly from the formulas for β and γ that

$$\delta = \pm 2\sqrt{1 - \alpha}.$$

In either case, recalling that $r_v(\alpha) = r_v(\beta) = r_v(\gamma) = 1$, we have $|\delta|_v < 1$, and we conclude that f has good reduction at v. □

It follows from Proposition 7.3 that, in the absence of potential good reduction, there will be at least one multiplier with absolute value greater than 1. The first case of Theorem 7.1 is already formulated in terms of the fixed points. If there is an attracting fixed point, then the other two multipliers must have absolute value greater than 1, by (7.1). The remaining cases of Theorem 7.1 hold when two of the multipliers, say α and β, have absolute value 1, and their residue classes are given by $r_v(\alpha) = \lambda$ and $r_v(\beta) = 1/\lambda$. The value of λ will determine whether we are in case (2) or in cases (3) and (4). Cases (3) and (4) are distinguished by the residue class

$$\tau^2 := r_v\left(\frac{\alpha^p - 1}{1 - \alpha\beta}\right) = r_v\left(\frac{\beta^p - 1}{1 - \alpha\beta}\right). \tag{7.2}$$

For $\tau^2 \neq \infty$, the value of τ in case (3) is either choice of square root of τ^2; otherwise we are in case (4). See [**De1**, §5 and Propositions 6.1 and 6.4].

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7.3. Attracting fixed point. We now work towards the proof of Theorem 1.5, which will be completed in §7.4. Suppose that f has an attracting fixed point, and assume that it does not have potential good reduction. From Theorem 7.1(1), the Julia set of f is a Cantor set in $\mathbb{P}^1(\mathbb{C}_v)$. From Proposition 7.3 and equation (7.1), the other two fixed points have multipliers of absolute value greater than 1. Suppose the three fixed-point multipliers are α, β, γ with $|\gamma|_v < 1$ and $|\alpha|_v \geq |\beta|_v > 1$. Again from (7.1), we compute that $|\alpha|_v > |\beta|_v = 1/|\gamma|_v > 1$ if $|\alpha\gamma|_v > 1$; otherwise, we have $|\beta|_v = |\alpha|_v$. We may assume we have chosen coordinates so that the attracting fixed point lies at ∞.

In the proof of Theorem 1.5, we use the following result.

PROPOSITION 7.4. *Suppose f has an attracting fixed point at ∞ and two repelling fixed points in $\mathbb{P}^1(\mathbb{C}_v)$ with multipliers α and β. Then the local canonical height $\widehat{\lambda}_{f,v}$ satisfies*

$$\widehat{\lambda}_{f,v}(a) \in \mathbb{Q} \quad \textit{for all } a \in \mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$$

if and only if $|\alpha|_v = |\beta|_v$.

For convenience, we will assume that 0 is a repelling fixed point. Assume further that the other zero of f is at 1. We can express our map in homogeneous coordinates as

$$F(z, w) = (c\, z(z - w),\ u\, w(p_2 z - p_1 w))$$

where $\min\{v(c), v(u)\} = \min\{v(p_1), v(p_2)\} = 0$, with $p = [p_1 : p_2]$ being the other pole of f in $\mathbb{P}^1(\mathbb{C}_v)$. We compute that the attracting fixed point at ∞ has multiplier up_2/c, and therefore

$$|p_2 u|_v < |c|_v.$$

Since 0 is repelling with multiplier $c/(p_1 u)$, we deduce that

$$|p_1 u|_v < |c|_v.$$

Independent of p, then, we see that $|u|_v < |c|_v$, so we may take $c = 1$ and $|u|_v < 1$. Thus, we may write

$$F(z, w) = (z(z - w),\ u\, w(p_2 z - p_1 w)) \tag{7.3}$$

where $v(u) > 0$ while $\min\{v(p_1), v(p_2)\} = 0$. Computing directly, the triple of fixed-point multipliers at the three fixed points ∞, 0, and $(1 - up_1)/(1 - up_2) \approx 1$ is

$$\left\{ up_2,\ \frac{1}{up_1},\ \frac{1 - 2up_1 + u^2 p_1 p_2}{u(p_2 - p_1)} \right\}.$$

LEMMA 7.5. *The two repelling fixed-point multipliers have the same absolute value if and only if $v(p) = v(p - 1)$, where $p = p_1/p_2$.*

