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Cyclicity in Poletsky–Stessin weighted Bergman spaces

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Abstract. We study the cyclicity of polynomials in Poletsky–Stessin weighted Bergman spaces on various domains in $\mathbb{C}^2$, including the unit ball, the bidisk, and the complex ellipsoid. To this end, we introduce a natural extension of the parameter range for Poletsky–Stessin weighted Bergman spaces on complete Reinhardt domains, yielding a family of spaces that resemble Dirichlet-type spaces on the unit ball. We highlight the differences in the cyclicity behavior of polynomials in these spaces on the bidisk compared to those studied by Bénéteau et al. Finally, we propose several open problems concerning the structure of cyclic polynomials in these spaces.

1 Introduction

Given a Banach space $\mathcal{A}$ of holomorphic functions on a domain in $\mathbb{C}^n$ and a function $f \in \mathcal{A}$, f is called *cyclic* when the span of polynomial multiples of f is dense in $\mathcal{A}$. If polynomials are dense in $\mathcal{A}$ and act as multipliers for $\mathcal{A}$ – that is, if for any polynomial p, we have $ph \in \mathcal{A}$ whenever $h \in \mathcal{A}$ – an equivalent characterization of cyclicity is that there exists a sequence of polynomials p_n such that $\|p_n f - 1\|_{\mathcal{A}} \to 0$. The problem of characterizing cyclic functions is highly space dependent and remains challenging – even in Hilbert spaces of holomorphic functions on the unit disk. Complete characterizations of cyclic functions are rare. A classical example is given by Beurling's theorem [5]: every nonzero closed shift-invariant subspace of $H^2(\mathbb{D})$ is of the form $\theta H^2(\mathbb{D})$, where θ is an inner function. In particular, the cyclic functions in $H^2(\mathbb{D})$ are precisely the outer functions. For the Dirichlet space, Brown and Shields [7] proved that a cyclic function must be outer and its boundary zero set must have logarithmic capacity zero. They conjectured that these two conditions are also sufficient. This conjecture, posed in 1984, remains open. Recent studies on cyclicity in spaces of holomorphic functions on the unit disk – especially Dirichlet-type spaces – as well as partial results related to the Brown–Shields conjecture, appear in [1, 9, 11]. Additionally, [10] provides a comprehensive textbook treatment of Dirichlet-type spaces.

In several variables, a complete characterization of cyclic functions in Dirichlet-type spaces remains elusive for classical domains, such as the unit ball and the

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polydisk. However, significant progress has been made in identifying cyclic polynomials, with explicit characterizations obtained in the unit ball of dimension two and the bidisk [2, 15, 16]. On the unit ball $\mathbb{B}_2$, the characterization in [16] determines the cyclicity range of a polynomial with no zeros in the ball according to whether its boundary zero set is empty, has real dimension zero (finitely many points), or has real dimension one.

On the bidisk, the characterization in [2] for the classical Dirichlet-type spaces concerns zeros on the distinguished boundary $\mathbb{T}^2$. For polynomials with no zeros in $\mathbb{D}^2$, a similar distinction applies, with one important difference: factors of the form $\zeta - z_i$, $\zeta \in \mathbb{T}$, have the same cyclicity range as polynomials with a nonempty zero-dimensional zero set, although they vanish on one-dimensional subsets of $\mathbb{T}^2$. The remaining irreducible polynomials with one-dimensional zero sets have a smaller cyclicity range.

Weighted Bergman spaces on the unit ball are initially defined via an integral condition and for a limited range of parameters. However, they can be naturally extended to cover the Dirichlet-type spaces of the unit ball, as discussed in [22]. In this work, we adopt the definition introduced by Poletsky and Stessin for weighted Bergman spaces on a hyperconvex domain using an exhaustion function, and we demonstrate that this definition can be extended naturally to a wider range of parameters whenever the domain is complete Reinhardt. On the unit ball, this construction recovers the classical Dirichlet-type spaces, and the corresponding norm varies continuously under continuous changes in the domain, in the sense of varying the exhaustion function. Although it can be shown that on the bidisk this construction does not reproduce the Dirichlet-type spaces studied in [2, 3], an important observation is that our results concerning the cyclicity of polynomials on the bidisk depend solely on the dimension of the zero set on the distinguished boundary – similar to the case of the unit ball [16]. This contrasts with the spaces in [2], where polynomials of the form $\zeta - z_i$ must be treated separately when categorizing cyclic polynomials whose zero set has dimension one on the distinguished boundary.

Our main cyclicity results are Theorems 34 and 38. The first extends the characterization for polynomials with a nonempty finite boundary zero set from the unit ball to the complex ellipsoids Ω_p. The second determines the cyclicity range for polynomials whose zero set on the distinguished boundary of the bidisk has dimension one and gives sufficient and necessary conditions for cyclicity when this zero set is nonempty and finite.

The structure of the article is as follows. In Section 2, we recall the original definition of Poletsky and Stessin and apply it to several examples of domains, while also providing the necessary background for the rest of the article. A precise definition and first properties of cyclicity are given in Section 2.3. Section 3 establishes some general results and extends Poletsky–Stessin weighted Bergman spaces to more general indices. The motivation here is to construct a family of spaces on complete Reinhardt domains that inherit the properties of Dirichlet-type spaces. In Sections 4 and 5, we study the cyclicity problem in Poletsky–Stessin weighted Bergman spaces on the complex ellipsoid and the bidisk, respectively. Finally, in Section 6, we propose several open problems for future research.

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2 Preliminaries

2.1 Poletsky–Stessin Hardy spaces

Throughout this article, we denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^n$. We write $A \lesssim B$ if $A \leq CB$ for some constant $C > 0$, and $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold. The dependence of the constants on parameters will be specified whenever necessary. For $z, w \in \mathbb{C}^n$, we use the standard Hermitian inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. We denote the unit ball by $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ and its boundary by $\mathbb{S}_n = \partial\mathbb{B}_n$.

Definition 1 Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $u : \Omega \to [-\infty, 0)$ is called an *exhaustion function* of Ω if

$$\{z \in \Omega : u(z) < c\} \Subset \Omega$$

for every $c < 0$. In particular, $u(z) \to 0$ as $z \to \partial\Omega$.

For a smooth function u on $\mathbb{C}^n$, we denote

$$d = \partial + \bar{\partial}, \qquad d^c = \frac{i}{2\pi}(\bar{\partial} - \partial).$$

In particular,

$$dd^c = \frac{i}{\pi}\partial\bar{\partial}.$$

Definition 2 Let u be a $\mathcal{C}^2$ function on a domain $\Omega \subset \mathbb{C}^n$. The *complex Monge–Ampère operator* of u is the (n, n)-form

$$(dd^c u)^n = \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n \text{ times}}.$$

Definition 3 Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $u : \Omega \to [-\infty, +\infty)$ is called *plurisubharmonic* if u is upper semicontinuous, and for every $z \in \Omega$ and every $\xi \in \mathbb{C}^n$, the function

$$t \mapsto u(z + t\xi)$$

is subharmonic on the set $\{t \in \mathbb{C} : z + t\xi \in \Omega\}$.

Remark 1 Let u be a plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$, locally bounded outside a compact subset of Ω. The Monge–Ampère operator $(dd^c u)^n$ can be defined as a positive measure on Ω (see [6, Theorem 1.2 and the subsequent paragraph]). We refer to [14, Chapter 3] for further discussions.

Definition 4 Let $\Omega \subset \mathbb{C}^n$ be a domain. A plurisubharmonic function u on Ω is called *maximal* if for every relatively compact open set $U \Subset \Omega$ and every plurisubharmonic function v on U such that

$$v \leq u \quad \text{on } \partial U,$$

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we have

$$v \le u \quad \text{on } U.$$

Remark 2 If $u \in \mathcal{C}^2(\Omega)$ is plurisubharmonic, then u is maximal if and only if

$$(dd^c u)^n = 0$$

on Ω (see [14, Chapter 3]).

Definition 5 A domain $\Omega \subset \mathbb{C}^n$ is called hyperconvex if there exists a negative plurisubharmonic exhaustion function $u : \Omega \to [-\infty, 0)$; that is, for every $c < 0$,

$$\{z \in \Omega : u(z) < c\} \Subset \Omega$$

(see [14, Chapters 2 and 4]).

Definition 6 A domain $\Omega \subset \mathbb{C}^n$ is called complete Reinhardt if, for every $z \in \Omega$ and $w \in \mathbb{C}^n$,

$$|w_j| \le |z_j|, \quad j = 1, \dots, n, \qquad \text{implies} \qquad w \in \Omega.$$

Remark 3 If $\Omega \subset \mathbb{C}^n$ is a pseudoconvex Reinhardt domain, then its logarithmic image

$$\{(\log|z_1|, \dots, \log|z_n|) : z \in \Omega \cap (\mathbb{C}\backslash\{0\})^n\}$$

is convex. In particular, this holds for the hyperconvex complete Reinhardt domains (see [23, Proposition 2]).

Definition 7 If $\Omega \subset \mathbb{C}^n$ is a hyperconvex domain equipped with a plurisubharmonic exhaustion function $u : \Omega \to [-\infty, 0)$, we first define for $r < 0$,

$$B_r(u) = \{z \in \Omega : u(z) < r\}, \qquad S_r(u) = \{z \in \Omega : u(z) = r\}.$$

Let $u^{(r)} = \max\{u, r\}$. The associated Monge–Ampère measure is defined by

$$\mu_{u^{(r)}} = (dd^c u^{(r)})^n - \chi_{\Omega\backslash B_r(u)}(dd^c u)^n. \tag{1}$$

We then define μ_u as the weak-$*$ limit of $\mu_{u^{(r)}}$ as $r \to 0^-$. The *Poletsky–Stessin Hardy space* on Ω with respect to u is

$$H^2_u = \{f \in \mathcal{O}(\Omega) : \mu_u(|f|^2) < +\infty\}. \tag{2}$$

Notation 8 *Let $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ be a multi-index and let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. We write*

$$|k| = k_1 + \cdots + k_n, \qquad k! = k_1! \dots k_n!, \qquad z^k = z_1^{k_1} \dots z_n^{k_n}.$$

If $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, then with a slight abuse of notation, we write

$$|z^\gamma| = |z_1|^{\gamma_1} \dots |z_n|^{\gamma_n}.$$

The following lemma will be used to construct several examples.

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Lemma 9 [20] *Let σ and v be the normalized surface and volume measures for $\mathbb{S}_n$ and $\mathbb{B}_n$, respectively. Assume that $\gamma \in (-1, +\infty)^n$. Then, we have*

$$\int_{\mathbb{S}_n} |\zeta^\gamma|^2 d\sigma(\zeta) = \frac{\Gamma(n)\Gamma(\gamma_1+1)\dots\Gamma(\gamma_n+1)}{\Gamma(\gamma_1+\dots+\gamma_n+n)}, \tag{3}$$

$$\int_{\mathbb{B}_n} |z^\gamma|^2 dv(z) = \frac{\Gamma(n+1)\Gamma(\gamma_1+1)\dots\Gamma(\gamma_n+1)}{\Gamma(\gamma_1+\dots+\gamma_n+n+1)}. \tag{4}$$

Here, Γ denotes the Gamma function.

