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G-DIMENSIONS FOR DG-MODULES OVER COMMUTATIVE DG-RINGS

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Abstract We define and study a notion of G-dimension for DG-modules over a non-positively graded commutative noetherian DG-ring A. Some criteria for the finiteness of the G-dimension of a DG-module are given by applying a DG-version of projective resolution introduced by Minamoto [Israel J. Math. 245 (2021) 409-454]. Moreover, it is proved that the finiteness of G-dimension characterizes the local Gorenstein property of A. Applications go in three directions. The first is to establish the connection between G-dimensions and the little finitistic dimensions of A. The second is to characterize Cohen-Macaulay and Gorenstein DG-rings by the relations between the class of maximal local-Cohen-Macaulay DG-modules and a special G-class of DG-modules. The third is to extend the classical Buchweitz-Happel Theorem and its inverse from commutative noetherian local rings to the setting of commutative noetherian local DG-rings. Our method is somewhat different from classical commutative ring.

Keywords: Buchwtweiz-Happel Theory, G-dimension; Gorenstein DG-algebra; little finitistic dimension; maximal local-Cohen-Macaulay DG-module

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1. Introduction

The Gorenstein dimension, or G-dimension, for finitely generated modules over a commutative noetherian ring was introduced by Auslander [1] and was developed deeply by Auslander and Bridger [2]. The reason for the name is that G-dimension characterizes Gorenstein local rings exactly as projective dimension characterizes regular local rings. With that as a start, G-dimension has played an important role in singularity theory [18, 37], cohomology theory of commutative rings [5, 10] and representation theory of Artin algebras [3, 25]. Over a general ring, Enochs and Jenda [12] defined Gorenstein projective dimension for arbitrary modules. For finitely generated modules over commutative noetherian rings it coincides with the Auslander and Bridger's G-dimension. For

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left modules over arbitrary associative rings, Holm [16] proved that the new concept has the desired properties.

In a different direction, Yassemi [34] studied G-dimension for complexes over a commutative noetherian local ring through a consistent use of the RHom-functor of complexes and the related category of reflexive complexes, while Christensen [10] went straight for the throat and gave the definition in terms of resolutions. Thus, the two definitions are equivalent, and they are both rooted in a result — due to Foxby — saying that a finitely generated module has finite G-dimension if and only if it is reflexive as a complex in the sense defined in [34, Definition 2.4] (see also [10, Definition 2.1.6]).

Despite the great success of the G-dimension in commutative noetherian rings, until now it was completely missing from higher algebra. The aim of this paper is to introduce and study G-dimensions in derived commutative algebra. More specifically, we work with non-positively graded commutative DG-rings $A = \bigoplus_{i=-\infty}^{0} A^i$ with a differential of degree $+1$. These include the normalizations of simplicial commutative rings.

Given a commutative DG-ring A, the derived category of DG-modules over A will be denoted by $\mathrm{D}(A)$. Its full subcategory consisting of DG-modules with bounded cohomology will be denoted by $\mathrm{D}^{\mathrm{b}}(A)$. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(A)$ consisting of DG-modules with finitely generated cohomology. For a DG-module M, we set $\inf M = \inf\{n \mid \mathrm{H}^n(M) \neq 0\}$, $\sup M = \sup\{n \mid \mathrm{H}^n(M) \neq 0\}$ and $\mathrm{amp} M = \sup M - \inf M$.

Motivated by the G-dimension for complexes over a commutative noetherian ring defined in [10, 34], we introduce the definition of G-dimension for DG-modules.

Definition 1.1. *Let* A *be a noetherian DG-ring with* $\mathrm{amp} A < \infty$.

(1) *A DG-module* $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *is said to be* reflexive *if* $\mathrm{RHom}_A(X, A) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *and the morphism* $X \to \mathrm{RHom}_A(\mathrm{RHom}_A(X, A), A)$ *is an isomorphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

(2) *For a reflexive DG-module* X*, we define the* G-dimension *of* X*, denoted by* $\mathrm{G\text{-}dim}_A X$*, by the formula*

$$\mathrm{G\text{-}dim}_A X = \sup \mathrm{RHom}_A(X, A).$$

If X *is not reflexive, we say that it has infinite G-dimension and write* $\mathrm{G\text{-}dim}_A X = \infty$.

Let A be an ordinary ring. If X is an A-module, then one can show that this definition coincides with the usual definition of the G-dimension of X defined by Auslander in [1]; moreover, if X is an A-complex, then the G-dimension of X defined here is just the definition defined by Yassemi in [34, Definition 2.8] or by Christensen in [10, Definition 2.3.3 and Theorem 2.3.7].

We denote by $\mathcal{R}(A)$ for the full subcategory of $\mathrm{D}(A)$ consisting of reflexive DG-modules. A DG-module $X \in \mathcal{R}(A)$ is said to be in the *G-class* $\mathcal{G}$ if either $\mathrm{G\text{-}dim}_A X = -\sup X$, or $X = 0$, and denote by $\mathcal{G}_0$ the full subcategory of $\mathcal{G}$ consisting of objects G such that either $\mathrm{amp} G \geq \mathrm{amp} A$ and $\mathrm{G\text{-}dim}_A G = -\sup G = 0$, or $G = 0$ (see Definition 3.5). If we write $\mathcal{P} \subseteq \mathrm{D}(A)$ for the full subcategory of direct summands of a finite direct sums of A, i.e. $\mathcal{P} = \mathrm{add} A$, then it is easy to check $\mathcal{P} \subseteq \mathcal{G}_0$. If A is an ordinary ring and X an A-module, then $\mathcal{G} = \mathcal{G}_0$ is exactly the class of modules of G-dimension zero.

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Following [23], for any $0 \not\simeq X \in \mathrm{D}^+(A)$, a *sppj morphism* $f : P \to X$ is a morphism in $\mathrm{D}(A)$ such that $P \in \mathrm{Add}A[-\sup X]$ and the morphism $\mathrm{H}^{\sup X}(f)$ is surjective, where $\mathrm{Add}A$ is the full subcategory of direct summands of a direct sum of A. Moreover, a *sppj resolution* $P_\bullet$ of X is a sequence of exact triangles $X_{i+1} \overset{g_{i+1}}{\to} P_i \overset{f_i}{\to} X_i \rightsquigarrow$ such that f_i is a sppj morphism for $i \geq 0$ with $X_0 := X$. If A is noetherian and $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ then we can choose P_i in $\mathcal{P}$ by [23, Proposition 2.26].

We set $injdim_A A := \{n \in Z | \mathrm{Ext}^i_A(N, A) = 0 \ for\ any\ N \in \mathrm{D}^b(A)\ and\ i > n - \inf N\}$. Recall from [13, 14] that a commutative noetherian local DG-ring $(A, \bar{\mathfrak{m}}, \bar{k})$ is called *Gorenstein* if $\mathrm{amp}A < \infty$ and $\mathrm{injdim}_A A < \infty$.

