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Intermittency and random matrices

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A spectacular phenomenon of intermittency, i.e. a progressive growth of higher statistical moments of a physical field excited by an instability in a random medium, attracted the attention of Zeldovich in the last years of his life. At that time, the mathematical aspects underlying the physical description of this phenomenon were still under development and relations between various findings in the field remained obscure. Contemporary results from the theory of the product of independent random matrices (the Furstenberg theory) allowed the elaboration of the phenomenon of intermittency in a systematic way. We consider applications of the Furstenberg theory to some problems in cosmology and dynamo theory.

1. Introduction

One of many areas of scientific activity of Zeldovich was a spectacular phenomenon known as *intermittency* that arises from an instability developing in a random media. We recall briefly an illustrative example of intermittency taken from Zeldovich et al. (1987).

Consider a sequence of statistically independent random quantities c_n each of which is equal to 0 or 2 with probabilities $p = 1/2$, and let $a_n = c_1 c_2 \dots c_i \dots c_n$. We are interested in what happens to the statistical properties of a_n for $n \to \infty$. Of course, each realization of a_n tends to zero in any appropriate sense (e.g. with probability 1) just because sooner or later a c_i in the definition of a_n becomes zero. Nevertheless, for each finite n there is just one non-vanishing realization of a_n that is equal to 2^n. Probability of such realization is 2^{-n}. The point is that this realization gives a finite contribution to the mathematical expectation of a_n,

$$\langle a_n \rangle = 0 \times (1 - (1/2)^n) + 2^n \times (1/2)^n = 1, \tag{1.1}$$

and leads to an exponential growth of the rms value

$$\langle a_n^2 \rangle = 0 \times (1 - (1/2)^n) + 2^{2n} \times (1/2)^n = 2^n. \tag{1.2}$$

Higher statistical moments grow even faster than $\langle a_n^2 \rangle$.

The above example could be considered as a mathematical trick; however, it can be straightforwardly generalized to a more realistic problem (Zeldovich et al. 1985). Consider a bacterial population distributed in space with density $\phi(\mathbf{x})$. Let the growth (or decay) rate of the population be a random field $U(\mathbf{x})$. For the sake of definiteness, we suppose that at a given point U is a Gaussian random quantity with zero mean

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and dispersion σ^2, and ignore any spatial motion of the bacteria. Then,

$$\frac{\partial \phi}{\partial t} = U\phi. \tag{1.3}$$

If U is time-independent, then $\phi = \exp(Ut)$ (to simplify the notation, we take the initial distribution to be $\phi(0, \mathbf{x}) = 1$). A straightforward calculation shows that $\langle\phi\rangle = \exp(\sigma^2 t^2/2)$, i.e. the mean density of the bacterial population grows superexponentially. If U is time-dependent, say, constant in the time bins τ and statistically independent in different bins, then

$$\phi(n\tau, \mathbf{x}) = \exp\sum_{i=1}^{n} U_i(\mathbf{x})\tau \propto \exp(\sqrt{n\tau^2}\zeta) \propto \exp(\sqrt{t\tau}\zeta). \tag{1.4}$$

Here, U_i are realizations of the random field U in specific time-space bins, ζ is a Gaussian quantity with a mean zero and dispersion unity, and we use the central limit theorem. The bacterial population grows subexponentially. However, a straightforward calculation shows that the higher statistical moments grow exponentially and their growth rate normalized to the power k of statistical moment $\langle\phi^k\rangle$ grows with the power k. This phenomenon is known as intermittency.

Of course, the problem becomes more realistic if diffusion is included in (1.3) and the intermittency phenomena becomes even richer and more spectacular (e.g. Zeldovich et al. 1985; Sokoloff and Rubashny 2013). We however here focus our attention on just one detail of the problem – what kind of intermittency occurs for vectorial quantities?

Consider in the Lagrangian frame of reference the magnetic field $\mathbf{H}(t, \mathbf{x})$ in a random flow $\mathbf{v}(t, \mathbf{x})$ of a conducting media, neglecting Ohmic losses. The magnetic field evolution is governed by

$$\frac{\mathrm{d}\mathbf{H}}{\mathrm{d}t} = \mathbf{H}\hat{\boldsymbol{A}}, \tag{1.5}$$

where the matrix $\hat{\boldsymbol{A}}$ is constructed from the quantities $\partial v_i/\partial x_j$ and we consider the vector $\mathbf{H}$ as a row rather than as a column. Let the random flow $\mathbf{v}$ of incompressible fluids lose its memory at prescribed times $\tau, 2\tau, 3\tau, \ldots$. Then,

$$\mathbf{H}(n\tau) = \mathbf{H}(0)\hat{\boldsymbol{B}}_1\hat{\boldsymbol{B}}_2 \ldots \hat{\boldsymbol{B}}_n, \tag{1.6}$$

where $\hat{\boldsymbol{B}}_i$ are independent random matrices that give the magnetic field stretching by the flow in the memory time τ. If $\hat{\boldsymbol{A}}$ is time-independent during the memory time and $\hat{\boldsymbol{A}}(t) = \hat{\boldsymbol{A}}_i$ at the i-th time interval, then $\hat{\boldsymbol{B}}_i = \exp(\hat{\boldsymbol{A}}_i\tau)$. In a more realistic case of the time-dependent $\hat{\boldsymbol{A}}_i$ the matrices $\hat{\boldsymbol{B}}_i$ can be presented as ordered matrix exponents (multiplicative integrals), e.g. Zeldovich et al. (1988). The matrices $\hat{\boldsymbol{B}}_i$ are unimodular, i.e. $\det\hat{\boldsymbol{B}}_i = 1$. This follows from the incompressibility of the flow. We assume that the flow is statistically homogeneous in time, so the probability distribution for the matrices $\hat{\boldsymbol{B}}_i$ does not depend on i.