Here, we compute the norm of monomials in certain function spaces.

Example 10 Let $\Omega_P = \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \dots + |z_n|^{2p_n} < 1\}$, where p_i's are positive integers and define

$$u(z) = \frac{1}{2(p_1 \dots p_n)^{1/n}} \log(|z_1|^{2p_1} + \dots + |z_n|^{2p_n}).$$

Note that since u is maximal plurisubharmonic on $\Omega_P \backslash \{0\}$, we have $\mu_{u^{(r)}} = (dd^c u^{(r)})^n$, which is supported on $S_r(u)$. Consequently, one can compute the norm of a monomial z^L, where $L = (l_1, \dots, l_n)$ is a multi-index, as follows:

$$\begin{aligned}\mu_u(|z^L|^2) &= \lim_{r\to 0^-} \mu_{u^{(r)}}(|z^L|^2) \\ &= \lim_{r\to 0^-} \int_{S_r(u)} |z^L|^2 (dd^c u^{(r)})^n(z).\end{aligned}$$

To compute this limit, we use the $(p_1 \dots p_n)$-sheeted change of variables

$$z_i \mapsto z_i^{p_i}$$

and the fact that $\mu_v = \sigma$, the Lebesgue measure on $\mathbb{S}_n$ (see [8, Corollary 7.4]), for

$$v(z) = \log|z|.$$

This gives

$$\begin{aligned}\mu_u(|z^L|^2) &= \frac{p_1 \cdots p_n}{p_1 \dots p_n} \lim_{r\to 0^-} \int_{S_r(v)} |z_1|^{2\frac{l_1}{p_1}} \dots |z_n|^{2\frac{l_n}{p_n}} (dd^c v^{(r)})^n(z) \\ &= \int_{\mathbb{S}_n} |z_1|^{2\frac{l_1}{p_1}} \dots |z_n|^{2\frac{l_n}{p_n}} d\sigma(z) \\ &= \frac{\Gamma(n)\Gamma\left(\frac{l_1}{p_1}+1\right)\dots\Gamma\left(\frac{l_n}{p_n}+1\right)}{\Gamma\left(\frac{l_1}{p_1}+\dots+\frac{l_n}{p_n}+n\right)}.\end{aligned}$$

In the previous example, we observe that when $p_i = 1$ for all i, we obtain the same norm as in the classical Hardy space on the ball. Similarly, as $p_i \to \infty$ for all i, we recover the monomial norm for the classical Hardy space on the polydisc.

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Example 11 Let $\Omega_\lambda = \{z \in \mathbb{C}^2 : |z_1| < 1,\ |z_2| < 1,\ \lambda|z_1^m z_2^n| < 1\}$ and define

$$u(z) = C \log \max\{|z_1|, |z_2|, \lambda|z_1^m z_2^n|\},$$

where $\lambda > 1$ and $C > 0$ is chosen such that $\mu_u(1) = 1$.

By a similar argument to Example 6.11 in [8], μ_u is supported on $T_1 \cup T_2$, where

$$T_1 = \{z \in \mathbb{C}^2 : |z_1| = (1/\lambda)^{1/m},\ |z_2| = 1\}, \quad T_2 = \{z \in \mathbb{C}^2 : |z_1| = 1,\ |z_2| = (1/\lambda)^{1/n}\}.$$

We have

$$\mu_u = C_1\mu_1 + C_2\mu_2,$$

where μ_i is the uniform probability measure supported on T_i. Note that

$$\mu_u(1) = 1 = C_1\mu_1(1) + C_2\mu_2(1) = C_1 + C_2.$$

Now, when $z \in \Omega_\lambda$ is sufficiently close to T_1, we have

$$u(z) = u^1(z) = C \log \max\{|z_2|, \lambda|z_1^m z_2^n|\}.$$

Hence, if we set $\tilde{z}_1 = z_1^m z_2^n$ and $\tilde{z}_2 = z_2$, then

$$\begin{aligned} C_1 = \lim_{r\to 0^-} \mu_{(u^1)^{(r)}}(S_r(u^1)) &= \lim_{r\to 0^-} \int_{S_r(u^1)} (dd^c(u^1)^{(r)})^2(z) \\ &= \lim_{r\to 0^-} mC^2 \int_{S_{\tilde{r}}(V)} (dd^c V^{(\tilde{r})})^2(\tilde{z}) = m\lambda C^2. \end{aligned}$$

Here, $V(z) = \log \max\{\lambda|z_1|, |z_2|\}$, and $\tilde{r} = r/C$. Similarly, we obtain

$$C_2 = n\lambda C^2.$$

Hence, we have $C_1 = \frac{m}{m+n}$ and $C_2 = \frac{n}{m+n}$, and the norm of the monomial $z_1^k z_2^l$ then can be computed as

$$\|z_1^k z_2^l\|^2 = \frac{m\lambda^{-\frac{2k}{m}} + n\lambda^{-\frac{2l}{n}}}{m+n}.$$

2.2 Poletsky–Stessin weighted Bergman spaces

Definition 12 If $\Omega \subset \mathbb{C}^n$ is a hyperconvex domain equipped with a plurisubharmonic exhaustion function u, then for $\alpha > -1$, the Poletsky–Stessin weighted Bergman space on Ω with respect to u is defined as follows [19]:

$$A^2_{u,\alpha} = \left\{ f \in \mathcal{O}(\Omega) \ :\ \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u^{(r)}}(|f|^2)\, dr < +\infty \right\}. \tag{5}$$

The following examples illustrate the definition and motivate the extension of the parameter range in Section 3 through explicit formulas for monomial norms. They

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also introduce the spaces on the complex ellipsoids and the polydisk that we will study in Sections 4 and 5.

Example 13 Let $\Omega_P = \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\}$ and define

$$u(z) = \frac{1}{2(p_1 \dots p_n)^{1/n}} \log\left(|z_1|^{2p_1} + \cdots + |z_n|^{2p_n}\right).$$

Similar to Example 10, we use the $p_1 \dots p_n$-sheeted change of variables $z_i \mapsto z_i^{p_i}$ to compute the norm of a monomial z^L. Let $p^n = p_1 \dots p_n$, $v(z') = \frac{1}{2}\log(|z_1|^2 + \cdots + |z_n|^2)$, $r' = pr$, and $R = e^{r'}$. We have

$$\begin{aligned}
\|z^L\|^2_{A^2_{u,\alpha}} &= \int_{-\infty}^{0} |r|^\alpha e^r \mu_{u^{(r)}}(|z^L|^2)\, dr \\
&= \int_{-\infty}^{0} |r|^\alpha e^r \int_{S_r(u)} |z^L|^2 (dd^c u^{(r)})^n(z)\, dr \\
&= \frac{p_1 \dots p_n}{p_1 \dots p_n} \int_{-\infty}^{0} |r|^\alpha e^r \int_{S_{r'}(v)} |z_1|^{\frac{2l_1}{p_1}} \dots |z_n|^{\frac{2l_n}{p_n}} (dd^c v^{(r')})^n(z)\, dr.
\end{aligned}$$

Again, similar to Example 10, $(dd^c v^{(r')})^n = \sigma_R$, the normalized surface measure on $S_{r'}(v) = \{z \in \mathbb{C}^n \mid |z| = R\}$. Therefore,

$$\begin{aligned}
\|z^L\|^2_{A^2_{u,\alpha}} &= \int_{-\infty}^{0} |r|^\alpha e^r \int_{S_{r'}(v)} |z_1|^{\frac{2l_1}{p_1}} \dots |z_n|^{\frac{2l_n}{p_n}}\, d\sigma_R(z')\, dr \\
&= \int_{-\infty}^{0} |r|^\alpha e^r \int_{\mathbb{S}_n} |R\zeta_1|^{\frac{2l_1}{p_1}} \dots |R\zeta_n|^{\frac{2l_n}{p_n}}\, d\sigma(\zeta)\, dr \\
&= \frac{\Gamma(n)\Gamma\left(\frac{l_1}{p_1}+1\right)\dots\Gamma\left(\frac{l_n}{p_n}+1\right)}{\Gamma\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}+n\right)} \int_{-\infty}^{0} |r|^\alpha e^r R^{2\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}\right)}\, dr.
\end{aligned}$$

After the change of variable $r \mapsto -r$, we continue as

$$\begin{aligned}
\|z^L\|^2_{A^2_{u,\alpha}} &= \frac{\Gamma(n)\Gamma\left(\frac{l_1}{p_1}+1\right)\dots\Gamma\left(\frac{l_n}{p_n}+1\right)}{\Gamma\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}+n\right)} \int_0^{\infty} r^\alpha e^{-(2p(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n})+1)r}\, dr \\
&= \frac{\Gamma(n)\Gamma\left(\frac{l_1}{p_1}+1\right)\dots\Gamma\left(\frac{l_n}{p_n}+1\right)\Gamma(\alpha+1)}{\Gamma\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}+n\right)} \left(2p\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}\right)+1\right)^{-(\alpha+1)}.
\end{aligned}$$

To make a comparison, we compute the Bergman norm of a monomial in the next example. One may compare the case $\alpha = 0$ in Example 13 with the following.

Example 14 (Bergman space) Let Ω_P be as in Example 13. We may compute the norm of z^L as follows:

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$$\begin{aligned}\|z^L\|^2_{A^2(\Omega_P)} &= \int_{\Omega_P} |z^L|^2 \, dv_P(z) \\ &= \frac{\Gamma\left(\frac{1}{p_1}+\cdots+\frac{1}{p_n}+1\right)}{\Gamma(n+1)\Gamma\left(\frac{1}{p_1}\right)\ldots\Gamma\left(\frac{1}{p_n}\right)} \int_{\mathbb{B}_n} |z_1|^{2\left(\frac{l_1+1}{p_1}-1\right)} \ldots |z_n|^{2\left(\frac{l_n+1}{p_n}-1\right)} \, dv(z) \\ &= \frac{\Gamma\left(\frac{1}{p_1}+\cdots+\frac{1}{p_n}+1\right)}{\Gamma\left(\frac{1}{p_1}\right)\ldots\Gamma\left(\frac{1}{p_n}\right)} \frac{\Gamma\left(\frac{l_1+1}{p_1}\right)\ldots\Gamma\left(\frac{l_n+1}{p_n}\right)}{\Gamma\left(\frac{l_1+1}{p_1}+\cdots+\frac{l_n+1}{p_n}+1\right)}.\end{aligned}$$

Here, ν_P and ν denote the normalized volume measures for Ω_P and $\mathbb{B}_n$, respectively.

Definition 15 From this point onward, we use the following modified definition of the Poletsky–Stessin weighted Bergman norm. For $\alpha > -1$, we set

$$\|f\|^2_{A^2_{u,\alpha}} = \frac{(2n)^{\alpha+1}}{\Gamma(\alpha+1)} \int_{-\infty}^{0} |r|^{\alpha} e^{2nr} \mu_{u^{(r)}}(|f|^2) \, dr. \tag{6}$$

The subsequent extension to other parameter values is based on this norm.