Recall that for full subcategories $\mathcal{X}, \mathcal{Y} \subseteq \mathrm{D}(A)$, we define $\mathcal{X} * \mathcal{Y}$ to be the full subcategory consisting $Z \in \mathrm{D}(A)$ which fits into an exact triangle $X \to Z \to Y \rightsquigarrow$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Now, our main result can be stated as follows, which coveys that the standard techniques for classical commutative algebra can be generalized to commutative DG-algebras with appropriate modifications.

Theorem 1.2. *Let* $(A, \bar{\mathfrak{m}}, \bar{k})$ *be a commutative noetherian local DG-ring with* $\mathrm{amp}A < \infty$.

(1) *If* $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *with* $\mathrm{amp}X \geq \mathrm{amp}A$ *and* n *is a natural number, then the following are equivalent:*
   (i) $\mathrm{G\text{-}dim}_A X \leq n - \sup X$;
   (ii) X *has a sppj resolution* $P_\bullet$ *such that* $X_e \in \mathcal{G}_0[-\sup X_e]$;
   (iii) X *belongs to* $\mathcal{P}[-\sup X] * \cdots * \mathcal{P}[-\sup X + n - 1] * \mathcal{G}_0[-\sup X + n]$;
   (iv) X *belongs to* $\mathcal{G}_0[-\sup X] * \mathcal{P}[-\sup X + 1] * \cdots * \mathcal{P}[-\sup X + n]$.

(2) *If* $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *with* $\mathrm{amp}X < \mathrm{amp}A$ *and* n *is a natural number, then the following are equivalent:*
   (i) $\mathrm{G\text{-}dim}_A X \leq n + \inf A - \inf X$;
   (ii) $X \oplus X[\mathrm{amp}A - \mathrm{amp}X]$ *belongs to* $\mathcal{P}[-\sup X] * \cdots * \mathcal{P}[-\sup X + n - 1] * \mathcal{G}_0[-\sup X + n]$.

(3) *The following conditions are equivalent:*
   (i) A *is local Gorenstein;*
   (ii) $\mathrm{G\text{-}dim}_A \bar{k} < \infty$;
   (iii) $\mathrm{G\text{-}dim}_A X < \infty$ *for any* $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

A few comments on Theorem 1.2 are in order. First, Theorem 1.2 provides a criteria for a natural number to be an upper bound of the G-dimension of a DG-module over a commutative noetherian local DG-ring. One major difference from the case of rings is that $\mathrm{G\text{-}dim}_A X \geq -\sup X$ need not hold in the DG-setting (see Example 3.4), it leads to the proof of two key results, Lemmas 3.8 and 3.9, for obtaining such a characterization is rather different with the ordinary commutative rings.

Second, it should be pointed out that the inequality $\mathrm{amp}X < \mathrm{amp}A$ in Theorem 1.2(2) is very often met. For instance, assume that $(A, \bar{\mathfrak{m}}, \bar{k})$ is a commutative noetherian local DG-ring with $0 < \mathrm{amp}A < \infty$. If we choose $X = \bar{k}$ then $\mathrm{amp}X = 0 < \mathrm{amp}A$, as desired.

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Finally, if $(A, \bar{\mathfrak{m}}, \bar{k})$ is a local Gorenstein DG-ring with $0 < \operatorname{amp} A < \infty$, then $\bar{k}$ has finite G-dimension by Theorem 1.2(3), but by [20, Theorem 0.2], $\bar{k}$ never has finite projective dimension introduced by Bird, Shaul, Sridhar and Williamson in [9]. For more explanations, we refer to Example 4.2. In combination with Proposition 4.1, we conclude that G-dmension is a finer invariant than projective dimension for DG-modules. We refer to Corollaries 1.3, 1.5 and 1.6 for more evidence about this.

Let A be an Artin algebra. The *little finitistic dimension* of A, $\operatorname{fpd} A$, is the supremum of projective dimensions of finitely generated A-modules of finite projective dimension. It is conjectured that $\operatorname{fpd} A < \infty$ holds for Artin algebras A; see Bass [6] and [4, Conjectures]. This is the Finitistic Dimension Conjecture. Recently, this invariant has been generalized by Bird, Shaul, Sridhar and Williamson in [9] to the setting of non-positive commutative noetherian DG-rings with bounded cohomology. More precisely, they considered the number

$$\operatorname{fpd} A = \sup\{\operatorname{projdim}_A X + \inf X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with } \operatorname{projdim}_A X < \infty\}.$$

It should be noted that the little finitistic dimensions of an algebra can alternatively be computed by G-dimension; see for instance [31, Lemma 4.4]. As the first application of Theorem 1.2, we have the following corollary which shows that the little finitistic dimension $\operatorname{fpd}(A)$ over a DG-ring A can be computed by G-dimension defined in Definition 1.1 above.

Corollary 1.3. *If* A *is a commutative noetherian local DG-ring with* $\operatorname{amp} A < \infty$*, then we have*

$$\operatorname{fpd} A = \sup\{\operatorname{G-dim}_A X + \inf X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with } \operatorname{G-dim}_A X < \infty\}.$$

As a direct consequence of Theorem 1.2(3) and Corollary 1.3, it follows that a local Gorenstein DG-ring A has finite little finitistic dimension, which gives a new proof of a very particular case of [9, Theorem A].

Let A be a commutative noetherian local ring (not a DG-ring). A close relation between the class of maximal Cohen-Macaulay modules and the class of modules of G-dimension zero over A can be shown in the following fact; see for instance [11, 3.3].

Fact 1.4. Let A be a commutative noetherian local ring. Denote by $\mathcal{M}$ the class of maximal Cohen-Macaulay modules over A. Then the following hold:

(1) A is Cohen-Macaulay if and only if $\mathcal{G} \subseteq \mathcal{M}$.
(2) A is Gorenstein if and only if $\mathcal{G} = \mathcal{M}$.

Recently, the theory of Cohen-Macaulay rings and Cohen-Macaulay modules has been extended by Shaul [28] to the setting of commutative noetherian DG-rings. There are many examples of local Cohen-Macaulay DG-rings, in particular local Gorenstein DG-rings are Cohen-Macaulay. As the second application of Theorem 1.2, we can generalize the above fact to the setting of commutative noetherian local DG-rings.

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Corollary 1.5. *Let* $(A, \bar{\mathfrak{m}})$ *be a commutative noetherian local DG-ring with* $\mathrm{amp}A < \infty$. *Denote by* $\mathcal{M}$ *the class of maximal local-Cohen-Macaulay DG-modules over* A. *Set*

$$\mathcal{H} = \{X \in \mathcal{G} \mid \mathrm{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}X \geq \mathrm{amp}X = \mathrm{amp}A\}.$$

Then the following hold:

(1) A *is local-Cohen-Macaulay if and only if* $\mathcal{H} \subseteq \mathcal{M}$.
(2) A *is Gorentein if and only* $\mathcal{H} = \mathcal{M}$.