Proof. For $v(p) < 0$, necessarily $v(p - 1) = v(p)$, and we can take $p_1 = 1$ and $|p_2|_v < 1$. In this case both repelling multipliers have absolute value $= 1/|u|_v$. For $v(p) = 0$, we can again take $p_1 = 1$, and the condition that the two multipliers have the same absolute value, $1/|u|_v$, is equivalent to $|p_2 - p_1|_v = 1$, which is equivalent to $v(p - 1) = 0$. For $v(p) > 0$, so that $v(p - 1) = 0 \neq v(p)$, taking $p_2 = 1$, we see from the formulas that the repelling multipliers will have absolute values $1/|up_1|_v > 1/|u|_v$. □

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Now take any point $a \in \mathbb{P}^1(\mathbb{C}_v)$. We apply Theorem 4.2 and Proposition 4.1 to the F of (7.3) to deduce that $\sigma(F, a) = 0$ unless $v(a) > 0$ or $v(a-1) > 0$. From (4.2), we compute that

$$\sigma(F, a) = \min\{v(a) + v(a-1),\ v(u) + \max\{0, v(a-p)\}\} \tag{7.4}$$

for all $|a|_v \le 1$, where $p = p_1/p_2$. The term involving p will vanish if $|p|_v > 1$. The shape of the spine will depend on the location of p and the value of u. Nevertheless, we have the following lemma.

LEMMA 7.6. *There exist open disks $D_0 \ni 0$ and $D_1 \ni 1$ such that $f(D_i) \supset D_0 \cup D_1$ and $\sigma(F, \cdot)$ is constant on D_i for $i = 0, 1$. A conjugacy from $f|J(f)$ to the shift map on $\{0, 1\}^{\mathbb{N}_0}$ is given by the itinerary of points in these two disks. The two repelling fixed-point multipliers have the same absolute value if and only if $\sigma(F, D_0) = \sigma(F, D_1)$.*

Proof. First suppose that the two fixed-point multipliers have the same absolute value. From Lemma 7.5, we know that $v(p) = v(p-1)$. Assume that $v(p) = 0$, and let $r = |u|_v < 1$. Then we can compute directly that the two preimages of the Gauss point ζ_0 are $\zeta_r := D(0, r)$ and $\zeta_{1,r} := D(1, r)$. Indeed, taking $p_1 = p$ and $p_2 = 1$, we have

$$f(z) = \frac{z(z-1)}{u(z-p)},$$

so that

$$f(z\,u) = z/p + O(u)$$

and

$$f(1 + z\,u) = z/(1-p) + O(u).$$

It follows that the Julia set is the nested intersection of the preimages of the disk $D(0, 1)$; it lies in the union of the open disks $D_0 = \{|z|_v < r\}$ and $D_1 = \{|z-1|_v < r\}$; and the itinerary of an orbit with respect to D_0 and D_1 determines the conjugacy from $f|J(f)$ to the 2-shift. It is straightforward to see from (7.4) that

$$\sigma(F, a) = r = |u|_v$$

for all $a \in D_0 \cup D_1$.

The case where $v(p) = v(p-1) < 0$ is similar. In this case, we set $p_1 = 1$ and $p_2 = 1/p$ so that

$$f(z) = \frac{z(z-1)}{u(z/p-1)} = \frac{pz(z-1)}{u(z-p)}.$$

We compute that

$$f(z\,u) = z + O(u)$$

and

$$f(1 + z\,u) = pz/(1-p) + O(u),$$

so that, again, the preimages of the Gauss point ζ_0 are the Type II points ζ_r and $\zeta_{1,r}$. As before, we have

$$\sigma(F, a) = r = |u|_v$$

for all $a \in D_0 \cup D_1$.

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Now assume the two fixed-point multipliers do not have the same absolute value. From Lemma 7.5, either $v(p) > 0$ or $v(p-1) > 0$. Suppose that $v(p) > 0$. From (7.4), we compute that

$$\sigma(F, a) = v(u)$$

for all $a \in \{z : |z-1|_v < |u|_v\}$. On the other hand,

$$\sigma(F, a) = v(u) + v(p) > v(u)$$

for all $a \in \{z : |z|_v < |up|_v\}$. Moreover, the closed disk $D(0, |up|_v)$ is sent to the Gauss point ζ_0, since

$$f(upz) = z + O(u)$$

and the disk $D(1, |u|_v)$ is also sent to ζ_0, since

$$f(uz) = z + O(p).$$

It follows that the Julia set lies in the union of $D_1 = \{z : |z-1|_v < |u|_v\}$ and $D_0 = \{z : |z|_v < |up|_v\}$. The Julia set lies in the union of the two disks D_0 and D_1, with the orbit itineraries determined by these disks providing the conjugacy to the 2-shift. However, now we have that $\sigma(F, D_0) = v(u) + v(p) > v(u) = \sigma(F, D_1)$. The case where $v(p-1) > 0$ is similar. □