We see that (1.6) reduces the magnetic field evolution along a Lagrangian trajectory to a product of many independent random unimodular matrices. The relevant mathematical theory was developed by Furstenberg (1963), and presentation of this high-brow mathematical theory in terms of fluid mechanics can be found in Zeldovich et al. (1984).

The basic message from the Furstenberg theory is that the product of random matrices (1.6) grows exponentially in contrast to subexponential growth of a product of scalar random quantities (1.4). More precisely, when n becomes large, a spontaneous

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breaking of the symmetry occurs, a random basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ arises such that the vector $\mathbf{H}$ tends to line up in direction of $\mathbf{e}_1$ and grows exponentially with n with growth rate λ (Lyapunov exponent). Interpretation of the Furstenberg theory as a manifestation of a spontaneous breaking of symmetry was suggested by Zeldovich during preparation of the paper Zeldovich et al. (1984) and rejected that time by his younger however more conservative co-authors.

Apart from the growth of $\mathbf{H}$ according to (1.6), intermittency excites exponential growth of statistical moments $\langle|\mathbf{H}|^p\rangle$, their growth rate exceeds λ and (after appropriate normalization to the number of the moments) the normalized growth rates λ_p become larger and larger with p.

Furstenberg (1963) demonstrated how the growth rate λ can be calculated from an invariant measure and suggested an integral equation for this quantity. The point however is that this equation can be solved explicitly in some special cases only, and these cases are quite remote from physical problems. Numerical attempts to solve the equation were absent, which obviously limited physical applications of the theory.

The integral equation for the invariant measure is actually quite specific. Now numerical approaches to its solution have been developed, at least for the simplest cases. Below we review corresponding results, which tend to be hidden in mathematical papers.

2. Light propagation in a statistically inhomogeneous Universe

Evolution of magnetic field frozen in to a random flow of electrically conducting fluid is far from being the only physical problem that can be reduced to the product of independent random matrices. Remarkably, Zeldovich (1964) recognized another such problem from the field of cosmology that provides a simple and instructive mathematical illustration of the general problem.

Consider a Universe which is spatially homogeneous with $\bar{K}$ is curvature of spatial section which corresponds to the mean density. This curvature can be obtained from astronomical observational data based on the relation between the angular size of a light source and the distance to this source (e.g. Zeldovich and Novikov 1983). Zeldovich (1964) found that for a Universe homogeneous on average only and with average curvature $\bar{K}$, the same observational tests give for the curvature a quantity which is systematically lower than $\bar{K}$. In other words, due to curvature fluctuations (say, density fluctuations) the Universe looks closer to an open model than it really is. If, say, the average density of the Universe is equal to the critical density, as it is probably true, an observer would conclude that the Universe is slightly hyperbolic. The paper Zeldovich (1964) played an important role in the early investigations in gravitation lensing and is recognized by contemporary investigators; however, the effect discussed is very small quantitatively, in the sense that it is much easier to get lensing from a compact object than to get it as a result of the cumulative action of many weak density perturbations, the lensing effect of each of which individually is negligible (e.g. Ivanova and Khovanskaya 2005). Nevertheless, Zeldovich liked this paper, and included it in his books (e.g. Zeldovich and Novikov 1983) and in his selected papers (Zeldovich 1964). From the contemporary viewpoint, the paper seems to be an instructive contribution to Riemannian geometry.

Let $\gamma(\theta)$ be a family of geodesic lines on a Riemannian manifold which pass through a point P, and x be a distance measured from the point P along geodesics. We are interested in behaviour of the family near the geodesic with $\theta = 0$. For the sake of simplicity, we consider a 2D manifold only (for a general case, see Gromol

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et al. 1968). Then,

$$y(x) = \frac{\partial \gamma(x,\theta)}{\partial \theta}\Big|_{\theta=0} \tag{2.1}$$

is known as the Jacobi field and is governed by the following equation:

$$y'' + K(x)y = 0, \tag{2.2}$$

the Jacobi equation or equation for geodesic deviations. Here, prime means derivative in respect to x and $K(x)$ is the Gaussian curvature at the point x.

Classical observational cosmological tests can be presented in terms of scaling of y with respect to x. In particular, if K is negative, y grows exponentially with x, and the growth rate gives K. If K is positive, y oscillates with x and the oscillation frequency gives K.

Zeldovich (1964)'s intuition means that if $K(x)$ is a random process with a zero average, then y still grows (due to the curvature fluctuations) exponentially with x and the growth rate (Lyapunov exponent) gives the effective curvature, which is negative. The main bulk of ideas in Riemannian geometry deals with manifolds with curvature that are positive or negative everywhere and is based on the above behaviour of the Jacobi field. At the moment, Zeldovich's idea is the only one available to suggest how to investigate manifolds where curvature is somewhere positive and somewhere negative.

Zeldovich (1964) also recognized that intermittent behaviour of y is expected. At that time, however, the mathematical tools to investigate equations with random coefficients were almost unknown, and he exploited an argument adopting some particular features of cosmology. We demonstrate how to reduce the problem to a product of random matrices.