In the study of stable homological algebra and Tate cohomology for two sided noetherian rings A, Buchweitz [18] introduced the Verdier quotient $\mathrm{D}_{\mathrm{sg}}(A) := \mathrm{D}^{\mathrm{b}}(A)/\mathrm{K}^{\mathrm{b}}(A)$, where $\mathrm{K}^{\mathrm{b}}(A)$ is the bounded homotopy category of finitely generated projective left A-modules. Later on, this category was reconsidered by Orlov [24]. Since it measures the homological singularity of the category of finitely generated left A-modules in sense that $\mathrm{D}_{\mathrm{sg}}(A) = 0$ if and only if the global dimension of A is finite, we call it the *singularity category*. Buchweitz-Happel Theorem ([18, Theorem 4.4.1], see also [15, Theorem 4.6]) says that there is a fully faithful triangle functor $F : \underline{\mathcal{G}} \to \mathrm{D}_{sg}(A)$ provided that A is a commutative local Gorenstein ring. Also, it was shown in [8] that the inverse of this theorem holds true (see also [7, 21]). Recently, Buchweitz-Happel Theorem has been generalized by Jin to a proper noncommutative Gorenstein DG-algebra over a field k (see [19, Theorem 0.3 and Assumption 0.1]).

As the third application of Theorem 1.2, we extend the classical Buchweitz-Happel Theorem and its inverse from commutative noetherian local rings to the setting of commutative noetherian local DG-rings. For notation and notions, we refer the reader to Section 4.3.

Corollary 1.6. *Assume that* A *is a commutative noetherian local DG-ring with* $\mathrm{amp}A < \infty$. *Let* $\mathcal{A} = \{X \in \mathfrak{R}(A) | \mathrm{sup}X \leq 0$ *and* $\mathrm{G} - \mathrm{dim}_A X \leq 0\}$, *and let* $\mathrm{D}_{sg}(A) := \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle \mathcal{P} \rangle$ *be the singularity category over* A. *Then the functor*

$$F : \underline{\mathcal{A}} \to \mathrm{D}_{sg}(A)$$

is a triangle equivalence if and only if A *is a local Gorenstein DG-ring.*

Remark that if A is a commutative noetherian local DG-algebra over a field k, a Gorenstein ring in our setting is a proper Gorenstein DG-algebra defined by Jin in [19, Assumption 0.1]. Therefore, over a commutative noetherian local non-positive DG algebra over a field k, Corollary 1.6 conveys that the inverse of Buchweitz-Happel Theorem obtained by Jin [19, Theorem 0.3] is also true.

The article is organized as follows. In Section 2, we introduce notation, definitions and basic facts needed for proofs. Section 3 is devoted to giving the proof of Theorem 1.2. In Section 4, we apply Theorem 1.2 to establish the connection between G-dimensions and the little finitistic dimensions of A, to obtain a DG version of Fact 1.4, and to show a DG-version of Buchweitz-Happel Theorem and its inverse over A, and therefore prove the three corollaries.

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2. Preliminaries

An *associative DG-ring* A is a $\mathbb{Z}$-graded ring $A = \bigoplus_{i\in\mathbb{Z}} A^i$ equipped with a $\mathbb{Z}$-linear map $d : A \to A$ of degree $+1$ such that $d \circ d = 0$, and such that

$$d(a \cdot b) = d(a) \cdot b + (-1)^i a \cdot d(b) \tag{2.1}$$

for all $a \in A^i$ and $b \in A^j$. A DG-ring A is called *commutative* if for all $a \in A^i$ and $b \in A^j$, one has $b \cdot a = (-1)^{i\cdot j} a \cdot b$, and $a^2 = 0$ if i is odd. A DG-ring A is called *non-positive* if $A^i = 0$ for all $i > 0$. A non-positive DG-ring A is called *noetherian* if the ring $\mathrm{H}^0(A)$ is noetherian and the $\mathrm{H}^0(A)$-modules $\mathrm{H}^i(A)$ are finitely generated for all $i < 0$. **All DG-rings in this paper will be assumed to be commutative and non-positive.**

A DG-module M over A is a graded A-module together with a differential of degree $+1$ satisfying the Leibniz rule similar to (2.1). The DG-A-modules form an abelian category, and by inverting the quasi-isomorphisms we obtain the derived category $\mathrm{D}(A)$ which is triangulated. We refer the reader to [36] for more details about DG-rings and their derived categories.

If A is a noetherian DG-ring, we say that $M \in \mathrm{D}(A)$ has finitely generated cohomology if for all $n \in \mathbb{Z}$, the $\mathrm{H}^0(A)$-modules $\mathrm{H}^n(M)$ are finitely generated. We denote by $\mathrm{D_f}(A)$ the full triangulated subcategory of $\mathrm{D}(A)$ consisting of DG-modules with finitely generated cohomology. We also set $\mathrm{D_f^-}(A) = \mathrm{D_f}(A) \cap \mathrm{D}^-(A)$. Similarly we will consider $\mathrm{D_f^+}(A)$ and $\mathrm{D_f^b}(A)$. All these are full triangulated subcategories of $\mathrm{D}(A)$. If A is a noetherian DG-ring, and if the noetherian ring $\mathrm{H}^0(A)$ is local with maximal ideal $\bar{\mathfrak{m}}$ and residue field $\bar{k}$, we will say that $(A, \bar{\mathfrak{m}})$ (or $(A, \bar{\mathfrak{m}}, \bar{k})$) is a noetherian local DG-ring.

We recall the definition of the projective and injective dimensions of DG-modules introduced by Bird, Shaul, Sridhar and Williamson.

Definition 2.1. ([9, Definition 2.1]) *Let* A *be a DG-ring and* $M \in \mathrm{D}(A)$.

(1) *The* projective dimension *of* M *is defined by*

$$projdim_A M = \inf\{n \in Z | \mathrm{Ext}^i_A(M, N) = 0 \text{ for any } N \in \mathrm{D}^b(A) \text{ and } i > n + \sup N\}$$

(2) *The* injective dimension *of* M *is defined by*

$$injdim_A M = \inf\{n \in Z | \mathrm{Ext}^i_A(N, M) = 0 \text{ for any } N \in \mathrm{D}^b(A) \text{ and } i > n - inf N\}$$

Following [27], let us recall the notion of local cohomology and local homology functors over commutative DG-rings. Let A be a commutative DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be a finitely generated ideal. The category of derived $\bar{\mathfrak{a}}$-torsion DG-modules over A, denoted by $\mathrm{D}_{\bar{\mathfrak{a}}\text{-tor}}(A)$, is the full triangulated subcategory of $\mathrm{D}(A)$ consisting of DG-modules M such that the $\mathrm{H}^0(A)$-modules $\mathrm{H}^n(M)$ are $\bar{\mathfrak{a}}$-torsion for all $n \in \mathbb{Z}$. One can show that the inclusion functor $F : \mathrm{D}_{\bar{\mathfrak{a}}\text{-tor}}(A) \to \mathrm{D}(A)$ has a right adjoint $G : \mathrm{D}(A) \to \mathrm{D}_{\bar{\mathfrak{a}}\text{-tor}}(A)$, and composing this right adjoint with the inclusion, one obtains a triangulated functor

$$\mathrm{R}\Gamma_{\bar{\mathfrak{a}}} : \mathrm{D}(A) \to \mathrm{D}(A),$$

which we call the *derived torsion* or *local cohomology functor* of A with respect to $\bar{\mathfrak{a}}$.

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We recall the following definitions introduced by Shaul in [28].