This definition yields the classical weighted Bergman spaces in the case of the unit ball when one chooses $u(z) = \frac{1}{2}\log\left(|z_1|^2+\cdots+|z_n|^2\right)$ as the exhaustion function.

Example 16 Let Ω_P and u be as in Example 13. Then, the norm of z^L with the modified definition is given by

$$\|z^L\|^2_{A^2_{u,\alpha}} = \frac{n^{\alpha+1}\Gamma(n)\Gamma\left(\frac{l_1}{p_1}+1\right)\ldots\Gamma\left(\frac{l_n}{p_n}+1\right)}{\Gamma\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}+n\right)} \left(p\left(\frac{l_1}{p_1}+\cdots+\frac{l_n}{p_n}\right)+n\right)^{-(\alpha+1)}. \tag{7}$$

In this way, one may define $A^2_{u,\alpha}(\Omega_P)$ for $\alpha < -1$ using the norm expressed as a series. Moreover, in the case of the unit ball and $\alpha = -(\beta+1)$, these spaces coincide with the Dirichlet-type space $D_\beta(\mathbb{B}_n)$ studied in [16, 21].

Example 17 (Poletsky–Stessin weighted Bergman spaces on the polydisk) Let $f \in \mathcal{O}(\mathbb{D}^n)$ be a holomorphic function on the polydisk $\mathbb{D}^n$, written in the form of a series as

$$f(z) = \sum_L a_L z^L.$$

Also, let $u(z) = \log\max\{|z_1|,\ldots,|z_n|\}$ be the exhaustion function for the domain. Up to a constant normalization, the norm of f in $A^2_{u,\alpha}(\mathbb{D}^n)$ is given by

$$\|f\|^2_{A^2_{u,\alpha}(\mathbb{D}^n)} = \sum_L |a_L|^2 (|L|+n)^{-(\alpha+1)}.$$

Note that other than direct computation this norm can also be obtained by letting $p_i = p \to \infty \; \forall \; 1 \le i \le n$ in (7). For the case of $\alpha \le -1$, we can take the norm in the

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form of a power series as the definition of the space. This is clearly different than the classical weighted Bergman spaces $A^2_\alpha(\mathbb{D}^n)$ given by the norm

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^n)} = \sum_L |a_L|^2 (l_1+1)^{-(\alpha+1)} \dots (l_n+1)^{-(\alpha+1)}.$$

2.3 Basic results of cyclicity

In this section, we present the definition of cyclicity along with fundamental results that are essential for understanding the rest of the article as we use them several times throughout the article. While these results hold in more general settings, we formulate them in the context of the function spaces studied here for convenience.

Definition 18 A function $\varphi \in \mathcal{O}(\Omega)$ is called a multiplier of $A^2_{u,\alpha}(\Omega)$ if the operator

$$M_\varphi : A^2_{u,\alpha}(\Omega) \to A^2_{u,\alpha}(\Omega), \qquad M_\varphi f = \varphi f,$$

is bounded. We denote the space of multipliers by $\mathrm{Mult}(A^2_{u,\alpha})$, equipped with the norm

$$\|\varphi\|_{\mathrm{Mult}(A^2_{u,\alpha})} = \|M_\varphi\|.$$

Remark 4 For $\alpha \in \mathbb{R}$, let $A^2_{u,\alpha}(\Omega)$ denote the spaces obtained by the parameter extension described in Section 3.1. If g is holomorphic on a neighborhood of $\overline{\Omega}$, then

$$g \in \mathrm{Mult}(A^2_{u,\alpha}(\Omega)).$$

The corresponding statement for the bidisk spaces is proved in [4, Proposition 3.4]. The proof is similar for the spaces considered here.

Definition 19 Let $A^2_{u,\alpha}$ be a Poletsky–Stessin weighted Bergman space on a domain $\Omega \subset \mathbb{C}^n$; a function $f \in A^2_{u,\alpha}$ is called a *shift-cyclic vector* if

$$[f] = \overline{\mathrm{span}\{pf : p \in \mathbb{C}[z_1, \dots, z_n]\}} = A^2_{u,\alpha}. \tag{8}$$

Remark 5 In this article, we choose rotation-invariant exhaustion functions on complete Reinhardt domains. The associated measures are therefore rotation invariant, and the monomials are mutually orthogonal. Consequently, every function in $A^2_{u,\alpha}(\Omega)$ can be approximated in norm by its Taylor polynomials.

These spaces are also reproducing kernel Hilbert spaces. For $\alpha \geq -1$, the boundedness of point evaluations follows from [19, Theorem 3.6]. For $\alpha < -1$, it follows from the continuous inclusion $A^2_{u,\alpha}(\Omega) \subset H^2_u(\Omega)$, obtained by comparing the monomial norms in (10) for the exhaustions considered here.

Polynomial multipliers are dense in $\mathrm{Mult}(A^2_{u,\alpha})$ in the strong operator topology. Indeed, rotation invariance and polynomial density imply that $U_\theta h(z) = h(e^{i\theta}z)$ defines a strongly continuous unitary group. For a multiplier $\varphi(z) = \sum_L a_L z^L$, consider

$$p_N(z) = \sum_{|L| \leq N} \left(1 - \frac{|L|}{N+1}\right) a_L z^L.$$

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Let

$$F_N(\theta) = \frac{1}{N+1}\left|\sum_{k=0}^{N} e^{ik\theta}\right|^2$$

be the Fejér kernel [13, Chapter I, Section 2.5]. For every $h \in A^2_{u,\alpha}(\Omega)$,

$$p_N h = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(\theta)\, U_\theta M_\varphi U_{-\theta} h \, d\theta.$$

Positivity and normalization of F_N give

$$\|p_N\|_{\mathrm{Mult}(A^2_{u,\alpha})} \le \|\varphi\|_{\mathrm{Mult}(A^2_{u,\alpha})}.$$

Since $\theta \mapsto U_\theta M_\varphi U_{-\theta} h$ is norm-continuous, [13, Chapter I, Lemma 2.2] gives $p_N h \to \varphi h$ in norm. Consequently, for every $f \in A^2_{u,\alpha}(\Omega)$,

$$[f]_{\mathrm{mult}} := \overline{\{\varphi f : \varphi \in \mathrm{Mult}(A^2_{u,\alpha})\}} = [f]. \tag{9}$$

Theorem 20 *If $f \in A^2_{u,\alpha}(\Omega)$ vanishes at $\zeta \in \Omega$, then f is not cyclic.*

Proof If we define $\Gamma_\zeta : A^2_{u,\alpha} \to \mathbb{C}$ by $\Gamma_\zeta(f) = f(\zeta)$, then Γ_ζ is a bounded linear functional and consequently has a closed kernel. For any polynomial p, we have $pf \in \mathrm{Ker}\Gamma_\zeta$. This implies

$$[f] \subseteq \mathrm{Ker}\Gamma_\zeta \subsetneq A^2_{u,\alpha}.$$ ∎

Remark 6 In the proof of Theorem 20, we used the fact that the point evaluation functional Γ_ζ is continuous for any point ζ in the domain. In fact, the same proof applies whenever the point evaluation functional is bounded at some point in the closure of the domain, which occurs for certain values of α in the spaces we study.

The following theorem offers a useful sufficient criterion to check whether a function is cyclic. A corresponding result is proved in [15, Theorem 4]; we include the proof here for completeness.

Theorem 21 *Let $f \in A^2_{u,\alpha}(\Omega)$ and assume that f does not vanish on Ω. For $0 < r < 1$, define the radial dilation*

$$f_r(z) = f(rz).$$

If

$$\sup_{0<r<1}\left\|\frac{f}{f_r}\right\|_{A^2_{u,\alpha}} < +\infty,$$

then f is cyclic in $A^2_{u,\alpha}(\Omega)$.

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Proof Since $1/f_r$ is holomorphic on a neighborhood of $\overline{\Omega}$, it is a multiplier by Remark 4, and (9) gives

$$\left\{\frac{f}{f_r} : 0 < r < 1\right\} \subseteq [f].$$

By the hypothesis, this family is bounded in $A^2_{u,\alpha}(\Omega)$. Choose $r_j \uparrow 1$. The Banach–Alaoglu theorem and the reflexivity of $A^2_{u,\alpha}$ yield a weakly convergent sequence $\{\frac{f}{f_{r_{j_k}}}\}_{k\geq 0}$. Since point evaluations are bounded and

$$\frac{f(z)}{f(r_{j_k}z)} \longrightarrow 1 \qquad (z \in \Omega),$$

the weak limit is 1. Thus,

$$1 \in \overline{\left\{\frac{f}{f_r} : 0 < r < 1\right\}}^{\text{weak}}.$$

Finally, $[f]$ is a norm-closed linear subspace, and hence it is weakly closed by Mazur's lemma. Therefore, $1 \in [f]$, and f is cyclic. ∎

Lemma 22 *Let $f, g \in A^2_{u,\alpha}(\Omega)$ be cyclic. If f is a multiplier of $A^2_{u,\alpha}(\Omega)$, then fg is also cyclic.*

Proof Since g is cyclic, there exist polynomials p_j such that $p_j g \to 1$ in $A^2_{u,\alpha}(\Omega)$. Thus,

$$\|p_j f g - f\|_{A^2_{u,\alpha}} \leq \|f\|_{\mathrm{Mult}(A^2_{u,\alpha})} \|p_j g - 1\|_{A^2_{u,\alpha}} \longrightarrow 0.$$

Hence, $f \in [fg]$, and therefore $A^2_{u,\alpha}(\Omega) = [f] \subseteq [fg]$. ∎

Remark 7 Let $\alpha \geq -1$ and $1 \in A^2_{u,\alpha}(\Omega)$. If $f, 1/f \in H^\infty(\Omega)$, then both are multipliers by the norm definition, so (9) gives $1 = (1/f)f \in [f]$.

If f is a polynomial with no zeros on $\overline{\Omega}$, the same argument applies for every $\alpha \in \mathbb{R}$ in the extended family since $1/f$ is holomorphic on a neighborhood of $\overline{\Omega}$ and is therefore a multiplier by Remark 4.

3 General results

In this section, we establish some general results that provide a basis for studying cyclicity in these spaces. We first give conditions under which the spaces $A^2_{u,\alpha}(\Omega)$ can be naturally extended to all $\alpha \in \mathbb{R}$. We then introduce a capacity condition for noncyclicity to be used later.

3.1 Parameter extension

Here, our goal is to extend the definition of Poletsky–Stessin weighted Bergman spaces $A^2_{u,\alpha}(\Omega)$, originally defined for $\alpha \geq -1$, to any $\alpha \in \mathbb{R}$. We introduce the notion of the parameter shift equivalence (PSE) property (Definition 25), which is a sufficient

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condition for this extension to be well defined. Our main result here is the following, which will be proved at the end of this section.