Proof of Proposition 7.4. For $a \notin J(f)$, we have $f^n(a) \to \infty$ under iteration by Theorem 7.1(1). Thus, $f^n(a)$ will lie in the direction of ∞ from the Gauss point ζ_0 for all n sufficiently large, and therefore, from Theorem 4.2 we have $\sigma(F, f^n(a)) = 0$ for all n sufficiently large. Thus, the sum

$$\eta(F, a) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\sigma(F, f^n(a))}{2^n}$$

is rational, and $\widehat{\lambda}_{f,v}(a)$ will be rational by Lemma 3.1.

For $a \in J(f)$, we have $f^n(a) \in J(f)$ for all n. From Lemma 7.6, we know that $\sigma(F, \cdot)$ is constant on $J(f)$ if the two repelling fixed-point multipliers have the same absolute value. In that case, $\eta(F, a)$ is clearly rational, and Lemma 3.1 shows that $\widehat{\lambda}_{f,v}(a)$ will be rational. On the other hand, also from Lemma 7.6, if $\sigma(F, \cdot)$ is non-constant, then it takes distinct rational values on the two disks D_0 and D_1 that determine the conjugacy to the 2-shift. Let a be any point in $J(f)$ with an aperiodic itinerary with respect to the disks D_0 and D_1 of Lemma 7.6; then the sequence $\sigma_n = \sigma(F, f^n(a))$ is not eventually periodic. Consequently, the value of $\eta(F, a)$ will be irrational. (We may apply our rationality criterion of Proposition 3.2 after transforming the values by $\sigma_n \mapsto (\sigma_n - \sigma(F, D_0))/(\sigma(F, D_1) - \sigma(F, D_0))$ so the terms in the series are all 0 and 1.) We conclude from Lemma 3.1 that $\widehat{\lambda}_{f,v}(a)$ is irrational. □

7.4. Proof of Theorem 1.5. Fix a quadratic map f and a place v of k, and fix a normalized homogeneous presentation F of f.

Suppose first that f has potential good reduction at v. It follows from Proposition 3.3 that the local canonical height of f takes only rational values.

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Assume now that f does not have potential good reduction at v. Then we are in one of the four cases of Theorem 7.1. Case (1) of Theorem 7.1 is treated by Proposition 7.4, showing that if there exists an attracting fixed point, then all points have rational local canonical height if and only if the two repelling fixed points have multipliers of the same absolute value. In other words, all points have rational canonical height if and only if f is strongly polynomial-like.

It remains to show that if we are in cases (2), (3) or (4) of Theorem 7.1, then there must exist at least one point $a \in \mathbb{P}^1(\mathbb{C}_v)$ for which $\widehat{\lambda}_{f,v}(a)$ is an irrational number.

Assume that we are in case (2) of Theorem 7.1. The result will follow from Proposition 6.1. Choose coordinates so that the fixed Type II point of Theorem 7.1(2) is the Gauss point ζ_0. From Theorem 7.1(2), there is a unique preimage $\zeta \neq \zeta_0$ of the Gauss point ζ_0 in the direction of $z = 1$ from ζ_0. The action on directions from ζ_0 sends 1 to $z = \lambda$ (respectively, to $z = 2$ when the reduction map is $z + 1$); it follows that the direction from ζ containing ζ_0 must also be sent towards $z = \lambda$ (respectively, towards $z = 2$); otherwise, by continuity, there would be another preimage of ζ_0 lying between the two points, which is impossible. Thus, the map satisfies all the hypotheses of Proposition 6.1, taking the point 1 to play the role of 0, taking b_n in the direction of λ^{-n} (respectively, $1 - n$) for all $n > 0$, and setting $x = \infty$. Theorem 4.2 shows that the order function $\sigma(F, \cdot)$ has the necessary properties for Proposition 6.1. Therefore, choosing any sequence $s \in \{0, \sigma_F(\zeta)\}^{\mathbb{N}}$ which is not eventually periodic, we obtain a point $a \in \mathbb{P}^1(\mathbb{C}_v)$ with the given order itinerary $\sigma(F, f^n(a)) = s_n$. It follows that

$$\eta(F, a) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\sigma(F, f^n(a))}{2^n}$$

will be irrational by Proposition 3.2, and therefore also the local height by Lemma 3.1.