Definition 2.2. *Let* $(A, \bar{\mathfrak{m}})$ *be a noetherian local DG-ring.*

(1) *We define the* local cohomology Krull dimension *of* $M \in \mathrm{D}^-(A)$ *to be*

$$\mathrm{lc.dim}_A M := \sup_{l\in\mathbb{Z}}\{\dim(\mathrm{H}^l(M)) + l\}.$$

(2) *We define the* depth *of* $N \in \mathrm{D}^+(A)$ *to be the number*

$$\mathrm{depth}_A N := \inf RHom_A(\bar{k}, N).$$

Definition 2.3. *Let* $(A, \bar{\mathfrak{m}})$ *be a noetherian local DG-ring with* $\mathrm{amp}A < \infty$.

(1) A *is called* local Cohen-Macaulay *if* $\mathrm{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(A) = \mathrm{amp}A$.
(2) *We say that* $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *is a* local-Cohen-Macaulay DG-module *if there are equalities*

$$\mathrm{amp}M = \mathrm{amp}A = \mathrm{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}(M).$$

(3) *A local-Cohen-Macaulay DG-module* M *is* maximal local-Cohen-Macaulay *if*

$$\mathrm{lc.dim}_A M = \sup M + \dim \mathrm{H}^0(A).$$

We denote by $\mathcal{M}$ *the full subcategory of maximal local-Cohen-Macaulay DG-modules.*

From now until the end of the paper, we always assume that $(A, \bar{\mathfrak{m}}, \bar{k})$ **is a commutative noetherian local DG-ring with** $\mathrm{amp}A < \infty$.

3. Proof of Theorem 1.2

This section is devoted to a proof of the statements of Theorem 1.2. We start with the following proposition.

Proposition 3.1. *Let* $X' \to X \to X'' \to X'[1]$ *be an exact triangle in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. *If two of the DG-modules belong to* $\mathcal{R}(A)$, *then so is the third.*

Proof. If two of $\mathrm{RHom}_A(X', A), \mathrm{RHom}_A(X, A)$ and $\mathrm{RHom}_A(X'', A)$ are in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, then so is the third. Using the abbreviated notation $((-, A), A) := \mathrm{RHom}_A(\mathrm{RHom}_A(-, A), A)$, we have the following commutative diagram of exact triangles in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$:

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$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & X'[1] \\ {\scriptstyle \delta^{X'}}\downarrow & & {\scriptstyle \delta^{X}}\downarrow & & {\scriptstyle \delta^{X''}}\downarrow & & {\scriptstyle \delta^{X'}[1]}\downarrow \\ ((X',A),A) & \longrightarrow & ((X,A),A) & \longrightarrow & ((X'',A),A) & \longrightarrow & ((X',A),A)[1]. \end{array}$$

Therefore, if two of the morphisms $\delta^{X'}, \delta^{X}$ and $\delta^{X''}$ are isomorphisms, then so is the third. The rest follows from the fact that $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is the full triangulated subcategory of $\mathrm{D}(A)$. □

The following lemma was proven in a greater generality by Jorgensen in [20].

Lemma 3.2. ([20], Theorem 0.2). *Let* $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *with* $\mathrm{projdim}_A X < \infty$. *Then* $\mathrm{amp}X \geq \mathrm{amp}A$.

Lemma 3.3. *If* $X \in \mathcal{R}(A)$ *with* $\mathrm{amp}X \geq \mathrm{amp}A$, *then* $\text{G-dim}_A X \geq -\sup X$.

Proof. By assumption, one has the following (in)equalities

$$\begin{aligned} \sup X &\geq \inf X - \inf A \\ &= \inf \mathrm{RHom}_A(\mathrm{RHom}_A(X,A),A) - \inf A \\ &\geq \inf A - \sup \mathrm{RHom}_A(X,A) - \inf A \\ &= -\sup \mathrm{RHom}_A(X,A), \end{aligned}$$

where the first inequality is by assumption and the second one is by [9, Lemma 3.2(i)], which implies that $\text{G-dim}_A X = \sup \mathrm{RHom}_A(X,A) \geq -\sup X$. □

We thank Dong Yang for communicating to us the next example which implies that $\text{G-dim}_A X \geq -\sup X$ need not hold in the DG-setting in general.

Example 3.4. Let $A = k[x]/(x^2)$ considered as a DG k-algebra, where k is a field and $\deg(x) = -1$. Let $S = A/(x)$, which concentrated in degree 0. Then S has a sppj resolution

$$\cdots \to A[2] \xrightarrow{x} A[1] \xrightarrow{x} A \to S \to 0.$$

Its total complex F is a minimal semi-free resolution of S. Now we can deduce that the DG-module $\mathrm{Hom}(F,A)$ is the total complex of

$$0 \to A \xrightarrow{x} A[-1] \xrightarrow{x} A[-2] \to \cdots.$$

Then $\mathrm{RHom}(S,A) \simeq \mathrm{Hom}(F,A) \simeq S[1]$ and $S \in \mathcal{R}(A)$. So in this case, $\sup S = 0$, but $\text{G-dim}_A S = \sup \mathrm{RHom}(S,A) = -1$.

Definition 3.5. *A DG-module* $X \in \mathcal{R}(A)$ *is said to be in the G-class* $\mathcal{G}$ *if either* $\text{G-dim}_A X = -\sup X$, *or* $\mathrm{G} = 0$, *and we denote by* $\mathcal{G}_0$ *the full subcategory of* $\mathcal{G}$ *consisting of objects* G *such that either* $\mathrm{amp}G \geq \mathrm{amp}A$ *and* $\text{G-dim}_A G = -\sup G = 0$, *or* $\mathrm{G} = 0$.

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Remark 3.6. (1) Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathcal{R}(A)$. The exact triangle $\mathrm{RHom}_A(Z,A) \to \mathrm{RHom}_A(Y,A) \to \mathrm{RHom}_A(X,A) \rightsquigarrow$ yields that

$$\text{G-dim}_A Y \leq \max\{\text{G-dim}_A X, \text{G-dim}_A Z\}.$$

(2) For $0 \not\simeq X \in \mathcal{R}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$, it follows from Lemma 3.3 that $\text{G-dim}_A X = -\sup X$ if and only if $X \in \mathcal{G}_0[-\sup X]$.

(3) For any $i \in \mathbb{Z}$, one has $\mathcal{P}[i] \subseteq \mathcal{G}_0[i]$ by [23, Lemma 2.14] and Lemma 3.2.

(4) For any $X \in \mathcal{R}(A)$ and $n \in \mathbb{Z}$, one has $\text{G-dim}_A X[n] = \text{G-dim}_A X + n$.

(5) For any $X \in \mathrm{D^b}(A)$, there is a sppj $f : P \to X$ and an exact triangle $Y \to P \to X \rightsquigarrow$. If $\mathrm{amp}Y < \mathrm{amp}A$, then $(f,0) : P \oplus P \to X$ is also a sppj morphism such that the triangle $Y \oplus P \to P \oplus P \to X \rightsquigarrow$ is exact and $\mathrm{amp}(Y \oplus P) \geq \mathrm{amp}A$ by Lemma 3.2.

We give several useful properties about sppj morphisms.