Theorem 23 *Let $\Omega \subseteq \mathbb{D}^n$ be a hyperconvex complete Reinhardt domain. Then, there exists an exhaustion function u for Ω such that $A^2_{u,\alpha}(\Omega)$ satisfies the PSE property.*

Definition 24 Let $-1 \le s, t$, and let $\Omega \subseteq \mathbb{C}^n$ be a hyperconvex domain equipped with an exhaustion function u. Then, the operator $R^{s,t} : A^2_{u,t}(\Omega) \to A^2_{u,s}(\Omega)$ is defined as follows:

$$(R^{s,t} f)(w) = \sum_L a_L \frac{\|z^L\|_{A^2_{u,t}}}{\|z^L\|_{A^2_{u,s}}} w^L.$$

Here, $f(z) = \sum_L a_L z^L \in A^2_{u,t}(\Omega)$.

It is straightforward to verify that $R^{s,t}$ is a continuous operator on $A^2_{u,t}(\Omega)$ with a continuous inverse.

Definition 25 We say that the spaces $A^2_{u,\alpha}(\Omega)$, for $-1 \le \alpha$, satisfy the PSE property if for all $\alpha, \alpha', \gamma, \gamma' \ge -1$ satisfying $\alpha - \alpha' = \gamma - \gamma'$, we have

$$\frac{\|z^L\|_{A^2_{u,\alpha}}}{\|z^L\|_{A^2_{u,\alpha'}}} \asymp \frac{\|z^L\|_{A^2_{u,\gamma}}}{\|z^L\|_{A^2_{u,\gamma'}}}, \quad \text{as } |L| \to \infty.$$

Definition 26 Let $\{A^2_{u,\alpha}(\Omega)\}_{\alpha \ge -1}$ be a family of Poletsky–Stessin weighted Bergman spaces satisfying the PSE property. For $\gamma < -1$, we say that $f \in A^2_{u,\gamma}(\Omega)$ if

$$R^{-(1+\gamma)-1,-1} f \in A^2_{u,-1}(\Omega) = H^2_u(\Omega).$$

The norm of a function f in $A^2_{u,\gamma}(\Omega)$ is defined as

$$\|f\|_{A^2_{u,\gamma}} = \left\|R^{-(1+\gamma)-1,-1} f\right\|_{A^2_{u,-1}}. \tag{10}$$

Example 27 On the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for $f(z) = \sum_{l \ge 0} a_l z^l$, the classical α-weighted Bergman norm of f, $\alpha > -1$, can be obtained by

$$\|f\|^2_{A^2_\alpha(\mathbb{D})} = \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dv(z) \asymp \sum_{l \ge 0} (l+1)^{-(\alpha+1)} |a_l|^2.$$

It can be verified using Example 16 that, for $u(z) = \log|z|$, $A^2_{u,\alpha}(\mathbb{D}) = A^2_\alpha(\mathbb{D})$. Let $s, t > -1$, we have

$$(R^{s,t} f)(z) = \sum_{l \ge 0} a_l (l+1)^{\frac{s-t}{2}} z^l.$$

Also, for $\alpha, \alpha', \gamma, \gamma' \ge -1$ satisfying $\alpha - \alpha' = \gamma - \gamma'$, we can verify

$$\frac{\|z^l\|_{A^2_\alpha(\mathbb{D})}}{\|z^l\|_{A^2_{\alpha'}(\mathbb{D})}} = (l+1)^{\frac{\alpha'-\alpha}{2}} = (l+1)^{\frac{\gamma'-\gamma}{2}} = \frac{\|z^l\|_{A^2_\gamma(\mathbb{D})}}{\|z^l\|_{A^2_{\gamma'}(\mathbb{D})}}.$$

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So, the extension in the sense of (10) is well defined and one may define for any $\alpha \in \mathbb{R}$,

$$\|f\|^2_{A^2_\alpha(\mathbb{D})} = \sum_{l\geq 0} (l+1)^{-(\alpha+1)} |a_l|^2.$$

Proposition 28 *Let $\Omega \subseteq \mathbb{D}^n$ be the polyhedral domain defined by*

$$p_j(z) = \lambda_j |z_1|^{\beta_1^j} \dots |z_n|^{\beta_n^j} < 1, \quad j = 1, \dots, m,$$

where $\lambda_j \geq 1$, $\beta_i^j \in [0, +\infty)$, and $\sum_i \beta_i^j = 1$ for all j. If $m \geq n$, then the family of spaces $A^2_{u,\alpha}(\Omega)$ satisfies the PSE property when

$$u(z) = \log \max\{p_j(z) : j = 1, \dots, m\}.$$

Proof Fix $r < 0$. Following [8, Chapter III, Example 6.11], we work in local logarithmic coordinates. At $z \in S_r(u)$, only the terms satisfying $p_j(z) = e^r$ contribute locally to $u^{(r)}$. If the corresponding vectors β^j have rank less than n, a linear change of coordinates shows that $u^{(r)}$ depends on fewer than n complex variables, so $(dd^c u^{(r)})^n = 0$ there.

Thus, $\mu_{u^{(r)}}$ is supported on finitely many tori $S_{r,q}$, each determined by n linearly independent equations $p_j = e^r$. By rotation invariance, $\mu_{u^{(r)}}|_{S_{r,q}} = C_q(r)\mu_q^r$, where μ_q^r is the uniform probability measure on $S_{r,q}$. Since $\sum_i \beta_i^j = 1$, we have $p_j(tz) = t p_j(z)$. Consequently, for $r, s < 0$, the dilation $T(z) = e^{s-r} z$ maps $S_{r,q}$ onto $S_{s,q}$ and satisfies

$$u^{(s)} \circ T = u^{(r)} + s - r$$

near $S_r(u)$. Since the Monge–Ampère operator commutes with biholomorphic pullback, the definition of $\mu_{u^{(r)}}$ gives

$$T_* \mu_{u^{(r)}} = \mu_{u^{(s)}}, \qquad T_* \mu_q^r = \mu_q^s.$$

Hence, $C_q(r) = C_q(s)$, and we may write

$$\mu_{u^{(r)}} = \sum_{q=1}^{N} C_q \mu_q^r, \tag{11}$$

where $C_q > 0$ is independent of r. The fact that $\mu_{u^{(r)}}(S_r(u)) = 1$ implies

$$\sum_{q=1}^{N} C_q = 1.$$

Given $S_{r,q}$, choose indices $\sigma_q(j)$, $j = 1, \dots, n$, corresponding to linearly independent equations defining this torus. To determine $S_{r,q}$, we solve the following linear system:

$$\begin{bmatrix} \beta_1^{\sigma_q(1)} & \dots & \beta_n^{\sigma_q(1)} \\ \vdots & \ddots & \vdots \\ \beta_1^{\sigma_q(n)} & \dots & \beta_n^{\sigma_q(n)} \end{bmatrix} \begin{bmatrix} \log|z_1| \\ \vdots \\ \log|z_n| \end{bmatrix} = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix} - \begin{bmatrix} \log \lambda_{\sigma_q(1)} \\ \vdots \\ \log \lambda_{\sigma_q(n)} \end{bmatrix}.$$

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A direct application of Cramer's rule shows that for every $z \in S_{r,q}$,

$$|z_i| = C'_{i,q} e^r,$$

where each $C'_{i,q}$ is independent of r. Thus, for a monomial z^L and $\alpha > -1$,

$$\begin{aligned}\int_{-\infty}^{0} |r|^\alpha e^{2nr} \mu_q^r(|z^L|^2)\, dr &= \prod_{i=1}^{n} (C'_{i,q})^{2l_i} \int_{-\infty}^{0} |r|^\alpha e^{2(n+|L|)r}\, dr \\ &= \prod_{i=1}^{n} (C'_{i,q})^{2l_i} \frac{\Gamma(\alpha+1)}{[2(|L|+n)]^{\alpha+1}}.\end{aligned}$$

Consequently, recalling (6) and (11),

$$\|z^L\|^2_{A^2_{u,\alpha}(\Omega)} = C_L \left(\frac{n}{|L|+n}\right)^{\alpha+1}, \qquad C_L = \sum_{q=1}^{N} C_q \prod_{i=1}^{n} (C'_{i,q})^{2l_i}.$$

Since $\|z^L\|^2_{H^2_u} = C_L$, this formula also holds for $\alpha = -1$. As C_L is independent of α, taking ratios establishes the PSE property. ∎

Proof of Theorem 23 Consider the mapping $L : \Omega \cap (\mathbb{C}\backslash\{0\})^n \to \Omega' \subset \mathbb{R}^n$ defined by

$$L(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|).$$

For $x = (x^{(1)}, \ldots, x^{(n)}) \in \Omega'$, let

$$F_0(x) = \max\{x^{(1)}, \ldots, x^{(n)}\}.$$

Given a dense sequence $\{x_i\}_{i\in\mathbb{N}}$ in $\partial\Omega'$, define the iterative sequence

$$F_i = \max\{F_{i-1}, f_i\},$$

where $f_i(x) = x \cdot \nu_i + \lambda_i$ with $(1, \ldots, 1) \cdot \nu_i = 1$, and $\nu_i \in [0,1]^n$ is such that $f_i(x_i) = 0$ and $f_i(x) < 0$ for all $x \in \Omega'$. Then, the sequence $g_i = F_i \circ L$ consists of increasing plurisubharmonic functions on Ω converging to an exhaustion function u for Ω. So, by Theorem 3.6.1 in [14], we have

$$\|z^L\|_{A^2_{g_i,\alpha}} \to \|z^L\|_{A^2_{u,\alpha}}$$

for any multi-index L and $\alpha \in [-1, 0]$. Applying Proposition 28 completes the proof. ∎

Notation 29 *Theorem 23 allows us to define the Poletsky–Stessin weighted Bergman spaces on a wide range of domains and parameter values of α. From now on, when there is no ambiguity in the choice of the exhaustion function, we denote the space $A^2_{u,\alpha}(\Omega)$ by $D_\beta(\Omega)$, where $\beta = -(\alpha+1)$.*

3.2 Kernels and capacity

We denote the reproducing kernel of $A^2_{u,\alpha}(\Omega)$ by k_α. For the complete Reinhardt domains and exhaustion functions considered here, the monomials form an

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orthogonal basis, and therefore,

$$k_\alpha(w,\zeta) = \sum_L \frac{w^L \overline{\zeta}^L}{\|z^L\|^2_{A^2_{u,\alpha}}}, \qquad w,\zeta \in \Omega.$$

Definition 30 Let $\mu \in M(\operatorname{supp}(\mu_u))$, the set of Borel measures on $\operatorname{supp}(\mu_u)$. The Cauchy transform of μ, denoted by $C[\mu]$, is defined as follows:

$$C[\mu](z) = \int_{\operatorname{supp}(\mu_u)} k_{-1}(z,\zeta)\, d\mu(\zeta).$$

Definition 31 Let $\alpha \in \mathbb{R}$, $E \subseteq \operatorname{supp}(\mu_u)$, and $\mathcal{P}(E)$ be the set of Borel probability measures supported on E. $I_\alpha[\mu]$ denotes α-energy of μ and is obtained by

$$I_\alpha[\mu] = \iint_{\operatorname{supp}(\mu_u)\times\operatorname{supp}(\mu_u)} k_\alpha(z,\zeta)\, d\mu(z)\, d\mu(\zeta).$$

The α-capacity of E is defined as follows:

$$\operatorname{Cap}_\alpha(E) = \inf\{I_\alpha[\mu] : \mu \in \mathcal{P}(E)\}^{-1}.$$

The following proposition is the analog of the duality discussed in [7, p. 289, Equation (17)]. We include a proof for completeness.