Assume that the random process $K(x)$ is constant at the intervals between the points $x = (n-1)l$ and $x = nl$, where l can be referred to as a memory scale, $K(x) = K_n$. The constants at each of such intervals are independent random quantities with zero mean and dispersion σ^2. Their statistical properties are supposed to be identical. Introduce a 2D row vector $\mathbf{z} = (y, ly')$, then

$$\mathbf{z}' = \mathbf{z}\hat{\boldsymbol{A}}, \tag{2.3}$$

where $\hat{\boldsymbol{A}} = \begin{pmatrix} 0 & -lK(x) \\ 1/l & 0 \end{pmatrix}$. Solving (2.3), we obtain

$$\mathbf{z}(nl) = \mathbf{z}(0)\hat{\boldsymbol{B}}_1\hat{\boldsymbol{B}}_2 \ldots \hat{\boldsymbol{B}}_n, \tag{2.4}$$

where $\hat{\boldsymbol{B}}_i = \exp \hat{\boldsymbol{A}}_i$. The explicit form of the matrix exponent depends on the sign of K_i. If $K_i \geqslant 0$, then

$$\hat{\boldsymbol{B}}_i = \begin{pmatrix} \cos\sqrt{K_i} & -l\sqrt{K_i}\sin\sqrt{K_i} \\ \sin\sqrt{K_i}/l\sqrt{K_i} & \cos\sqrt{K_i} \end{pmatrix}.$$

In the case $K_i < 0$

$$\hat{\boldsymbol{B}}_i = \begin{pmatrix} \cosh\sqrt{-K_i} & l\sqrt{-K_i}\sinh\sqrt{-K_i} \\ \sinh\sqrt{-K_i}/l\sqrt{-K_i} & \cosh\sqrt{-K_i} \end{pmatrix}.$$

Below we use l as a length unit for the Jacobi equation.

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3. Invariant measure for the Jacobi equation

Further analysis of the problem deals with a product $\hat{\boldsymbol{B}}(n) = \hat{\boldsymbol{B}}_1\hat{\boldsymbol{B}}_2 \ldots \hat{\boldsymbol{B}}_n$ of independent random matrices from the group of real unimodular matrices $SL(2, \mathbb{R})$. We follow here a method (Furstenberg 1963; Tutubalin 1977) which is applicable after obvious adjustment to the general case rather than to the Jacobi equation.

A straightforward calculation shows that each matrix $\hat{B}$ from the group $SL(2, \mathbb{R})$ can be presented as a product $\hat{\boldsymbol{Z}}\hat{\boldsymbol{D}}\hat{\boldsymbol{O}}$, where $\hat{\boldsymbol{Z}}$ is an upper triangular matrix with units on the diagonal, $\hat{\boldsymbol{D}}$ is a diagonal matrix $\mathrm{diag}(\lambda^{-1}, \lambda)$ with positive elements on the diagonal, and $\hat{\boldsymbol{O}}$ is an orthogonal matrix that corresponds to a rotation in 2D by some angle $0 \leqslant \varphi < 2\pi$.

Let

$$\hat{\boldsymbol{B}}(n) = \hat{\boldsymbol{Z}}(n)\hat{\boldsymbol{D}}(n)\hat{\boldsymbol{O}}(n)\,. \tag{3.1}$$

First, we see what happens with this representation after multiplication by the matrix $\hat{\boldsymbol{B}}_{n+1}$, given by

$$\hat{\boldsymbol{B}}(n+1) = \hat{\boldsymbol{B}}(n)\hat{\boldsymbol{B}}_{n+1} = \hat{\boldsymbol{Z}}(n)\hat{\boldsymbol{D}}(n)(\hat{\boldsymbol{O}}(n)\hat{\boldsymbol{B}}_{n+1})\,. \tag{3.2}$$

Applying the decomposition of $\hat{\boldsymbol{O}}(n)\hat{\boldsymbol{B}}_{n+1}$ in the form $\hat{\boldsymbol{Z}}'\hat{\boldsymbol{D}}'\hat{\boldsymbol{O}}(n+1)$ to (3.2), we get finally

$$\hat{\boldsymbol{B}}(n+1) = (\hat{\boldsymbol{Z}}(n)\hat{\boldsymbol{Z}}'')(\hat{\boldsymbol{D}}(n)\hat{\boldsymbol{D}}')\hat{\boldsymbol{O}}(n+1), \tag{3.3}$$

where $\hat{\boldsymbol{Z}}'' = \hat{\boldsymbol{D}}(n)\hat{\boldsymbol{Z}}'(\hat{\boldsymbol{D}}(n))^{-1}$.

Detailed investigation (Tutubalin 1977) shows that the sequence $\hat{\boldsymbol{O}}(n)$ generates an ergodic Markovian chain on a special space, which is a unit sphere S^1 with identified diametrically opposite points, so its probability distribution tends to a stationary distribution that is independent of n, while the matrix $\hat{\boldsymbol{Z}}''$ is close to the identity matrix, and the sequence $\hat{\boldsymbol{Z}}(n)$ converges to random limit. In turn, the growth rate of the lower diagonal element $n^{-1}\log\lambda(n)$ tends to a positive deterministic quantity λ known as the leading Lyapunov exponent. According to the central part of the Furstenberg theory, we expect to obtain $\lambda > 0$. The last statement in the background of fluid mechanics is presented by Zeldovich et al. (1984) and implies that solution of Jacobi equation grows exponentially with probability 1. Furstenberg theory can be considered as a generalization of the law of large numbers.