Lemma 3.7. Let $0 \not\simeq X \in \mathcal{R}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$ and $f : P \to X$ a sppj morphism with $Y = \mathrm{cn}(f)[-1]$. If $\text{G-dim}_A X + \sup X \geq 1$, then $\text{G-dim}_A Y = \text{G-dim}_A X - 1$.

Proof. By Proposition 3.1, $Y \in \mathcal{R}(A)$. Set $n = \text{G-dim}_A X + \sup X$ and $g : Y \to P$ be the canonical morphism. We may assume that $\sup X = 0$ by shifting the degree. Then $n \geq 1$.

Firstly, we need to prove $\mathrm{H}^i(\mathrm{RHom}_A(Y,A)) = 0$ for all $i > n-1$. By assumption, we have $\mathrm{H}^i(\mathrm{RHom}_A(X,A)) = 0$ for all $i > n$ and $\mathrm{H}^i(\mathrm{RHom}_A(P,A)) = 0$ for all $i > 0$. Then the following exact sequence

$$\mathrm{H}^i(\mathrm{RHom}_A(P,A)) \to \mathrm{H}^i(\mathrm{RHom}_A(Y,A)) \to \mathrm{H}^{i+1}(\mathrm{RHom}_A(X,A)) \to \mathrm{H}^{i+1}(\mathrm{RHom}_A(P,A))$$

yields that $\mathrm{H}^i(\mathrm{RHom}_A(Y,A)) = 0$ for all $i > n-1$. Next, since $\mathrm{H}^n(\mathrm{RHom}_A(X,A)) \neq 0$, the exact sequence $\mathrm{H}^{n-1}(\mathrm{RHom}_A(Y,A)) \to \mathrm{H}^n(\mathrm{RHom}_A(X,A)) \to 0$ implies that $\mathrm{H}^{n-1}(\mathrm{RHom}_A(Y,A)) \neq 0$. Therefore, one has $\text{G-dim}_A Y = n-1$. □

Lemma 3.8. Let $0 \not\simeq X \in \mathcal{G}$ with $\mathrm{amp}X \geq \mathrm{amp}A$. Then there exists a sppj morphism $f : P \to X$ such that the cocone $Y = \mathrm{cn}(f)[-1]$ *is in* $\mathcal{G}_0[-\sup X]$.

Proof. Let $f : P \to X$ be a sppj morphism with $P \in \mathcal{P}[-\sup X]$. One has an exact triangle $Y \to P \to X \rightsquigarrow$. By Remark 3.6(5), we may assume that $\mathrm{amp}Y \geq \mathrm{amp}A$. Then $\text{G-dim}_A P = -\sup X$ by [23, Lemma 2.14] and [33, Proposition 4.4(3)], so $\sup\mathrm{RHom}_A(Y,A) \leq -\sup X \leq -\sup Y$ since $\sup Y \leq \sup X$. Hence Remark 3.6(2) implies that $Y \in \mathcal{G}_0[-\sup X]$. □

Lemma 3.9. Let $0 \not\simeq X \in \mathcal{G}$ with $\mathrm{amp}X \geq \mathrm{amp}A$. Then there exists a morphism $f : X \to P$ such that $Y = \mathrm{cn}(f) \in \mathcal{G}_0[-\sup X]$.

Proof. Since $\sup\mathrm{RHom}_A(X,A) = -\sup X$ and $\inf\mathrm{RHom}_A(X,A) \geq \inf A - \sup X$, it follows that $\mathrm{amp}\mathrm{RHom}_A(X,A) \leq \mathrm{amp}A$. Set $n = \mathrm{amp}A - \mathrm{amp}\mathrm{RHom}_A(X,A)$ and $X' = \mathrm{RHom}_A(X,A) \oplus \mathrm{RHom}_A(X,A)[n]$. Then $\mathrm{amp}X' = \mathrm{amp}A$ and $\sup X' =$

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$\mathrm{supRHom}_A(X,A)$. By Lemma 3.8, there exists a sppj morphism $g : P \to X'$ such that $P \in \mathcal{P}[-\mathrm{sup}X']$, so we have a morphism

$$g^* : X \oplus X[-n] \simeq \mathrm{RHom}_A(X', A) \to \mathrm{RHom}_A(P, A)$$

and an exact triangle $X \xrightarrow{f} \mathrm{RHom}_A(P,A) \to Y \rightsquigarrow$, where f is the composition of $X \to X \oplus X[-n]$ and g^*. Since $\mathrm{RHom}_A(P,A) \in \mathcal{P}[-\mathrm{sup}X]$, $Y \in \mathcal{R}(A)$. By Remark 3.6(5), we may assume that $\mathrm{amp}Y \geq \mathrm{amp}A$. We need to show that $Y \in \mathcal{G}_0[-\mathrm{sup}X]$. Note that $\mathrm{H}^{-\mathrm{sup}X}(\mathrm{RHom}(f,A))$ is surjective, $\text{G-dim}_A Y = \mathrm{supRHom}_A(Y,A) \leq \mathrm{supRHom}_A(X,A) = \text{G-dim}_A X = -\mathrm{sup}X$. If $\text{G-dim}_A \mathrm{Y} + \mathrm{sup}Y \geq 1$, then $\mathrm{sup}X \leq \mathrm{supY} - 1$, and so $\mathrm{H}^{\mathrm{supY}}(Y) = 0$, this is a contradiction. Therefore, $Y \in \mathcal{G}_0[-\mathrm{sup}X]$. □

With the above preparations, now we prove Theorem 1.2.

Proof of Theorem 1.2. (1) Let X be a DG-module in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$ and n a natural number. Then $(ii) \Longrightarrow (iii)$ follows from Lemma 3.8 as $e - \mathrm{sup}X_e \leq n - \mathrm{sup}X$. $(iii) \Longrightarrow (i)$ and $(iv) \Longrightarrow (i)$ hold by the definition.

To show $(i) \Longrightarrow (ii)$, we assume that $P_\bullet$ is a sppj resolution of X and $X_{i+1} \xrightarrow{g_{i+1}} P_i \xrightarrow{f_i} X_i \rightsquigarrow$ are the corresponding exact triangles such that f_i is a sppj morphism and $\mathrm{amp}X_i \geq \mathrm{amp}A$ for $i \geq 0$ with $X_0 := X$. Fix $i \geq 1$. If $\text{G-dim}_A X_{j-1} + \mathrm{sup}X_{j-1} > 0$ for $1 \leq j \leq i$, then $\text{G-dim}_A X_j = \text{G-dim}_A X_{j-1} - 1$ for $1 \leq j \leq i$ by Lemma 3.7, which implies

$$\text{G-dim}_A X_i + i = \text{G-dim}_A X.$$

Since $\mathrm{sup}X_i \leq \mathrm{sup}X$, it follows that the set $\{i \geq 1 \mid \text{G-dim}_A X_{i-1} + \mathrm{sup}X_{i-1} > 0\}$ is finite. Set $e := \max\{i \geq 1 \mid \text{G-dim}_A X_{i-1} + \mathrm{sup}X_{i-1} > 0\}$. Then $\text{G-dim}_A X_e + e = e - \mathrm{sup}X_e = \text{G-dim}_A X \leq n - \mathrm{sup}X$ and $X_e \in \mathcal{G}_0[-\mathrm{sup}X_e]$.