Proposition 32 *Let $\{A^2_{u,\alpha}(\Omega)\}_{\alpha\in\mathbb{R}}$ be a family of Poletsky–Stessin weighted Bergman spaces satisfying the PSE property. If $\alpha + \alpha' = -2$, then the dual space satisfies*

$$\left(A^2_{u,\alpha}(\Omega)\right)^* = A^2_{u,\alpha'}(\Omega).$$

Proof Since $\alpha + \alpha' = -2$, (10) gives

$$\|z^L\|^2_{A^2_{u,\alpha}} \|z^L\|^2_{A^2_{u,\alpha'}} = \|z^L\|^4_{H^2_u}.$$

For $f = \sum_L a_L z^L \in A^2_{u,\alpha}(\Omega)$ and $g = \sum_L b_L z^L \in A^2_{u,\alpha'}(\Omega)$, define

$$\langle f, g\rangle_u = \sum_L a_L \overline{b_L}\, \|z^L\|^2_{H^2_u}.$$

By Cauchy–Schwarz,

$$|\langle f, g\rangle_u| \le \|f\|_{A^2_{u,\alpha}} \|g\|_{A^2_{u,\alpha'}}.$$

Conversely, let $\Lambda \in (A^2_{u,\alpha}(\Omega))^*$. By the Riesz representation theorem, there exists $h = \sum_L c_L z^L \in A^2_{u,\alpha}(\Omega)$ such that

$$\Lambda(f) = \sum_L a_L \overline{c_L} \|z^L\|^2_{A^2_{u,\alpha}}, \qquad \|h\|_{A^2_{u,\alpha}} = \|\Lambda\|.$$

Set

$$b_L = c_L \frac{\|z^L\|^2_{A^2_{u,\alpha}}}{\|z^L\|^2_{H^2_u}}.$$

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Then,

$$\sum_L |b_L|^2 \|z^L\|^2_{A^2_{u,\alpha'}} = \sum_L |c_L|^2 \|z^L\|^2_{A^2_{u,\alpha}} = \|\Lambda\|^2.$$

Hence, $g = \sum_L b_L z^L \in A^2_{u,\alpha'}(\Omega)$ represents Λ under the above pairing. ∎

The proof of the following follows the same approach as [7, Theorem 5]; we give the details in our setting for the reader's convenience.

Theorem 33 *Let $f \in A^2_{u,\alpha}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and assume that Ω satisfies the PSE property. If $\mathrm{Cap}_\alpha(\mathcal{Z}(f) \cap \mathrm{supp}(\mu_u)) > 0$, then f is not cyclic in $A^2_{u,\alpha}(\Omega)$.*

Proof Put $E = \mathcal{Z}(f) \cap \mathrm{supp}(\mu_u)$. By the definition of capacity, there exists $\mu \in \mathcal{P}(E)$ such that

$$I_\alpha[\mu] = \iint_{E\times E} k_\alpha(z,\zeta)\, d\mu(z)\, d\mu(\zeta) < \infty.$$

Expanding the kernel, we obtain

$$I_\alpha[\mu] = \sum_L \frac{\left|\int_E z^L\, d\mu(z)\right|^2}{\|z^L\|^2_{A^2_{u,\alpha}}}.$$

Thus, by Cauchy–Schwarz, for every polynomial q,

$$\left|\int_E q\, d\mu\right|^2 \le \|q\|^2_{A^2_{u,\alpha}} I_\alpha[\mu].$$

By polynomial density, $\Lambda(q) = \int_E q\, d\mu$ extends to a bounded linear functional on $A^2_{u,\alpha}(\Omega)$, with $\Lambda(1) = 1$.

Let p be a polynomial and put $h = pf$. For each $0 < r < 1$, we have $\Lambda(h_r) = \int_E h_r\, d\mu$. Since $h \in \mathcal{C}(\overline{\Omega})$ and $h = 0$ on E,

$$\Lambda(pf) = \lim_{r\to 1^-} \Lambda(h_r) = \lim_{r\to 1^-} \int_E h(r\zeta)\, d\mu(\zeta) = 0.$$

Hence, $[f] \subseteq \ker\Lambda \neq A^2_{u,\alpha}(\Omega)$, and f is not cyclic. ∎

4 Some results on the complex ellipsoid in $\mathbb{C}^2$

Here, in this section, we study the cyclicity properties of some polynomials in $A^2_{u,\alpha}(\Omega_p) = D_\beta(\Omega_p)$, $\beta = -(\alpha+1)$, where $\Omega_p = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^{2p} + |z_2|^{2p} < 1\}$ and $u(z) = \frac{1}{2p}\log(|z_1|^{2p} + |z_2|^{2p})$. We recall

$$\|z_1^{j_1} z_2^{j_2}\|^2_{D_\beta(\Omega_p)} = \frac{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)}{\Gamma\left(\frac{j_1+j_2}{p}+2\right)}(j_1+j_2+2)^\beta.$$

For $p = 1$, the domain Ω_p is the unit ball in $\mathbb{C}^2$. The following theorem extends the finite boundary zero case of [16, Theorem 6] to Ω_p for $p \ge 1$ and is the main result of this section. We give its proof at the end of the section, after establishing the necessary preliminary results.

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Theorem 34 *Let q be a polynomial, non-vanishing on Ω_p; moreover, assume that q has a nonempty finite zero set on $\partial\Omega_p$; then, q is cyclic on $D_\beta(\Omega_p)$ if and only if $\beta \le 2$.*

4.1 Properties of the spaces

In this section, we establish the boundedness of boundary point evaluations for $\beta > 2$, which will be used to prove the necessity of $\beta \le 2$ in the main theorem. We first recall the following technical lemma from [12], which will be used in the proof of Proposition 36.

Lemma 35 (Laplace's method) *Let $I(\lambda) = \int_a^b e^{-\lambda h(x)} g(x)\,dx$. Assume that $h \in \mathcal{C}^2[a,b]$ and $h \ge 0$ has a unique global minimum at $x_0 \in (a,b)$, such that $h''(x_0) > 0$, $g(x_0) \ne 0$, and g is continuous on $[a,b]$. Then, as $\lambda \to \infty$,*

$$I(\lambda) \sim \sqrt{\frac{2\pi}{\lambda h''(x_0)}}\, e^{-\lambda h(x_0)} g(x_0).$$

Proposition 36 *Let $\zeta \in \partial\Omega_p$ for $p \ge 1$; then, the point evaluation functional $\Gamma_\zeta : D_\beta(\Omega_p) \to \mathbb{C}$, $f \mapsto f(\zeta)$ is bounded for $\beta > 2$.*

Proof Let J be a multi-index $f(z) = \sum_J a_J z^J \in D_\beta(\Omega_p)$; and applying the Hölder inequality on $|\sum_J a_J \zeta^J|^2$, we will have

$$\begin{aligned}
\left|\sum_J a_J \zeta^J\right|^2 &\le \left(\sum_J |a_J|^2 \frac{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)}{\Gamma\left(\frac{j_1+j_2}{p}+2\right)}(|J|+2)^\beta\right) \\
&\quad \cdot \left(\sum_J \frac{1}{(|J|+2)^\beta} \frac{\Gamma\left(\frac{j_1+j_2}{p}+2\right)}{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)} |\zeta^J|^2\right) \\
&= \|f\|^2_{D_\beta(\Omega_p)} \left(\sum_J \frac{1}{(|J|+2)^\beta} \frac{\Gamma\left(\frac{j_1+j_2}{p}+2\right)}{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)} |\zeta^J|^2\right). \\
&\asymp \|f\|^2_{D_\beta(\Omega_p)} \left(\sum_{j\ge 0} \frac{1}{(j+2)^{\beta-1}} \sum_{j_1+j_2=j} \frac{\Gamma\left(\frac{j_1+j_2}{p}+1\right)}{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)} |\zeta_1|^{2j_1}|\zeta_2|^{2j_2}\right).
\end{aligned}$$

Now, the claim is that

$$S(j) = \sum_{j_1+j_2=j} \frac{\Gamma\left(\frac{j_1+j_2}{p}+1\right)}{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)} |\zeta_1|^{2j_1}|\zeta_2|^{2j_2} \le C, \tag{12}$$

where $C > 0$ is a positive constant independent of j. Note that this finishes the proof as we will have

$$\left|\sum_J a_J \zeta^J\right|^2 \lesssim \|f\|^2_{D_\beta(\Omega_p)} \left(\sum_{j\ge 0} \frac{1}{(j+2)^{\beta-1}}\right).$$

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And the sum is finite when $\beta > 2$.

To prove the claim, we compare the sum in (12) with the following integral:

$$\begin{aligned} I(\lambda) &= \int_0^\lambda \frac{\Gamma(\lambda+1)}{\Gamma(x+1)\Gamma(\lambda-x+1)} r^x (1-r)^{\lambda-x}\, dx \\ &\lesssim \int_0^\lambda \frac{\lambda^\lambda}{x^x(\lambda-x)^{\lambda-x}} \sqrt{\frac{\lambda}{x(\lambda-x)}}\, r^x (1-r)^{\lambda-x}\, dx \\ &= \sqrt{\lambda} \int_0^1 \left(\frac{1}{y^y(1-y)^{1-y}} r^y (1-r)^{1-y} \right)^\lambda \sqrt{\frac{1}{y(1-y)}}\, dy \\ &= \sqrt{\lambda} \int_0^1 e^{-\lambda h(y)} g(y)\, dy, \end{aligned}$$

where $\lambda = \frac{j}{p}$, $r = |\zeta_1|^{2p}$, $h(y) = y\log(\frac{y}{r}) + (1-y)\log(\frac{1-y}{1-r})$, and $g(y) = \sqrt{\frac{1}{y(1-y)}}$. Note that in the case of $r = 0$ or $r = 1$, $S(j) = 1$ and we get what we want; so we assume $r \in (0,1)$. Let a_λ denote the integrand defining $I(\lambda)$. By convexity of $\log\Gamma$, a_λ is unimodal, and so

$$\frac{1}{p}\sum_{k=0}^{j} a_\lambda(k/p) \le \int_0^\lambda a_\lambda(x)\, dx + \frac{2}{p}\max_{[0,\lambda]} a_\lambda.$$

By Stirling's inequalities, uniformly for $\lambda \ge 1$,

$$a_\lambda(x) \lesssim \sqrt{\frac{\lambda+1}{(x+1)(\lambda-x+1)}}\, e^{-\lambda h(x/\lambda)} \le 1$$

since $h \ge 0$, with its continuous endpoint values. Thus, $S(j) \lesssim I(\lambda) + 1$.