The leading Lyapunov exponent λ can be obtained as follows. Let the initial distribution of φ coincide with the stationary distribution π, then all statistical properties of the Markovian chain become independent of n, so

$$\lambda = \mathrm{E}\log\|\mathbf{w}(\varphi)\hat{\boldsymbol{B}}\|, \tag{3.4}$$

where E is the mathematical expectation (mean value) taken over the distribution of unit vector $\mathbf{w}(\varphi) = (\cos\varphi, \sin\varphi)$ and the distribution of the matrix $\hat{\boldsymbol{B}}$, which coincides with the distribution of independent identically distributed (i.i.d.) random matrices $\hat{\boldsymbol{B}}_n$.

The stationary distribution π can be obtained as follows. According to its definition, the distribution π has to be invariant under action of the matrix $\hat{\boldsymbol{B}}$, so it has to satisfy the integral equation

$$\int_{-\pi/2}^{\pi/2} \pi(\varphi)\mathrm{p}(\varphi, \psi)\,\mathrm{d}\varphi = \pi(\psi), \tag{3.5}$$

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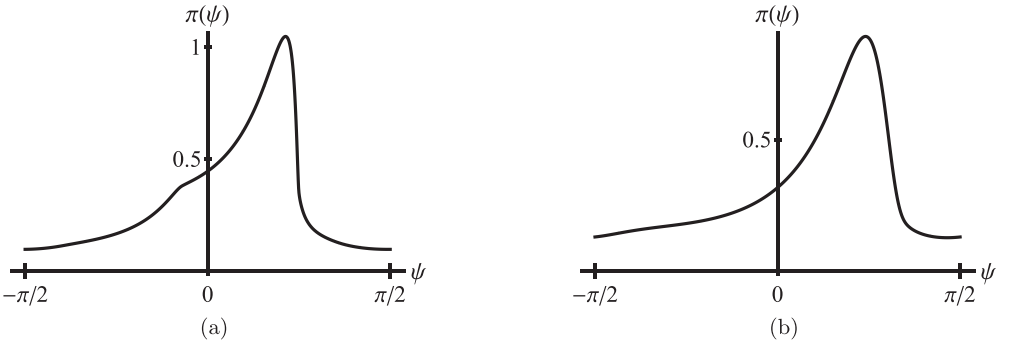


FIGURE 1. The stationary distribution for (a) uniform U[−1,−1] and (b) normal N(0,1) distributions of curvature parameter K.

where $\mathrm{p}(\varphi, \psi)$ is the transformation density of the Markovian chain under consideration, i.e. the probability density for going from the angle φ to the angle ψ after action of the matrix $\hat{\boldsymbol{B}}$.

We are going to learn how to solve (3.5) numerically and then obtain λ as a result of a straightforward integration in (3.4). As a by-product, we obtain that the leading Lyapunov exponent is positive, i.e. that solution of Jacobi equation grows exponentially with probability 1. In other words, the most sophisticated part of the Furstenberg theory that proves that $\lambda > 0$ is not very crucial for us.

Of course, one need some bulky, however, straightforward algebra to get $\mathrm{p}(\varphi, \psi)$ from probability density for random curvature K (see for details of calculations Illarionov et al. 2012; Illarionov 2013). Then, (3.5) can be solved numerically by iteration. We used for the numerical simulations the supercomputer of Moscow University (Sadovnichy et al. 2013).

We give here only some comments about this calculation. It is instructive to identify a point on the circle in 2D space under consideration with the angle φ as well as with the vector $\mathbf{w} = (\cos\varphi, \sin\varphi)$. Multiplying matrix $\hat{\boldsymbol{B}}$ by the vector $\mathbf{w}$, we obtain a new vector $\mathbf{w}' \equiv \mathbf{w}\hat{\boldsymbol{B}} = (\cos\psi, \sin\psi)$, where ψ depends on φ and the random quantity K, i.e. $\psi = \psi(\varphi, K)$. In order to get the probability density of a transition from the angle φ to the angle ψ, one have to consider the inverse mapping ψ to φ, fix φ and consider the result as a function of the random quantity K, and then sum the probability density of K over all values of φ that correspond to this given ψ.

The shape of the function $\mathrm{p}(\varphi, \psi)$ turns out to be to be quite complicated, it contains various sharp structures, so simulation requires a lot of mesh points and substantial computational facilities.

Nevertheless, the shape of the invariant density π obtained, say, for K homogeneously distributed in $[-1, 1]$ is rather simple (Fig. 1(a)). The corresponded Lyapunov exponents for K uniformly distributed in $[-1, 1]$ is $\lambda = 0.2132$ and for Gaussian K with zero mean and rms one $\lambda = 0.3137$. In all cases, the maximal or rms value of K is $1/l^2$.

The value of the Lyapunov exponent obtained is in agreement with corresponding estimate obtained by Mikhailov et al. (2010) in terms of direct numerical simulations of the Jacobi equation for a uniform distribution of K. The authors used for the simulation two numerical methods, on the one hand a standard Runge–Kutta method that gave $\lambda = 0.2133$, and on the other hand they obtained solutions of Jacobi equation by multiplying matrices $\hat{\boldsymbol{B}}_n$ that yielded $\lambda = 0.2131$.

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It should be noted that invariant measure is a useful tool for computation of Lyapunov exponent, but it is rather unclear how to obtain a finite expression for the growth rates γ_p of higher statistical moments. We discuss arising difficulties in Appendix A and show how to overcome them in several situations.

We remind that solution of Jacobi equation with random K provides a very simple illustrative example of intermittency. Simplicity of this example enables extensive simulations of a huge amount of statistically independent realizations of solution under consideration; however, it looks unrealistic for more complicated examples. It means that a numerical simulation of an intermittent solution becomes instructive only if a particular statistical property to be obtained from simulation is fixed in advance (Mogavero and Schekochihin 2014).