$(i) \Longrightarrow (iv)$ If $\text{G-dim}_A X = -\mathrm{sup}X$, then $X \in \mathcal{G}_0[-\mathrm{sup}X]$ by Remark 3.6(2). Assume that $\text{G-dim}_A X > -\mathrm{sup}X$. By analogy with the proof of Lemma 3.9, one can obtain an exact triangle $Y \to X \to P^* \rightsquigarrow$, which induces the exact triangle

$$\mathrm{RHom}_A(Y,A)[-1] \to P \xrightarrow{f} \mathrm{RHom}_A(X,A) \to \mathrm{RHom}_A(Y,A), \tag{3.1}$$

where $P^* = \mathrm{RHom}_A(P,A) \in \mathcal{P}[\mathrm{supRHom}_A(X,A)]$. By Remark 3.6(5), we may assume that $\mathrm{amp}Y \geq \mathrm{amp}A$. Since $\mathrm{sup}P^* = -\mathrm{supRHom}_A(X,A) = \mathrm{sup}X - n < \mathrm{sup}X$, one has $\mathrm{sup}Y = \mathrm{sup}X$. As $\mathrm{H}^{\mathrm{supRHom}_A(X,A)}(f)$ is surjective, it follows from the triangle (3.1) that

$$\text{G-dim}_A Y = \mathrm{supRHom}_A(Y,A) \leq \mathrm{supRHom}_A(X,A) - 1 = \text{G-dim}_A X - 1 = n - 1 - \mathrm{sup}Y.$$

Now by induction, one can obtain $X \in \mathcal{G}_0[-\mathrm{sup}X] * \mathcal{P}[-\mathrm{sup}X + 1] * \cdots * \mathcal{P}[-\mathrm{sup}X + n]$.

(2) Set $\bar{X} = X \oplus X[\mathrm{amp}A - \mathrm{amp}X]$. Then $\bar{X} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}\bar{X} = \mathrm{amp}A$ and $\mathrm{sup}\bar{X} = \mathrm{sup}X$. As $\text{G-dim}_A X + (\mathrm{amp}A - \mathrm{amp}X) = \text{G-dim}_A \bar{X}$, the statement follows from (1).

(3) Evidently, (iii) is stronger than (ii), so it is sufficient to prove that (i) implies (iii) and (ii) implies (i).

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$(i) \Rightarrow (iii)$ As A is local Gorenstein, $\operatorname{injdim}_A A < \infty$ and $\operatorname{supRHom}_A(X, A) \leq \operatorname{injdim}_A A - \inf X$, it follows that $\operatorname{RHom}_A(X, A) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ and $X \in \mathcal{R}(A)$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. Consequently, for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, we have $\text{G-dim}_A X + \inf X \leq \operatorname{injdim}_A A$, as desired.

$(ii) \Rightarrow (i)$ Assume that $\text{G-dim}_A \bar{k} < \infty$. Then $\operatorname{supRHom}_A(\bar{k}, A) < \infty$. Hence [33, Proposition 4.2 and Theorem 4.7] implies that $\operatorname{injdim}_A A < \infty$ and A is local Gorenstein. □

4. Applications

In this section, we will apply Theorem 1.2 to establish the connection between G-dimensions and the little finitistic dimensions of A, to examine the DG version of the Fact 1.4, and to establish a DG-version of Buchwtweiz-Happel Theorem and its inverse.

4.1. Relations between G-dimensions and the little finitistic dimensions

In this subsection we will apply Theorem 1.2 to show that the little finitistic dimension $\operatorname{fpd}(A)$ can be computed by G-dimension of DG-modules in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

Yekutieli [35] introduced a DG-version of projective dimension, $\operatorname{pd} M$, for a DG-module M over a DG-algebra which are different from the definition $\operatorname{projdim}_A M$ in Definition 2.1. By [23, Theorems 2.22 and 3.21], one has $\operatorname{projdim}_A M = \operatorname{pd} M - \sup M$.

Proposition 4.1. *Let $0 \not\simeq X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. One has*

$$\text{G-}\dim_A X \leq \operatorname{projdim}_A X,$$

and equality holds if $\operatorname{projdim}_A X < \infty$.

Proof. The inequality is trivial if X is of infinite projective dimension. Next, assume that $n = \operatorname{projdim}_A X + \sup X < \infty$. It follows from [23, Proposition 2.26] that $X \in \mathcal{P}[-\sup X] * \mathcal{P}[-\sup X + 1] * \cdots * \mathcal{P}[-\sup X + n]$. Hence $\text{G-dim}_A X \leq n - \sup X$ by Theorem 1.2(1) and Lemma 3.2. This yields that $\text{G-dim}_A X \leq \operatorname{projdim}_A X$. For the last part, one has

$$\operatorname{projdim}_A X = \operatorname{supRHom}_A(X, A) = \text{G-dim}_A X$$

by [22, Proposition 2.2] and [33, Proposition 4.4(3)], as desired. □

We employ an example in [29] to illustrate Proposition 4.1 that a DG-module with finite G-dimension may have infinite projective dimension.

Example 4.2. Let $(B, \mathfrak{m}, k)$ be a commutative noetherian local ring and D a dualizing complex over B with $\sup D < 0$. Then the trivial extension DG-ring $A := B \ltimes D$ is a local Gorenstein DG-ring with $0 < \operatorname{amp} A < \infty$ by [29, Example 7.2], and hence $\text{G-dim}_A k \leq \operatorname{injdim}_A A < \infty$, but $\operatorname{projdim}_A k = \infty$ by Lemma 3.2.

The next lemma is an immediate consequence of equivalence of (i) and (iv) in Theorem 1.2(1).

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Lemma 4.3. *Let* X *be in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *with* $\mathrm{amp}X \geq \mathrm{amp}A$. *If* $\text{G-dim}_A X = n$, *then there exists an exact triangle*

$$G \to X \to K \rightsquigarrow,$$

such that $G \in \mathcal{G}_0[-\mathrm{sup}X]$ *and* $\mathrm{projdim}_A K = n-1$. *For* $n = -\mathrm{sup}X$, *we understand that* $\mathrm{K} = 0$.

The following result conveys that G-dimensions and the little finitistic dimensions are closely related to each other, which contains Corollary 1.3 in the introduction.