Fix $0 < \varepsilon < \min\{r, 1-r\}$. Since h has its unique minimum at r, with $h(r) = 0$ and $h''(r) = g(r)^2$, Lemma 35 gives

$$\sqrt{\lambda} \int_\varepsilon^{1-\varepsilon} e^{-\lambda h(y)} g(y)\, dy \longrightarrow \sqrt{2\pi}.$$

On the remaining intervals, $h \ge c := \min\{h(\varepsilon), h(1-\varepsilon)\} > 0$; so,

$$\sqrt{\lambda}\left(\int_0^\varepsilon e^{-\lambda h(y)} g(y)\, dy + \int_{1-\varepsilon}^1 e^{-\lambda h(y)} g(y)\, dy \right) \le \sqrt{\lambda} e^{-c\lambda} \int_0^1 g(y)\, dy \longrightarrow 0.$$

Hence, $I(\lambda)$, and therefore $S(j)$ remains bounded. ∎

4.2 Proof of Theorem 34

We begin with the following special case, which will be used in the proof of the main theorem.

Proposition 37 *Given $p \ge 1$, the polynomial $f(z) = 1 - r_1^{2p-1} z_1 - r_2^{2p-1} z_2$, where $r_1, r_2 \ge 0$ and $r_1^{2p} + r_2^{2p} = 1$ is cyclic in $D_\beta(\Omega_p)$ whenever $\beta \le 2$.*

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Proof

$$\begin{aligned}
\sup_{0<r<1}\left\|\frac{f}{f_r}\right\|^2_{D_\beta(\Omega_p)} &= \sup_{0<r<1}\big\|(1-r_1^{2p-1}z_1-r_2^{2p-1}z_2)\big(1+r\big(r_1^{2p-1}z_1\\
&\quad + r_2^{2p-1}z_2\big)+r^2\big(r_1^{2p-1}z_1+r_2^{2p-1}z_2\big)^2+\cdots\big)\big\|^2_{D_\beta(\Omega_p)}\\
&= \sup_{0<r<1}\left\|1+(1-r)\sum_{j\geq 1}r^{j-1}\big(r_1^{2p-1}z_1+r_2^{2p-1}z_2\big)^j\right\|^2_{D_\beta(\Omega_p)}\\
&= \sup_{0<r<1}\left\|1+(1-r)\sum_{j\geq 1}r^{j-1}\sum_{j_1+j_2=j}\binom{j}{j_1}r_1^{(2p-1)j_1}z_1^{j_1}r_2^{(2p-1)j_2}z_2^{j_2}\right\|^2_{D_\beta(\Omega_p)}\\
&= 1+\sup_{0<r<1}(1-r)^2\sum_{j\geq 1}r^{2(j-1)}\sum_{j_1+j_2=j}\binom{j}{j_1}^2\frac{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)}{\Gamma\left(\frac{j}{p}+2\right)}\\
&\quad\times(j+2)^\beta r_1^{2(2p-1)j_1}r_2^{2(2p-1)j_2}.
\end{aligned}$$

After applying Stirling's approximation to

$$\binom{j}{j_1}\frac{\Gamma\left(\frac{j_1}{p}+1\right)\Gamma\left(\frac{j_2}{p}+1\right)}{\Gamma\left(\frac{j}{p}+2\right)}(j+2),$$

we continue as follows:

$$\begin{aligned}
\sup_{0<r<1}\left\|\frac{f}{f_r}\right\|^2_{D_\beta(\Omega_p)} &\asymp 1+\sup_{0<r<1}(1-r)^2\sum_{j\geq 1}r^{2(j-1)}(j+2)^{\beta-1}\\
&\quad\times\sum_{j_1+j_2=j}\binom{j}{j_1}r_1^{2pj_1}r_2^{2pj_2}\frac{j^{j\left(1-\frac{1}{p}\right)}}{j_1^{j_1\left(1-\frac{1}{p}\right)}j_2^{j_2\left(1-\frac{1}{p}\right)}}r_1^{2(p-1)j_1}r_2^{2(p-1)j_2}\\
&= 1+\sup_{0<r<1}(1-r)^2\sum_{j\geq 1}r^{2(j-1)}(j+2)^{\beta-1}\\
&\quad\times\sum_{j_1+j_2=j}\binom{j}{j_1}r_1^{2pj_1}r_2^{2pj_2}\left(\frac{r_1^{2(p-1)\frac{j_1}{j}}r_2^{2(p-1)\frac{j_2}{j}}}{\left(\frac{j_1}{j}\right)^{\frac{j_1}{j}\left(1-\frac{1}{p}\right)}\left(\frac{j_2}{j}\right)^{\frac{j_2}{j}\left(1-\frac{1}{p}\right)}}\right)^j.
\end{aligned}$$

Note that the function

$$F(x)=\frac{r_1^{2(p-1)x}r_2^{2(p-1)(1-x)}}{x^{x\left(1-\frac{1}{p}\right)}(1-x)^{(1-x)\left(1-\frac{1}{p}\right)}}$$

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in $[0,1]$ attains its maximum at $x = r_1^{2p}$. Therefore,

$$\begin{aligned}
&\sup_{0<r<1} \left\| \frac{f}{f_r} \right\|^2_{D_\beta(\Omega_p)} \\
&\lesssim 1 + \sup_{0<r<1} (1-r)^2 \sum_{j\geq 1} r^{2(j-1)} (j+2)^{\beta-1} \sum_{j_1+j_2=j} \binom{j}{j_1} r_1^{2pj_1} r_2^{2pj_2} \\
&= 1 + \sup_{0<r<1} (1-r)^2 \sum_{j\geq 1} r^{2(j-1)} (j+2)^{\beta-1} \\
&\leq 1 + \sup_{0<r<1} (1-r)^2 \sum_{j\geq 1} r^{2(j-1)} (j+2) = 1 + \sup_{0<r<1} \frac{3-2r^2}{(1+r)^2} < +\infty, \qquad \beta \leq 2.
\end{aligned}$$

Now, Theorem 21 finishes the proof. ∎

Proof of Theorem 34 Let $\{\zeta_j\}_{j=1}^k = \mathcal{Z}(q) \cap \partial\Omega_p$, and let

$$s_j(z) = 1 - \sum_{\ell=1}^{2} |\zeta_{j,\ell}|^{2p-2} \overline{\zeta_{j,\ell}} z_\ell,$$

with zero coefficients at zero coordinates. Up to coordinatewise rotations, these are the polynomials in Proposition 37, vanishing on $\partial\Omega_p$ only at ζ_j. Since $s_j(\zeta_j) = 0$, Cauchy–Schwarz gives

$$\operatorname{dist}(z, \zeta_j) \geq C_j |s_j(z)|, \qquad C_j = \left(\sum_{\ell=1}^{2} |\zeta_{j,\ell}|^{4p-2} \right)^{-1/2}.$$

On the other hand, the Łojasiewicz's inequality [17] gives a positive integer m and a constant $C > 0$ such that

$$|q(z)| \geq C \operatorname{dist}(z, \mathcal{Z}(q) \cap \partial\Omega_p)^m,$$

for any $z \in \partial\Omega_p$. Choosing ζ_j nearest to z and bounding the remaining factors by their supremum norms, we obtain

$$|q(z)| \geq C' \prod_{j=1}^{k} |s_j(z)|^m, \qquad C' = C \left(\min_{1\leq j\leq k} \frac{C_j}{\prod_{i\neq j} \|s_i\|_\infty} \right)^m > 0,$$

for any $z \in \partial\Omega_p$, where the supremum norms are taken over $\partial\Omega_p$. By increasing m sufficiently, we may assume that

$$Q(z) = \frac{\prod_{j=1}^{k} s_j(z)^m}{q(z)} \in D_\beta(\Omega_p).$$

Assume $\beta \leq 2$. By Proposition 37, each s_j is cyclic in $D_\beta(\Omega_p)$. Since the s_j are cyclic polynomials and hence multipliers, Lemma 22 implies that $qQ = \prod_{j=1}^{k} s_j^m$ is cyclic. Fix $\varepsilon > 0$, and let p_1 and p_2 be polynomials such that

$$\|p_1 qQ - 1\|_{D_\beta} < \frac{\varepsilon}{2}$$

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and

$$\|p_2 - Q\|_{D_\beta} < \|p_1 q\|^{-1}_{\mathrm{Mult}(D_\beta)} \frac{\varepsilon}{2},$$

where $\|p_1 q\|_{\mathrm{Mult}(D_\beta)}$ denotes the multiplier norm of $p_1 q$, and the inequality follows from the density of polynomials in D_β. Then, we have

$$\|p_1 p_2 q - 1\|_{D_\beta} \le \|p_1 p_2 q - p_1 q Q\|_{D_\beta} + \|p_1 q Q - 1\|_{D_\beta} < \varepsilon.$$

Conversely, if $\beta > 2$, choose $\zeta \in \mathcal{Z}(q) \cap \partial\Omega_p$. By Proposition 36, evaluation at ζ is bounded on $D_\beta(\Omega_p)$. Therefore, q is not cyclic by Remark 6. ∎

5 Some results on the bidisk

In this section, we study the cyclicity properties of polynomials in $A^2_{u,\alpha}(\mathbb{D}^2) = D_\beta(\mathbb{D}^2)$, $\beta = -(\alpha+1)$, where $u(z) = \log\max\{|z_1|, |z_2|\}$. We recall

$$\|z_1^{j_1} z_2^{j_2}\|^2_{D_\beta(\mathbb{D}^2)} = (j_1 + j_2 + 2)^\beta.$$

The following is the main result of this section.

Theorem 38 *Let f be a polynomial that does not vanish on $\mathbb{D}^2$. Then:*

(a) *If $\mathcal{Z}(f) \cap \mathbb{T}^2$ is nonempty and finite, then f is cyclic in $D_\beta(\mathbb{D}^2)$ for $\beta \le 3/2$, and noncyclic for $\beta > 2$.*

(b) *If $\dim(\mathcal{Z}(f) \cap \mathbb{T}^2) = 1$, then f is cyclic in $D_\beta(\mathbb{D}^2)$ if and only if $\beta \le 1$.*

(c) *For $\beta > 2$, f is cyclic in $D_\beta(\mathbb{D}^2)$ if and only if it does not vanish on $\overline{\mathbb{D}}^2$.*

The proof is given in Section 5.2.

On the polydisk, the classical weighted Bergman spaces $A^2_\alpha(\mathbb{D}^n)$, $\alpha \ge -1$, are defined by prescribing the norm of monomials z^L as

$$\|z^L\|^2_{A^2_\alpha(\mathbb{D}^n)} = (l_1 + 1)^{-(\alpha+1)} \dots (l_n + 1)^{-(\alpha+1)}.$$

We can also extend the definition to any $\alpha \in \mathbb{R}$ using the power series norm. In this case, we obtain the classical Dirichlet-type spaces $\mathfrak{D}_\beta(\mathbb{D}^n) = A^2_{-(\beta+1)}(\mathbb{D}^n)$. Bénéteau et al. [2] gave a full characterization of cyclic polynomials for these spaces of dimension 2.