We demonstrated that the Jacobi field on the geodesic line in a Riemannian manifold with a random curvature grows exponentially, i.e. the distance between two geodesic lines starting at the same point grows exponentially. One can say that an observer interested in the spatial curvature of the universe would say that the effective value of the curvature as it comes from cosmological tests is negative (we recall that the average value of K vanishes). In other words, the universe as a whole can be considered as a lens with imaginary focal distance. In particular, the absolute value of the focal distance d depends on the choice: either we are interested in the lensing with probability 1 (then $|d| = l/\lambda$) or in lensing in terms of the averaging quantities (then $|d| = l/\gamma_1$), etc. This ambiguity in definition of d is a result of intermittency. We note that the curvature fluctuation in our illustrating example is many orders of magnitude higher than in the real Universe. It comes from the fact that fluctuations of the relic radiation are very small. As a result, realistic values of $|d|$ are larger than the horizon distance (Ivanova and Khovanskaya 2005).

4. Intermittent growth of magnetic field in a random flow

We now return to the field of fluid mechanics and consider magnetic field growth in a statistically homogeneous and isotropic random flow of electrically conducting fluid. Here, we use the Lagrangian approach, and restrict our analysis to the initial stage of magnetic field evolution, during which Ohmic losses can be neglected (see a discussion on how to use the results obtained in this approach for the dynamo problem in Arnold et al. 1981; Molchanov et al. 1984). In other words, we are going to investigate the growth rates of magnetic field $\mathbf{H}$ governed by (1.5).

Let the correlation tensor V_{ij} of the velocity field be homogeneous, isotropic and mirror-symmetric

$$V_{ij} = \langle v_i(\mathbf{x}),\, v_j(\mathbf{y})\rangle = \frac{v^2}{3}\left(\mathrm{F}(r^2)\delta_{ij} + \frac{r}{2}\mathrm{F}'(r^2)\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\right), \tag{4.1}$$

where $r = \|\mathbf{y}-\mathbf{x}\|$. For the sake of definiteness, we choose $\langle v_i(\mathbf{x}), v_i(\mathbf{y})\rangle = v^2\mathrm{f}(r^2)$ and $\mathrm{f}(r^2) = \exp(-r^2)$. This gives $\mathrm{F}(r^2) = \exp(-3r^2/5l^2)$, i.e. we assume that the random vortices are of the same scale l, this scale is chosen as a unit of length, and the renewal time τ chosen as a time unit coincides with the turnover time l/v. We assume that the vector $\mathbf{v}$ at different intervals of renewal are independent, and that the matrices $\hat{\boldsymbol{A}}_n$ are independent Gaussian matrices with zero mean.

In order to go further in investigating (1.5), we need to calculate correlations of elements of matrices $\hat{\boldsymbol{A}}_n$. According to our assumption, the correlations are n-independent. As $\hat{\boldsymbol{A}}_n$ is a 3×3 matrix, the correlations can be presented as a matrix $\boldsymbol{B}$, which is a 9×9 matrix and $\mathrm{corr}(A_{ik}, A_{jp}) = B_{3(i-1)+k,3(j-1)+p}$.

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Taking derivatives of V_{ij} in respect to x_k and y_p for $r = \|\mathbf{y} - \mathbf{x}\| \to 0$, we obtain (we drop the common factor $v^2/5l^2$)

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 4 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{4.2}$$

Because of the symmetry arguments, the invariant measure is just the Haar measure (speaking in less mathematical terms it means that distribution of $\|\mathbf{w}\exp\hat{\boldsymbol{A}}\|$ and does not depend on the point $\mathbf{w}$ on the 2D sphere, which follows from the isotropy assumption). This can be easily verified: for any orthogonal matrix $\boldsymbol{Q}$, the following sequence of random variables have matching distributions: $\|\mathbf{w}\boldsymbol{Q}\exp\hat{\boldsymbol{A}}\| \stackrel{\mathrm{d}}{=} \|\mathbf{w}\boldsymbol{Q}(\exp\hat{\boldsymbol{A}})\boldsymbol{Q}^{\boldsymbol{T}}\| \stackrel{\mathrm{d}}{=} \|\mathbf{w}\exp(\boldsymbol{Q}\hat{\boldsymbol{A}}\boldsymbol{Q}^{\boldsymbol{T}})\| \stackrel{\mathrm{d}}{=} \|\mathbf{w}\exp\hat{\boldsymbol{A}}\|$. It is a rare case when (3.5) can be obtained in an explicit form.

Then, the leading Lyapunov exponent λ can be obtained as

$$\lambda = \frac{1}{2\tau}\mathrm{E}\log\|\mathbf{w}\exp\hat{\boldsymbol{A}}\tau\|, \tag{4.3}$$

where the density of the invariant measure is the probability density of the unite vector $\mathbf{w} = (x, y, z)$ on the 2D sphere with identified diametrically opposite points, and τ is the renewal time (which is twice the memory time).

The calculation of the mean (4.3) is a separate mathematical problem, which is a topic of discussions. The key physical issue here is how realistic is the hypothesis that the matrix $\hat{A}$ is Gaussian in the Lagrangian frame of reference. In any case, a straightforward approach based on expansion of expression (4.3) as a Taylor series with respect to the renovation time τ needs only laborious algebra. The first non-vanishing approximation reads

$$\lambda = \frac{1}{2\tau}\mathrm{E}\log\left\|\mathbf{w}\left(\hat{\boldsymbol{I}} + \hat{\boldsymbol{A}}\tau + \frac{1}{2}\hat{\boldsymbol{A}}^2\tau^2\right)\right\|, \tag{4.4}$$

where $\hat{\boldsymbol{I}}$ is the identity matrix.