Theorem 4.4. *Let* X *be an object in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *with* $\mathrm{amp}X \geq \mathrm{amp}A$. *If* $\text{G-dim}_A X = n < \infty$, *then there is a DG-module* $Y \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *with* $\mathrm{projdim}_A Y = n$. *Moreover, one has*

$$\mathrm{fpd}A = \mathrm{sup}\{\text{G-dim}_A X + \mathrm{inf}X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with } \mathrm{amp}X \geq \mathrm{amp}A,\ \text{G-dim}_A X < \infty\}$$

$$= \mathrm{sup}\{\text{G-dim}_A X + \mathrm{inf}X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with } \text{G-dim}_A X < \infty\}.$$

Proof. We may assume that $n > -\mathrm{sup}X$. By Lemma 4.3, there exists an exact triangle $G \to X \to K \rightsquigarrow$ with $\mathrm{projdim}_A K = n-1$ and $G \in \mathcal{G}_0[-\mathrm{sup}X]$, and there is an exact triangle $G \to P \to G' \rightsquigarrow$ with $P \in \mathcal{P}[-\mathrm{sup}G]$ and $G' \in \mathcal{G}_0[-\mathrm{sup}G]$ by Lemma 3.9. Thus we get a commutative diagram of exact triangles in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$:

$$\begin{array}{ccccccc}
 & & G'[-1] & = & G'[-1] & & \\
 & & \downarrow & & \downarrow & & \\
K[-1] & \longrightarrow & G & \longrightarrow & X & \longrightarrow & K \\
\| & & \downarrow & & \downarrow & & \| \\
K[-1] & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & K \\
 & & \downarrow & & \downarrow & & \\
 & & G' & = & G' & &
\end{array}$$

The exact triangle $\mathrm{RHom}_A(G', A) \to \mathrm{RHom}_A(Y, A) \to \mathrm{RHom}_A(X, A) \rightsquigarrow$ yields that $\text{G-dim}_A Y = n$. Note that $\mathrm{sup}X = \mathrm{sup}Y$, so $Y \notin \mathcal{G}_0[-\mathrm{sup}X]$. As $\mathrm{sup}K \leq \mathrm{sup}X - 1$, $P \to Y$ is a sppj morphism, it follows from the exact triangle $P \to Y \to K \rightsquigarrow$ and Proposition 4.1 that $\mathrm{projdim}_A Y = n$. Hence we have shown the first statement. This yields that $\mathrm{fpd}(A) \geq \mathrm{sup}\{\text{G-dim}_A X + \mathrm{inf}X \mid X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \text{ with } \text{G-dim}_A X < \infty\}$. Also for any $Z \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{projdim}_A Z < \infty$, one has $\text{G-dim}_A Z \geq \mathrm{projdim}_A Z$. This shows the first equality.

If $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X < \mathrm{amp}A$, then $\mathrm{amp}(X \oplus X[\mathrm{amp}A - \mathrm{amp}X]) = \mathrm{amp}A$ and $\text{G-dim}_A X = \text{G-dim}_A(X \oplus X[\mathrm{amp}A - \mathrm{amp}X]) - \mathrm{amp}A + \mathrm{amp}X$, and therefore $\text{G-dim}_A(X \oplus X[\mathrm{amp}A - \mathrm{amp}X]) + \mathrm{inf}(X \oplus X[\mathrm{amp}A - \mathrm{amp}X]) = \text{G-dim}_A X + \mathrm{inf}X$. Thus the second equality follows. □

4.2. Comparison to maximal local-Cohen-Macaulay DG-modules

The task of this subsection is to examine the DG version of Fact 1.4. We begin with the following AB formula for G-dimension of DG-modules.

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Theorem 4.5. (AB Formula) *Let* X *be a DG-module of finite G-dimension. One has an equality*

$$\text{G-dim}_A X = \text{depth}A - \text{depth}_A X.$$

Proof. Since $\text{G-dim}_A X < \infty$, we have the following equalities

$$\begin{aligned}\text{depth}_A X &= \text{depthRHom}_A(\text{RHom}_A(X, A), A)\\ &= -\text{supRHom}_A(X, A) + \text{depth}A\\ &= -\text{G-dim}_A X + \text{depth}A,\end{aligned}$$

where the second equality is by [33, Proposition 4.9] as $\text{RHom}_A(X, A) \in \text{D}^{\text{b}}_{\text{f}}(A)$, as required. □

Following [28], if $X \in \text{D}^{\text{b}}_{\text{f}}(A)$ then we set

$$\text{seq.depth}_A X = \text{depth}_A X - \text{inf}X,$$

and call it the *sequential depth* of X.

Corollary 4.6. *Let* X *be a DG-module of finite G-dimension with* $\text{amp}X = \text{amp}A$. *Then we have*

$$\text{G-dim}_A X + \text{sup}X = \text{seq.depth}A - \text{seq.depth}_A X.$$

Proof. One has the following equalities

$$\begin{aligned}\text{G-dim}_A X + \text{sup}X &= \text{depth}A - \text{depth}_A X + \text{sup}X\\ &= \text{depth}A - \text{inf}A - (\text{depth}_A X - \text{inf}X)\\ &= \text{seq.depth}A - \text{seq.depth}_A X,\end{aligned}$$

where the first one is by Theorem 4.5 and the second one is by $\text{amp}X = \text{amp}A$. □

We are now ready to prove Corollary 1.5.

Proof of Corollary 1.5. (1) The "only if" part holds by noting that $A \in \mathcal{H}$ by [28, Theorem 4.1]. For the "if" part, we assume that X is an object in $\mathcal{H}$. It follows that $\text{lc. dim}_A A \geq \text{lc.dim}_A X - \text{sup}X \geq \text{depth}_A X - \text{inf}X$. Since A is local Cohen-Macaulay, we have the following (in)equalities

$$\begin{aligned}\text{seq.depth}_A X &\leq \text{lc.dim}_A X - \text{sup}X\\ &\leq \text{lc.dim}A\\ &= \text{seq.depth}A\\ &= \text{seq.depth}_A X,\end{aligned}$$

where the second equality follows from Corollary 4.6, which implies that $\text{ampR}\Gamma_{\bar{\mathfrak{m}}} X = \text{amp}A = \text{amp}X$ and $\text{lc.dim}_A X = \text{dimH}^0(A) + \text{sup}X$. Thus $X \in \mathcal{M}$.

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(2) For the "only if" part, since A is local Gorentein, it follows from [26, Theorem 7.26] that $X \simeq \mathrm{RHom}_A(\mathrm{RHom}_A(X,A),A)$ and $\mathrm{ampR}\Gamma_{\bar{\mathfrak{m}}}X = \mathrm{ampRHom}_A(X,A)$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. Let $X \in \mathcal{M}$. Then $\mathrm{ampRHom}_A(X,A) = \mathrm{amp}A$, it follows from Corollary 4.6 that

$$\mathrm{G\text{-}dim}_A X + \sup X = \mathrm{seq.depth}A - \mathrm{seq.depth}_A X = 0.$$

Thus $X \in \mathcal{H}$ and $\mathcal{H} = \mathcal{M}$ by (1).