Comparing our main result on the bidisk with the one in [2], we observe that, in our case, the cyclicity of a polynomial is completely determined by the dimension of its zero set on the distinguished boundary. In contrast, in [2], polynomials of the form $\zeta - z_i$ exhibit a different cyclicity behavior compared to other polynomials that vanish at infinitely many points on $\mathbb{T}^2$.

5.1 Properties of the spaces

We compare these spaces with the classical weighted Bergman spaces and establish some properties needed in the proof of Theorem 38.

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Proposition 39 [19, Proposition 3.5] *Let u and v be two continuous plurisubharmonic exhaustion functions for the domain $\Omega \subset \mathbb{C}^n$ and assume that $(dd^c u)^n$ is compactly supported; then, $A^2_{v,\alpha}(\Omega) \subset A^2_{u,\alpha}(\Omega)$ for any $\alpha \geq -1$.*

As a corollary of Proposition 39, we show that one cannot obtain the classical weighted Bergman spaces on the polydisk using the Poletsky–Stessin method.

Corollary 40 *There exists no continuous plurisubharmonic exhaustion function v for $\mathbb{D}^n$, $n \geq 2$ such that $A^2_{v,\alpha}(\mathbb{D}^n) = A^2_\alpha(\mathbb{D}^n)$.*

Proof Let $\varepsilon > 0$ and take $u(z) = \log\max\{|z_1|, \dots, |z_n|\}$. Also, let $f(z) = \sum_{|L|\geq 0} (l_1+1)^{\frac{\alpha-\varepsilon}{2}} \dots (l_n+1)^{\frac{\alpha-\varepsilon}{2}} z^L$. Then, one can write

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^n)} = \sum_L \frac{1}{(l_1+1)^{1+\varepsilon}} \cdots \frac{1}{(l_n+1)^{1+\varepsilon}} < \infty.$$

So, $f \in A^2_\alpha(\mathbb{D}^n)$. Also,

$$\begin{aligned} \|f\|^2_{A^2_{u,\alpha}(\mathbb{D}^n)} = \sum_L \frac{(l_1+1)^{\alpha-\varepsilon} \dots (l_n+1)^{\alpha-\varepsilon}}{(|L|+n)^{\alpha+1}} \geq \\ \sum_{l\geq 0} \frac{(l+1)^{\alpha-\varepsilon} \dots (l+1)^{\alpha-\varepsilon}}{(nl+n)^{\alpha+1}} \gtrsim \\ \sum_{l\geq 0} (l+1)^{(n-1)\alpha - n\varepsilon - 1}. \end{aligned} \tag{13}$$

Note that the right-hand side of (13) diverges whenever $\alpha > 0$ and $\varepsilon = \frac{n-1}{n}\alpha$; so, for such a case, $f \notin A^2_{u,\alpha}(\mathbb{D}^n)$. Now, Proposition 39 finishes the proof. ∎

Proposition 41 *If $\zeta \in \mathbb{T}^n$, then the point evaluation functional $\Gamma_\zeta : A^2_{u,\alpha}(\mathbb{D}^n) = D_\beta(\mathbb{D}^n) \to \mathbb{C}$, $f \mapsto f(\zeta)$ is bounded for $\beta > n$.*

Proof Let J be a multi-index $f(z) = \sum_J a_J z^J \in D_\beta(\mathbb{D}^n)$, applying the Hölder inequality on $|\sum_J a_J \zeta^J|^2$, we will have

$$\begin{aligned} \left|\sum_J a_J \zeta^J\right|^2 &\leq \left(\sum_J |a_J|^2 (|J|+n)^\beta\right) \cdot \left(\sum_J \frac{1}{(|J|+n)^\beta} |\zeta^J|^2\right) \\ &= \|f\|^2_{D_\beta(\mathbb{D}^n)} \left(\sum_J \frac{1}{(|J|+n)^\beta}\right). \end{aligned}$$

The last sum on the right-hand side is finite. Indeed, grouping the terms according to $j = |J|$, we obtain

$$\sum_{J\in\mathbb{N}^n} \frac{1}{(|J|+n)^\beta} = \sum_{j\geq 0} \binom{j+n-1}{n-1} (j+n)^{-\beta} \lesssim \sum_{j\geq 1} j^{n-1-\beta} < +\infty,$$

since $\beta > n$. This completes the proof. ∎

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Proposition 42 *$A^2_{u,\alpha}(\mathbb{D}^n) = D_\beta(\mathbb{D}^n)$ is an algebra when $\beta = -(\alpha+1) > n$.*

Proof Let $f(z) = \sum_{|L|\geq 0} a_L z^L$ and $g(z) = \sum_{|L|\geq 0} b_L z^L$ be holomorphic functions in $D_\beta(\mathbb{D}^n)$. Their product is

$$f(z)g(z) = \sum_{|L|\geq 0} \sum_{L_1+L_2=L} a_{L_1} b_{L_2} z^L.$$

The squared norm of $f \cdot g$ is

$$\|f \cdot g\|^2_{D_\beta(\mathbb{D}^n)} = \sum_{|L|\geq 0} (|L|+n)^\beta \left| \sum_{L_1+L_2=L} a_{L_1} b_{L_2} \right|^2.$$

Applying Hölder's inequality,

$$\left| \sum_{L_1+L_2=L} a_{L_1} b_{L_2} \right|^2 \leq \left(\sum_{L_1+L_2=L} \frac{1}{(|L_1|+n)^\beta (|L_2|+n)^\beta} \right) \cdot \left(\sum_{L_1+L_2=L} |a_{L_1}|^2 (|L_1|+n)^\beta |b_{L_2}|^2 (|L_2|+n)^\beta \right).$$

Thus,

$$\begin{aligned}
\|f \cdot g\|^2_{D_\beta(\mathbb{D}^n)} &\leq \sum_L (|L|+n)^\beta \left(\sum_{L_1+L_2=L} \frac{1}{(|L_1|+n)^\beta (|L_2|+n)^\beta} \right) \\
&\quad \cdot \left(\sum_{L_1+L_2=L} |a_{L_1}|^2 (|L_1|+n)^\beta |b_{L_2}|^2 (|L_2|+n)^\beta \right) \\
&\leq \sum_{|L|\geq 0} \left(\frac{|L|+n}{|L|+2n} \right)^\beta \sum_{L_1+L_2=L} \left(\frac{1}{(|L_1|+n)} + \frac{1}{(|L_2|+n)} \right)^\beta \\
&\quad \cdot \left(\sum_{L_1+L_2=L} |a_{L_1}|^2 (|L_1|+n)^\beta |b_{L_2}|^2 (|L_2|+n)^\beta \right) \\
&\leq \sum_{|L|\geq 0} \sum_{L_1+L_2=L} \left(\frac{1}{(|L_1|+n)} + \frac{1}{(|L_2|+n)} \right)^\beta \\
&\quad \cdot \left(\sum_{L_1+L_2=L} |a_{L_1}|^2 (|L_1|+n)^\beta |b_{L_2}|^2 (|L_2|+n)^\beta \right) \\
&\leq C_\beta \sum_{|L|\geq 0} \sum_{L_1+L_2=L} \left(\frac{1}{(|L_1|+n)^\beta} + \frac{1}{(|L_2|+n)^\beta} \right) \\
&\quad \cdot \left(\sum_{L_1+L_2=L} |a_{L_1}|^2 (|L_1|+n)^\beta |b_{L_2}|^2 (|L_2|+n)^\beta \right)
\end{aligned}$$

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$$\leq 2C_\beta \left(\sum_{|L|\geq 0} \frac{1}{(|L|+n)^\beta} \right) \cdot \left(\sum_{|L|\geq 0} \sum_{L_1+L_2=L} |a_{L_1}|^2 (|L_1|+n)^\beta |b_{L_2}|^2 (|L_2|+n)^\beta \right)$$
$$\leq 2C_\beta \sum_{|L|\geq 0} \frac{1}{(|L|+n)^\beta} \|f\|^2_{D_\beta(\mathbb{D}^n)} \|g\|^2_{D_\beta(\mathbb{D}^n)},$$

where $C_\beta = 2^{\beta-1}$ is obtained from Jensen's inequality. So, for $\beta > n$, $f \cdot g \in A^2_{u,\alpha}(\mathbb{D}^n)$. ∎

5.2 Proof of Theorem 38

We first prove part (c), and then turn to parts (a) and (b).

Proposition 43 *For $\beta > 2$, a polynomial f is cyclic in $D_\beta(\mathbb{D}^2)$ if and only if f does not vanish on $\overline{\mathbb{D}}^2$.*

Proof Proposition 41 implies that for $\beta > 2$, the point evaluation functional is bounded at any point of $\overline{\mathbb{D}}^2$. Therefore, by Remark 6, a cyclic function in $D_\beta(\mathbb{D}^2)$ cannot have any zero on $\overline{\mathbb{D}}^2$.

Conversely, if f does not vanish on $\overline{\mathbb{D}}^2$, then f and $1/f$ are multipliers. Thus, $1 = (1/f)f \in [f]$, and f is cyclic (see Remark 7). ∎

Proposition 44 *The function $f_0(z) = 1 - \frac{z_1+z_2}{2}$ is cyclic in $A^2_{u,\alpha}(\mathbb{D}^2) = D_\beta(\mathbb{D}^2)$, when $\beta = -(\alpha+1) \leq \frac{3}{2}$.*

Proof Let $F : \mathbb{D}^2 \to \mathbb{D}$ be $(z_1, z_2) \mapsto \frac{z_1+z_2}{2}$ and define $C_F : D_{\beta-\frac{1}{2}}(\mathbb{D}) \to D_\beta(\mathbb{D}^2)$ as $f \mapsto f \circ F$. Then, for $f(z) = \sum_{j\geq 0} a_j z^j$, we may write

$$\begin{aligned}
\|C_F(f)\|^2_{D_\beta(\mathbb{D}^2)} &= \sum_{j\geq 0} |a_j|^2 \frac{1}{2^{2j}} \|(z_1+z_2)^j\|^2_{D_\beta(\mathbb{D}^2)} \\
&= \sum_{j\geq 0} |a_j|^2 \frac{1}{2^{2j}} (j+2)^\beta \sum_{j_1=0}^{j} \binom{j}{j_1}^2 \\
&= \sum_{j\geq 0} |a_j|^2 \frac{1}{2^{2j}} \binom{2j}{j} (j+2)^\beta \qquad (14) \\
&\asymp \sum_{j\geq 0} |a_j|^2 \frac{1}{2^{2j}} \frac{2^{2j}}{(j+2)^{\frac{1}{2}}} (j+2)^\beta \\
&= \sum_{j\geq 0} |a_j|^2 (j+2)^{\beta-\frac{1}{2}} \\
&\asymp \|f\|^2_{D_{\beta-\frac{1}{2}}(\mathbb{D})}.
\end{aligned}$$

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This proves that C_F is bounded. Moreover, $B = \mathrm{Im}(C_F)$ is a closed subspace of $D_\beta(\mathbb{D}^2)$. This implies the cyclicity of f_0 whenever $\beta \le \frac{3}{2}$ since it is the image of the function $g(z) = 1 - z$ under C_F, which is cyclic in $D(\mathbb{D})$. In this case, there exists a sequence of polynomials $\{p_n\}$ such that $p_n g \to 1$ in $D(\mathbb{D})$, and the corresponding sequence $\{q_n = C_F(p_n)\}$ can be used to approximate 1 via $q_n f_0$ in $D_\beta(\mathbb{D}^2)$. ∎

Remark 8 The function $f(z) = 1 - \frac{z_1+z_2}{2}$ is noncyclic whenever $\beta > 2$ since it vanishes at $(1,1)$ and point evaluations are bounded on $\overline{\mathbb{D}^2}$ by Proposition 41. Proposition 44 establishes its cyclicity for $\beta \le \frac{3}{2}$, while the case $\frac{3}{2} < \beta \le 2$ remains open.