If f falls into cases (3) or (4), choose coordinates so that the periodic Type II point of period $p > 1$ in Theorem 7.1 is the Gauss point ζ_0 with the coordinate w on the space of directions shown in the theorem. We work with the iterate f^p and the distinguished bad direction $w = 0$ from ζ_0. Unfortunately, we cannot apply Proposition 6.1 directly, because we may not be able to show that all the hypotheses are satisfied. Nevertheless, the arguments in the proof of Proposition 6.1 may be repeated to show there exists at least one point with an aperiodic itinerary. Specifically, observe that the spine of f^p lies in the directions of 0 and ∞ from ζ_0, because these are the unique bad directions by Theorem 7.1. Take any end $\zeta \neq \zeta_0$ of the spine for f^p lying in the direction of 0 from ζ_0, so that $f^p(\zeta) = \zeta_0$. (The existence of ζ is guaranteed by Theorem 4.2.)

Let $c = \sigma_{F^p}(\zeta) > 0$.

Importantly, since ζ is an end of the spine, the order function σ_{F^p} will be constant (equal to c) in all but one direction from ζ. Let U denote this direction from ζ containing ζ_0. Consider the map of directions from ζ to ζ_0; let V denote the direction from ζ_0 corresponding to the image of U. Choose an infinite backward orbit b_n of 0 satisfying the conditions of Proposition 6.1 for f^p. The forms of the reduction map, $\bar{f}^p(w) = w + \tau + 1/w$ or $\bar{f}^p(w) = w + 1$ presented in Theorem 7.1 (3) and (4), show that such infinite orbits exist. Take any point x with forward orbit under $\bar{f}^p$ that does not

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contain 0 or ∞; in fact, we can always take either $x = 1$ or $x = -1$. This x will satisfy the hypotheses of Proposition 6.1.

Note that the direction V contains at most one element of $\{0, x, b_1, b_2, b_3, \ldots\}$. If V contains none of these points, then we can apply Proposition 6.1 directly to conclude that for any sequence $s = (s_n) \in \{0, c\}^{\mathbb{N}}$, there exists a point a with order itinerary

$$\sigma(F^p, f^{np}(a)) = s_n.$$

Suppose V contains x. Let $s = (s_n) \in \{0, c\}^{\mathbb{N}}$ be any aperiodic sequence. Such sequences cannot end with an infinite string of 0s, so the proof of Proposition 6.1 goes through to show that for any such sequence, there exists a point a with the given order itinerary for f^p. Suppose instead that V contains 0. The proof of Proposition 6.1 shows that we can realize any sequence in $\{0, c\}^{\mathbb{N}}$ with no repeated cs. Finally, suppose that V contains b_n. The construction shows that we can realize any sequence in $\{0, c\}^{\mathbb{N}}$ that does not have strings of exactly n zeros that are preceded and followed by a c. Thus, in every case, there exist points with aperiodic order itineraries and therefore points with irrational local canonical height for f^p, applying Proposition 3.2 and Lemma 3.1 to f^p.

It remains to observe that a point with irrational local canonical height for an iterate f^p will also be of irrational local height for f. But this follows from the computation of (3.4), where we observed that

$$\sigma(F^p, a) = d^{p-1}\,\sigma(F, a) + d^{p-2}\,\sigma(F, f(a)) + \cdots + \sigma(F, f^{p-1}(a)).$$

This implies that

$$\eta(F, a) = \frac{1}{d}\sum_{n=0}^{\infty}\frac{\sigma(F, f^n(a))}{d^n} = \frac{1}{d^p}\sum_{k=0}^{\infty}\frac{\sigma(F^p, f^{kp}(a))}{d^{kp}} = \eta(F^p, a).$$

In particular, the irrationality of $\eta(F^p, a)$ (and therefore of the local canonical height for f^p, by Lemma 3.1) implies the irrationality of $\widehat{\lambda}_{f,v}(a)$.

This concludes the proof of Theorem 1.5.

8. *Orbit intersections are unlikely*

We work under the hypotheses of Theorem 1.7.

We assume $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ is infinite and derive a contradiction. In particular, our assumption yields that a (respectively, b) is not a preperiodic point for f (respectively, g).