The rotation-invariant distribution of matrices $\hat{\boldsymbol{A}}$ (i.e. invariant to transformation $\boldsymbol{Q}\hat{\boldsymbol{A}}\boldsymbol{Q}^{\boldsymbol{T}}$ for any orthogonal matrix $\boldsymbol{Q}$) provides an exceptional case when growth rates of the pth statistical moment can be expressed in a finite form:

$$\lambda_p = \frac{1}{2p\tau}\log(\mathrm{E}\|\mathbf{w}\exp\hat{\boldsymbol{A}}\tau\|^p), \tag{4.5}$$

or in the first non-vanishing approximation:

$$\lambda_p = \frac{1}{2p\tau}\log\left(\mathrm{E}\left\|\mathbf{w}\left(\hat{\boldsymbol{I}} + \hat{\boldsymbol{A}}\tau + \frac{1}{2}\hat{\boldsymbol{A}}^2\tau^2\right)\right\|^p\right). \tag{4.6}$$

We give a proof of the expression (4.5) in Appendix A. Due to the Lyapunov inequalities for the statistical moments, the growth rate λ_p becomes larger for larger p, i.e. the magnetic field growth is intermittent.

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The calculation of λ_p is performed as follows. Let $\|\mathbf{w}(\hat{\boldsymbol{I}} + \hat{\boldsymbol{A}}\tau + \hat{\boldsymbol{A}}^2\tau^2/2)\|^2 = 1 + \alpha\tau + \beta\tau^2$, where we keep terms up to τ^2 only, α contains various linear combinations of matrix elements of $\hat{\boldsymbol{A}}$, β contains products of different matrix elements of $\hat{\boldsymbol{A}}$, and 1 comes from condition $x^2 + y^2 + z^2 = 1$. In particular,

$$\begin{aligned}\alpha = 2(&A_{11}x^2 + A_{12}xy + A_{21}xy + A_{13}xz + A_{31}xz \\ &+ A_{22}y^2 + A_{23}yz + A_{32}yz + A_{33}z^2)\end{aligned} \tag{4.7}$$

and we avoid giving a lengthier expression for β. Using the correlation properties (4.2) and taking into account that $x^2 + y^2 + z^2 = 1$, we obtain $\mathrm{E}\alpha^2 = 8v^2/5l^2$ (while $\mathrm{E}\alpha = 0$) and $\mathrm{E}\beta = 2v^2/l^2$. Since integration of the invariant measure can be easily performed, we can collect terms up to the order τ^2:

$$\lambda = \frac{1}{4\tau}\mathrm{E}\log(1 + \alpha\tau + \beta\tau^2) = \frac{1}{4\tau}\mathrm{E}\left(\alpha\tau + \beta\tau^2 - \frac{1}{2}\alpha^2\tau^2\right) = \frac{3}{10}\frac{v}{l}. \tag{4.8}$$

Similarly, we obtain

$$\lambda_p = \frac{1}{2p\tau}\log \mathrm{E}(1 + \alpha\tau + \beta\tau^2)^{p/2} = \frac{1}{2p\tau}\log(1 + (3p + p^2)\tau^2) = \frac{(3+p)}{10}\frac{v}{l}. \tag{4.9}$$

Equation (4.9) gives the same estimate λ as (4.8) for $p = 0$. In Appendix B, we reconstruct distribution and characteristic function of random variable $\|\mathbf{w}\exp\hat{\boldsymbol{A}}\tau\|$ for small τ and confirm the estimates of the growth rates under discussion by direct numerical simulations.

In particular, we obtain

$$\lambda_2 = \frac{1}{2}\frac{v}{l} \tag{4.10}$$

for the second moment (note that τ is twice the memory time).

The estimate (4.10) is slightly lower than the $3v/4l$ that follows from the Kazantsev equation for the short-correlated flow in the inviscid limit (e.g. Zeldovich et al. 1988). We explain the discrepancy by the fact that we used Lagrangian correlations while the Kazantsev equation is based on the Eulerian correlations; however, we appreciate that the point deserves further clarification.

On the other hand, generalization of the Kazantsev equation to include finite memory effects is a highly non-trivial undertaking because it leads to integral equations and it is far from clear how to solve such equations (e.g. Dittrich et al. 1984; Kleeorin et al. 2002). Recently, Bhat and Subramanian (2014) presented an effective approach in this field and found that at least to the lowest order in respect to the memory time the growth rate of magnetic energy coincides with the results known for the Kazantsev equation. Our (4.6) for $p = 2$ explains the nature of this fact and (4.5) opens a way to obtain higher order terms in the Taylor series of the growth rate of magnetic energy with respect to the memory time.

5. Discussion and conclusions

We have presented several problems concerning intermittent evolution of physical fields in random media addressed by Zeldovich and his school in the 1980s, and recent progress in the problems. Of course, formal similarity of the problems originating in various branches of physics such as cosmology, MHD and fluid mechanics, that was not very clear in those days, becomes more and more visible now. Some technical aspects that were hidden in not easy accessible Soviet and East German

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journals, and also results that were presented in scientific literature addressed to pure mathematicians, seem to contain meaningful physical information that opens a new perspective on investigations of intermittency of physical fields. In particular, the first attempts to base calculation of Lyapunov exponents, as well as growth rates of statistical moments of physical fields on invariant measure for the product of many independent random matrices, provide a new approach that can support traditional methods based on direct numerical simulations, as well as various mean-field equations.