The proof of Proposition 37 cannot be applied to the bidisk by letting $p \to \infty$ since its estimates are not uniform in p.

Proof of Theorem 38*(a)* Let f be a polynomial with no zeros in $\mathbb{D}^2$ and a nonempty finite zero set $\{\zeta_j\}_{j=1}^k$ on $\mathbb{T}^2$. Put

$$s_j(z) = 1 - \frac{\overline{\zeta_{j,1}} z_1 + \overline{\zeta_{j,2}} z_2}{2}.$$

By rotation invariance and Proposition 44, each s_j is cyclic for $\beta \le \frac{3}{2}$.

As explained in [2, Section 2.2], a zero of f on $\partial\mathbb{D}^2 \backslash \mathbb{T}^2$ would force a factor $z_i - \zeta$, with $\zeta \in \mathbb{T}$, contradicting the finiteness of $\mathcal{Z}(f) \cap \mathbb{T}^2$. Thus,

$$\mathcal{Z}(f) \cap \overline{\mathbb{D}^2} = \{\zeta_j\}_{j=1}^k.$$

Łojasiewicz's inequality [17] gives

$$|f(z)| \gtrsim \mathrm{dist}(z, \{\zeta_j\}_{j=1}^k)^N, \qquad z \in \overline{\mathbb{D}^2},$$

for some $N > 0$. Since $|s_j(z)| \lesssim |z - \zeta_j|$, choosing m sufficiently large ensures that

$$Q = \frac{\prod_{j=1}^k s_j^m}{f}$$

and its first partial derivatives are bounded on $\mathbb{D}^2$, by the quotient rule and the preceding estimate. Hence, $Q \in D_2(\mathbb{D}^2) \subseteq D_\beta(\mathbb{D}^2)$ for $\beta \le \frac{3}{2}$.

By Lemma 22, $fQ = \prod_j s_j^m$ is cyclic. Choose polynomials $p_n \to Q$ in $D_\beta(\mathbb{D}^2)$. Since f is a multiplier, $f p_n \to fQ$, and so

$$D_\beta(\mathbb{D}^2) = [fQ] \subseteq [f].$$

For $\beta > 2$, evaluation at any ζ_j is a bounded nonzero functional annihilating $[f]$, by Proposition 41. Thus, f is not cyclic. ∎

Proposition 45 *If $f(z) = \zeta - z_i$, $i = 1, 2$, where $\zeta \in \mathbb{T}$, then f is cyclic in $D_\beta(\mathbb{D}^2)$ if and only if $\beta \le 1$.*

Proof Without loss of generality, we can prove the statement only for $f(z) = 1 - z_1$. Assume that $\beta \le 1$. To prove the cyclicity of f, as in Proposition 37, we need to estimate

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the norm of f/f_r and then apply Theorem 21:

$$\sup_{0<r<1}\left\|\frac{f}{f_r}\right\|^2_{D_\beta(\mathbb{D}^2)} = 1+\sup_{0<r<1}(1-r)^2\sum_{j\geq 1}(j+2)^\beta r^{2j-2} \asymp 1+\sup_{0<r<1}(1-r)^{1-\beta} < +\infty.$$

Now let $\beta > 1$; our goal is to show that the capacity of $\mathcal{Z}(f)\cap\mathbb{T}^2$ is positive to apply Theorem 33. Assuming that μ is the uniform probability measure on $\{1\}\times\mathbb{T} = \mathcal{Z}(f)\cap\mathbb{T}^2$, we will have

$$\begin{aligned} I_\beta[\mu] &= \iint_{\mathbb{T}^2\times\mathbb{T}^2} k_\beta(z,w)\,d\mu(z)\,d\mu(w) \\ &= \iint_{\mathbb{T}\times\mathbb{T}}\sum_{|J|\geq 0}\frac{z_2^{j_2}\bar{w}_2^{j_2}}{(|J|+2)^\beta}\,d\mu(z_2)\,d\mu(w_2) = \sum_{j_1\geq 0}\frac{1}{(j_1+2)^\beta} < +\infty. \end{aligned}$$

This implies that the capacity of $\mathcal{Z}(f)\cap\mathbb{T}^2$ is positive. ∎

Proof of Theorem 38 (b) Recall that the classical spaces $\mathfrak{D}_\alpha(\mathbb{D}^2)$ have squared monomial norms

$$\|z_1^{j_1}z_2^{j_2}\|^2_{\mathfrak{D}_\alpha} = (j_1+1)^\alpha(j_2+1)^\alpha.$$

By [2, Main Theorem], an irreducible polynomial depending on both variables, non-vanishing in $\mathbb{D}^2$ and vanishing at infinitely many points of $\mathbb{T}^2$, is cyclic in $\mathfrak{D}_\alpha$ if and only if $\alpha \leq \frac{1}{2}$. We adapt their argument to D_β.

Assume $\beta \leq 1$. Factors with a nonempty finite zero set on $\mathbb{T}^2$ are cyclic by part (a). Factors with no zeros on $\mathbb{T}^2$ have no zeros on $\overline{\mathbb{D}^2}$, by [2, Section 2.2], and are cyclic by part (c) and the inclusions between the spaces. Thus, by Lemma 22, it remains to consider irreducible factors with infinitely many zeros on $\mathbb{T}^2$. The case $\zeta - z_i$ has already been treated, so assume that f depends on both variables.

Let (n,m) be the bidegree of f. By [2, Section 2.2 and Proposition 4.2], there exist a unitary $(n+m)\times(n+m)$ matrix U and a column vector polynomial $B\in\mathbb{C}^{n+m}[z_1,z_2]$ such that

$$(I-A(z))B(z)\in f(z)\mathbb{C}^{n+m}[z_1,z_2], \qquad A(z) = U\begin{bmatrix} I_n z_1 & 0 \\ 0 & I_m z_2\end{bmatrix}.$$

Moreover, there exists a row vector polynomial a such that $p(z_2) = a(z)B(z)$ is a one-variable polynomial with no zeros in $\mathbb{D}$.

Let $g\in[f]^\perp$ and let $v\in\mathbb{C}^{n+m}[z_1,z_2]$ be a row vector polynomial. Since

$$(I-A^k)B = \sum_{\ell=0}^{k-1}A^\ell(I-A)B \in f\mathbb{C}^{n+m}[z_1,z_2],$$

we obtain

$$\langle g, vB\rangle = \langle g, vA^kB\rangle, \qquad k\geq 0.$$

Put $d = \deg v + \deg B + 1$, where degrees are total degrees, taking the maximum over the entries. Since A is homogeneous of degree one, the monomials in $vA^{kd}B$ have degrees in $[kd,(k+1)d)$. Hence, $\{vA^{kd}B\}_{k\geq 0}$ is orthogonal. Moreover, A is unitary

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on $\mathbb{T}^2$; so,

$$\|vA^{kd}B\|_{D_\beta}^2 \lesssim (k+1)^\beta \|vA^{kd}B\|_{H^2(\mathbb{D}^2)}^2 \lesssim (k+1)^\beta,$$

where the last estimate follows from $|vA^{kd}B| \le |v||B|$ on $\mathbb{T}^2$, with Euclidean norms on the vectors.

If some $vA^{kd}B = 0$, then $\langle g, vB\rangle = 0$. Otherwise, Bessel's inequality gives

$$\|g\|_{D_\beta}^2 \ge \sum_{k\ge 0} \frac{|\langle g, vA^{kd}B\rangle|^2}{\|vA^{kd}B\|_{D_\beta}^2} \gtrsim |\langle g, vB\rangle|^2 \sum_{k\ge 0} \frac{1}{(k+1)^\beta}.$$

Since the series diverges for $\beta \le 1$, $\langle g, vB\rangle = 0$. Taking $v = z_1^{j_1} z_2^{j_2} a$, we obtain $g \perp [p]$. Since the restriction of the norm to functions of z_2 alone is equivalent to the $D_\beta(\mathbb{D})$ norm, one-variable cyclicity gives polynomials $p_j(z_2)$ such that $p_j p \to 1$ in $D_\beta(\mathbb{D}^2)$. Thus, p is cyclic, and $g = 0$.

Conversely, assume $\beta > 1$. Some irreducible factor h of f has infinitely many zeros on $\mathbb{T}^2$. If h is a constant multiple of $\zeta - z_i$, it is noncyclic by the preceding proposition. Otherwise, by [2, Main Theorem], h is noncyclic in $\mathfrak{D}_{\beta/2}(\mathbb{D}^2)$. Since

$$(j_1+1)^{\beta/2}(j_2+1)^{\beta/2} \le (j_1+j_2+2)^\beta, \qquad \|\cdot\|_{\mathfrak{D}_{\beta/2}} \le \|\cdot\|_{D_\beta},$$

h is also noncyclic in $D_\beta(\mathbb{D}^2)$. Finally, $[f] \subseteq [h] \ne D_\beta(\mathbb{D}^2)$, and so f is noncyclic. ∎

6 Open problems

Theorem 38, part (a), leaves the range $3/2 < \beta \le 2$ undecided for polynomials with a nonempty finite zero set on $\mathbb{T}^2$. At the time of submission, this was open even for the following polynomial. Resolving this case would complete the characterization of cyclic polynomials in $D_\beta(\mathbb{D}^2)$.

Open Problem 1 *For $\frac{3}{2} < \beta \le 2$, determine whether the polynomial $f(z) = 1 - \frac{z_1+z_2}{2}$ is cyclic in $D_\beta(\mathbb{D}^2)$.*

While this manuscript was under consideration, Nailwal and Zalar [18] answered this question affirmatively. Together with the results of Section 5, their result completes the characterization of cyclic polynomials in $D_\beta(\mathbb{D}^2)$.

For complex ellipsoids, Theorem 34 settles the case of a nonempty finite boundary zero set. To complete the characterization of cyclic polynomials in $D_\beta(\Omega_p)$, it remains to treat boundary zero sets of real dimension one. The following question asks for an extension of the corresponding result on the unit ball [16].

Open Problem 2 *Let f be a polynomial that does not vanish on Ω_p and vanishes on a set of real dimension one on $\partial\Omega_p$. Determine under what conditions f is cyclic in $D_\beta(\Omega_p)$.*

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