Under hypothesis (2) in Theorem 1.7, it follows from [**De2**, Theorem 1.2, Proposition 2.3] that

$$H_1 := \widehat{\mathrm{h}}_f(a) > 0$$

because a is not preperiodic. Set $H_2 := \widehat{\mathrm{h}}_g(b)$. Since the difference between the usual Weil height and the canonical height $\widehat{\mathrm{h}}_f$ or $\widehat{\mathrm{h}}_g$ is uniformly bounded on $\mathbb{P}^1(\bar{k})$, there exists a constant $C > 0$ so that

$$|\,\widehat{\mathrm{h}}_f(f^m(a)) - \widehat{\mathrm{h}}_g(g^n(b))| \le C, \tag{8.1}$$

for all $m, n \in \mathbb{N}$ such that $f^m(a) = g^n(b)$. Then (8.1) yields that

$$|d^m \cdot H_1 - e^n \cdot H_2| \le C,$$

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for the infinite collection of pairs of positive integers (m, n) for which $f^m(a) = g^n(b)$. In particular, the positivity of H_1 implies that we also have $H_2 > 0$. Furthermore, since H_1 and H_2 are rational numbers (by hypothesis (3)), we conclude that there exists a finite set S of rational numbers such that for each $m, n \in \mathbb{N}$ satisfying $f^m(a) = g^n(b)$, we have that

$$d^m \cdot H_1 - e^n \cdot H_2 \in S. \tag{8.2}$$

By the pigeonhole principle, there exists a rational number $q \in S$ for which there exist infinitely many $(m, n) \in \mathbb{N} \times \mathbb{N}$ with

$$d^m \cdot H_1 - e^n \cdot H_2 = q. \tag{8.3}$$

This means that the affine line L given by the equation

$$H_1 \cdot x - H_2 \cdot y = q \tag{8.4}$$

contains infinitely many points in common with the finitely generated subgroup Γ of $K^* \times K^*$ which consists of all points (d^k, e^ℓ), for $k, \ell \in \mathbb{Z}$. Then Lang's theorem on subgroups of $\mathbb{G}_m^2$ [**La1**] yields that $L \cap (K^* \times K^*)$ is a coset of an algebraic torus, so that $q = 0$. Note that when $q = 0$, L is defined by the equation $y = (H_1/H_2) \cdot x$, which is the multiplicative translate by the point $(1, H_1/H_2)$ of the diagonal line $y = x$, which is isomorphic to $\mathbb{G}_m$, when we see L as a subset of $\mathbb{G}_m \times \mathbb{G}_m$. A non-zero value of q does not make L a coset of an algebraic torus.

So with $q = 0$, equation (8.3) reads

$$d^m \cdot H_1 = e^n \cdot H_2 \tag{8.5}$$

for infinitely many pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$. Taking two distinct pairs (m_1, n_1) and (m_2, n_2) satisfying (8.5), and using the fact that H_1 and H_2 are non-zero, we obtain that

$$d^{m_2 - m_1} = e^{n_2 - n_1}, \tag{8.6}$$

which contradicts hypothesis (1) and thus proves that the intersection $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ must be finite if hypotheses (1)–(3) are met.

Now assume that hypothesis (1) is replaced by the weaker hypothesis (1′). Then equation (8.6) yields that d and e must be multiplicative dependent, i.e. there exist coprime positive integers k and ℓ such that $d^k = e^\ell$. Furthermore, we get that (8.6) yields the existence of an integer s such that

$$m_2 - m_1 = ks \quad \text{and} \quad n_2 - n_1 = \ell s. \tag{8.7}$$

So, if $\mathcal{O}_f(a) \cap \mathcal{O}_g(b)$ is infinite, then (8.7) yields that there exist infinitely many $s \in \mathbb{N}$ such that $f^{m_1 + sk}(a) = g^{n_1 + s\ell}(b)$, i.e. the diagonal Δ of $\mathbb{P}^1 \times \mathbb{P}^1$ contains infinitely many points from the orbit of the point $(f^{m_1}(a), g^{n_1}(b))$ under the action of the endomorphism Φ of $\mathbb{P}^1 \times \mathbb{P}^1$ given by

$$(x, y) \mapsto (f^k(x), g^\ell(y)).$$

Then the dynamical Mordell–Lang conjecture (see [**GT2**]) yields that Δ is periodic under the action of Φ. This means that there exists $r \in \mathbb{N}$ such that $f^{kr} = g^{\ell r}$, thus contradicting hypothesis (1′), which completes the proof of Theorem 1.7.

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