Speaking broadly, we see that the papers of Zeldovich concerning the intermittent growth of physical fields continue to be a source for new physical ideas, and are not purely historic documents.

One more message from this review is the very broad way of thinking and combining ideas from various branches of physics typical of Zeldovich.

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Appendix A

Here, we obtain an expression for the growth rate of statistical moments in a special case when distribution of matrices is rotation independent and discuss difficulties arising in general situation. Let us denote $\xi_n = \|\hat{\boldsymbol{B}}(n)\mathbf{x}\|^p$ and $\eta_n = \log\|\hat{\boldsymbol{B}}(n)\mathbf{x}\|/n$, where $\hat{\boldsymbol{B}}(n) = \hat{\boldsymbol{B}}_n\hat{\boldsymbol{B}}_{n-1}\hat{\boldsymbol{B}}_1$. We remind the definition of the growth rate:

$$\gamma_p = \lim_{n\to\infty}\frac{1}{np}\log \mathrm{E}\,\xi_n, \tag{A 1}$$

and note how it may be expressed in terms of characteristic function $\chi_n(t)$ of random variable η_n defined as $\chi_n(t) = \mathrm{E}\exp(\mathrm{i}t\eta_n)$. Indeed, $\chi_n(t) = \mathrm{E}\exp(\mathrm{i}t\log\xi_n/np) = \mathrm{E}\,\xi_n^{\mathrm{i}t/np}$, hence

$$\gamma_p = \lim_{n\to\infty}\frac{1}{np}\log\chi_n(-\mathrm{i}np). \tag{A 2}$$

Then, we decompose η_n in a sum

$$\eta_n = \frac{1}{n}\left(\log\frac{\|\hat{\boldsymbol{B}}_n\hat{\boldsymbol{B}}_{n-1}\ldots\hat{\boldsymbol{B}}_1\mathbf{x}\|}{\|\hat{\boldsymbol{B}}_{n-1}\ldots\hat{\boldsymbol{B}}_1\mathbf{x}\|} + \log\frac{\|\hat{\boldsymbol{B}}_{n-1}\hat{\boldsymbol{B}}_{n-2}\ldots\hat{\boldsymbol{B}}_1\mathbf{x}\|}{\|\hat{\boldsymbol{B}}_{n-2}\ldots\hat{\boldsymbol{B}}_1\mathbf{x}\|} + \cdots + \log\|\hat{\boldsymbol{B}}_1\mathbf{x}\|\right). \tag{A 3}$$

Let us now assume that distribution of $\log\|\hat{\boldsymbol{B}}\mathbf{x}\|$, where the distribution of the matrix $\hat{\boldsymbol{B}}$ coincides with the distribution of matrices $\hat{\boldsymbol{B}}_n$, does not depend on choice of unit vector $\mathbf{x}$. Under this assumption, η_n is a sum of i.i.d. random variables with common distribution of $\eta_1 = \log\|\hat{\boldsymbol{B}}\mathbf{x}\|$. So characteristic function of η_n is just $\chi_n(t) = (\chi_1(t/n))^n$. Then,

$$\gamma_p = \lim_{n\to\infty}\frac{n}{np}\log\chi_1(-\mathrm{i}p) = \frac{1}{p}\log\mathrm{E}\|\hat{\boldsymbol{B}}\mathbf{x}\|^p. \tag{A 4}$$

The independence of distribution of $\log\|\hat{\boldsymbol{B}}\mathbf{x}\|$ on unit vector $\mathbf{x}$ is rather a degenerated situation. In general case, we should assume $\mathbf{x}$ to be a random vector under stationary distribution. This turns the sum in (A 3) into a sum of identically distributed variables, although they are not independent.

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The key point here is that as $n \to \infty$ a kind of the law of large numbers – the ergodic theorem – holds and η_n converges to $\lambda = \mathrm{E}\log\|\hat{\boldsymbol{B}}\mathbf{x}\|$ with probability 1. Furstenberg (1963) proved that in most cases λ is strictly positive, that implies an exponential growth of matrix norms.

To compute γ_p in general case, one needs to know the growth rate of

$$\chi_n(-\mathrm{i}np) = 1 + np\mathrm{E}\eta_n + \frac{(np)^2}{2!}\mathrm{E}\eta_n^2 + \cdots + \frac{(np)^k}{k!}\mathrm{E}\eta_n^k + \cdots . \tag{A 5}$$

For any k the limit with probability 1 of η_n^k as $n \to \infty$ is just λ^k and it looks attractive to estimate the convergence rate through the central limit theorem (CLT) or its generalizations, i.e. approximate η_n with normal distribution $\mathrm{N}(\lambda, \sigma^2/n)$. For example, if distribution of matrices $\hat{\boldsymbol{B}}_i$ is rotation invariant, CLT is obviously applicable to (A 3). We could conclude from this that $\chi_n(t) = \exp(-\mathrm{i}t\lambda - \sigma^2 t^2/2n)$ and

$$\gamma_p = \lim_{n\to\infty} \frac{1}{np}\left(\lambda np + \frac{\sigma^2 n^2 p^2}{2n}\right) = \lambda + \frac{\sigma^2 p}{2}. \tag{A 6}$$

Comparison of (A 6) with (A 4) gives us

$$\frac{1}{p}\log \mathrm{E}\|\hat{\boldsymbol{B}}\mathbf{x}\|^p = \lambda + \frac{\sigma^2 p}{2}. \tag{A 7}$$

This implies $\mathrm{E}\|\hat{\boldsymbol{B}}\mathbf{x}\|^p = \exp(\lambda p + \sigma^2 p^2/2)$ and random variable $\|\hat{\boldsymbol{B}}\mathbf{x}\|$ has a log-normal distribution $\mathrm{LogN}(\lambda, \sigma^2)$. So its logarithm $\log\|\hat{\boldsymbol{B}}\mathbf{x}\|$ has normal distribution $\mathrm{N}(\lambda, \sigma^2)$. The result looks to be in disagreement with arbitrary distribution of matrices $\hat{\boldsymbol{B}}$. The point is that replacing of (A 3) with normal distribution is an oversimplification.

So we came to conclusion, that even when usual CLT holds for the sum (A 3), approximation of (A 3) with normal distribution does not provide desired accuracy in determination of γ_p.

To understand why we can compute λ and have difficulties with γ_p, it is instructive to recall a definition on Lyapunov exponent:

$$\lambda = \lim_{n\to\infty} \mathrm{E}\eta_n, \tag{A 8}$$

and rewrite it in terms of characteristic functions:

$$\lambda = -\mathrm{i}\lim_{n\to\infty} \chi_n'(0). \tag{A 9}$$

We know from ergodic theorems that the sequence of χ_n converges to the characteristic function of Delta measure with mass point λ. Clearly, we can replace χ_n with its limit in (A 9) since the argument does not change; however, this trick does not work in (A 2), because we cannot separate taking limit of functions and its arguments. In other words,

$$\lim_{n\to\infty} \frac{1}{np}\log\chi_n(-\mathrm{i}np) \neq \lim_{n\to\infty} \frac{1}{np}\left(\lim_{m\to\infty}\log\chi_m\right)(-\mathrm{i}np). \tag{A 10}$$

We conclude from this short overview that investigation of higher statistical moments is a separate problem and we bear in mind that it needs to be addressed systematically elsewhere. However, we indicated an important case, when most of the problem can be overcome and the desired parameters take a finite form.

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Appendix B

Here, we give some notes on computational aspects. Recall that we decompose matrix exponent up to terms τ^2:

$$\|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|^2 = \|(\hat{\boldsymbol{I}} + \hat{\boldsymbol{A}}\tau + \frac{1}{2}\hat{\boldsymbol{A}}^2\tau^2)\|^2 = 1 + \alpha\tau + \beta\tau^2, \tag{B 1}$$

where $\mathrm{E}\alpha = 0$, $\mathrm{E}\alpha^2 = 8$ and $\mathrm{E}\beta = 10$. From this, we obtain

$$\mathrm{E}\|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|^p = 1 + (p^2 + 3p)\tau^2 \tag{B 2}$$

and

$$\mathrm{E}\log\|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\| = 3\tau^2. \tag{B 3}$$

We made a series of numerical simulations in order to reproduce distribution of $\xi^p = \|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|^p$ and extract its statistical properties when τ^2 is small (we tested $\tau = 10^{-1}, \ldots, 10^{-5}$). More exactly, we simulated 10^5 realizations of ξ^p for several first p's and computed mean value m_p and variance s_p^2 of the samples obtained. Since means are assumed to be in a form $m_p = 1 + b_p\tau^2$ and variances are $s_p^2 = c_p\tau^2$, we derive desired parameters as $b_p = (m_p - 1)/\tau^2$ and $c_p = s_p^2/\tau^2$. Obtained from simulations values of b_p were rather unstable (they varied from negative to positive values for repeated simulations); however, with a high accuracy we obtained $c_p = 2p^2$. This relation is exactly the same if we compute variance (Var) from (B 2). Indeed, $\mathrm{Var}\|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|^p = \mathrm{E}\|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|^{2p} - (\mathrm{E}\|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|^p)^2 = 1 + (4p^2 + 6p)\tau^2 - (1 + (p^2 + 3p)\tau^2)^2 = 2p^2\tau^2 + O(\tau^4)$. So we numerically confirm the approximation (B 2). The instability in values of b_p by contrast with c_p, in our opinion, is a result of some small systematic errors due to limiting computational accuracy, which vanish after centring of the sample.

Since we know all moments of $\xi = \|\exp(\hat{\boldsymbol{A}}\tau)\mathbf{x}\|$, we can reconstruct its characteristic function as

$$\chi_\xi(t) = 1 + \sum_{p=1}^{\infty} \frac{(\mathrm{i}t)^p}{p\,!}\mathrm{E}\xi^p. \tag{B 4}$$

Substitution of (B 2) into (B 4) results in

$$\chi_\xi(t) = 1 + \sum_{p=1}^{\infty} \frac{(\mathrm{i}t)^p}{p\,!}(1 + (p^2 + 3p)\tau^2) = e^{\mathrm{i}t}(1 + 4\mathrm{i}\tau^2 t - \tau^2 t^2). \tag{B 5}$$

Note that characteristic function obtained does not have a form of normal or log-normal laws. Fourier transform of $\chi_\xi(t)$ gives a quite unusual from of probability density function that contains Delta function and its derivatives:

$$\rho_\xi(x) = \delta(1 - x) - 4\tau^2\delta'(1 - x) + \tau^2\delta''(1 - x). \tag{B 6}$$

Now we can reproduce approximation (B 3) by integrating the density function (B 6):

$$\mathrm{E}\log\xi = \int_{-\infty}^{+\infty} \log(x)\rho_\xi(x)\,\mathrm{d}x = 3\tau^2. \tag{B 7}$$

We performed a numerical simulation of distribution of $\log\xi$ as well and observed sample variance close to $2\tau^2$ for small τ. Exactly the same approximation one can obtain from (B 3) or by integrating the density function (B 6).

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