For the "if" part, we assume that $\mathcal{H} = \mathcal{M}$. Then A is local-Cohen-Macaulay by (1). Consider the sppj resolution $P^{\bullet}$ of the DG-module $k := \bar{k} \oplus \bar{k}[-\inf A]$

$$k_d \to P_{d-1} \to \cdots P_1 \to P_0 \to k.$$

By the proof of [32, Proposition 4.4], one has $G = A[-\sup k_d] \oplus k_d \in \mathcal{M}$ and k han an sppj resolution

$$G \to A[-\sup k_d] \oplus P_{d-1} \to \cdots P_1 \to P_0 \to k.$$

Thus, $\mathrm{G\text{-}dim}_A k < \infty$ by Theorem 1.2(1) and hence A is Gorenstein. □

4.3. Buchweitz-Happel theory and its inverse

In this subsection we will employ Theorem 1.2 to prove Corollary 1.6 in the introduction.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{D} \subseteq \mathcal{C}$ be subcategories in $\mathcal{T}$. As usual, we denote by $\langle \mathcal{D} \rangle$ the smallest thick triangulated subcategory containing $\mathcal{D}$, and denote by $\mathcal{C}/\langle \mathcal{D} \rangle$ the Verdier's quotient triangulated category. In fact, $\langle \mathcal{P} \rangle$ is the thick subcategory of the derived category $\mathrm{D}^{\mathrm{b}}(A)$ generated by A. We call the Verdier quotient $\mathrm{D}_{\mathrm{sg}}(A) := \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle \mathcal{P} \rangle$ the *singularity category* of A. Each morphism $f : X \to Y$ in $\mathrm{D}_{\mathrm{sg}}(A)$ is given by an equivalence class of left fractions $s \setminus f$ as presented by $s \setminus f : X \xrightarrow{f} Z \xLeftarrow{s} Y$, where the doubled arrow means s lies in the compatible saturated multiplicative system corresponding to $\langle \mathcal{P} \rangle$.

In the following, we set

$$A[<0]^{\perp_f} = \{X \in \mathcal{R}(A) \mid \mathrm{Hom}_{\mathrm{D}(A)}(A[<0], X) = 0\};$$

$$^{\perp_f}A[>0] = \{X \in \mathcal{R}(A) \mid \mathrm{Hom}_{\mathrm{D}(A)}(X, A[>0]) = 0\}.$$

Recall from Corollary 1.5 that

$$\mathcal{H} = \{X \in \mathcal{G} \mid \mathrm{ampR}\Gamma_{\bar{\mathfrak{m}}}X \geq \mathrm{amp}X = \mathrm{amp}A\}.$$

Lemma 4.7. *Let $X \in \mathcal{R}(A)$. Assume that*

$$\mathrm{ampR}\Gamma_{\bar{\mathfrak{m}}}X \geq \mathrm{amp}X \geq \mathrm{amp}A.$$

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Then $X \in \mathcal{H}$ *if and only if* $\sup X[\sup X] \leq 0$ *and* $\text{G-dim}_A X[\sup X] \leq 0$ *if and only if* $X[\sup X] \in {}^{\perp_f}A[>0] \cap A[<0]^{\perp_f}$.

Proof. We will start by proving the first equivalence. Let $X \in \mathcal{H}$. Since $X \in \mathcal{G}$, $\text{G-dim}_A X = -\sup X$. Therefore $\text{G-dim}_A X[\sup X] = \text{G-dim}_A X + \sup X = 0$ and $\sup X[\sup X] \leq 0$. Conversely, suppose that $\sup X[\sup X] \leq 0$ and $\text{G-dim}_A X[\sup X] \leq 0$. Then $\text{G-dim}_A X \leq -\sup X$ as $\text{G-dim}_A X[\sup X] = \text{G-dim}_A X + \sup X$. By Corollary 3.3, we have $\text{G-dim}_A X \geq -\sup X$. It follows that $X \in \mathcal{G}$. In order to show $X \in \mathcal{H}$, it suffices to show $\text{amp}X \leq \text{amp}A$. Since $X[\sup X] \in \mathcal{R}(A)$, by [9, Lemma 3.2(i)], we have

$$\begin{aligned}\inf(X[\sup X]) = \inf&\text{RHom}_A(\text{RHom}_A(X[\sup X], A), A) \geq -\sup\text{RHom}_A(X[\sup X], A)\\ &+ \inf(A).\end{aligned}$$

Then $0 \geq \text{G-dim}_A X[\sup X] = \sup(\text{RHom}_A(X[\sup X], A)) \geq \inf(A) - \inf(X[\sup X])$, it follows that $-\inf(A) \geq -\inf(X[\sup X])$. On the other hand, since $\sup A = 0$ and $0 \geq \sup X[\sup X]$, we have $\text{amp}A = -\inf(A) \geq \sup X[\sup X] - \inf(X[\sup X]) = \text{amp}X[\sup X] = \text{amp}X$, as desired.

The second equivalence follows from the equalities ${}^{\perp_f}A[>0] = \{X \in \mathcal{R}(A) \mid \text{H}^i(\text{RHom}_A(X, A)) = 0 \text{ for } i > 0\}$ and $A[<0]^{\perp_f} = \{X \in \mathcal{R}(A) \mid \text{H}^i(\text{RHom}_A(A, X)) = 0 \text{ for } i > 0\} = \{X \in \mathcal{R}(A) \mid \text{H}^i(X) = 0 \text{ for } i > 0\}$. □

In the following, we set $\mathcal{A} = {}^{\perp_f}A[>0] \cap A[<0]^{\perp_f}$.

Remark 4.8. Let A be a non-positive DG-algebra over a field k satisfying the following conditions:

(i) A is proper, i.e., $\dim_k \bigoplus_{i\in\mathbb{Z}} \text{H}^i(A) < \infty$;
(ii) A is Gorenstein, i.e., the thick subcategory per A of the derived category $\text{D}(A)$ generated by A coincides with the thick subcategory generated by DA, where $D = \text{Hom}_k(-, k)$ is the k-dual.

Assume that A is a proper Gorenstein DG-algebra. Recently, Jin defined that a DG-A-module $X \in \text{D}^{\text{b}}(A)$ is called *Cohen-Macaulay* if $\sup M \leq 0$ and $\text{Hom}_{\text{D}^{\text{b}}(A)}(M, A[i]) = 0$ for all $i > 0$ (see [19, Definition 2.1]). If A is an ordinary Gorenstein algebra, then Cohen-Macaulay DG-modules defined by Jin is canonically equivalent to the usual (maximal) Cohen-Macaulay modules. Note that if a DG-module X belongs to $\mathcal{R}(A)$, then $X \in \mathcal{A}$ if and only if it is Cohen-Macaulay defined by Jin in the sense of [19].

On the other hand, from Corollary 1.5 we know that $\mathcal{H}$ is equal to the class of maximal local-Cohen-Macaulay DG-modules defined by Shaul [28] over local Gorenstein DG-rings, see Definition 2.3. These two definitions are completely different at first glance. For example, one can see that the class of maximal local-Cohen-Macaulay DG-modules defined by Shaul is closed under suspensions, but the class of Cohen-Macaulay DG-modules defined by Jin is not. Fortunately, by Lemma 4.7, these two definitions are the same over commutative local Gorenstein DG-algebra over a field k under the condition that $\text{amp}\text{R}\Gamma_{\bar{\mathfrak{m}}}X \geq \text{amp}X \geq \text{amp}A$ for all $X \in \mathcal{R}(A)$. This condition is natural and very often met. For example, this holds for any complex $X \in \text{D}^{\text{b}}_{\text{f}}(A)$ over a commutative noetherian

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local ring A by [28, Propositions 2.8 and 3.4]. Moreover, we assume that A is a DG-ring with $\dim \mathrm{H}^0(A) = 0$. Let $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{amp}X \geq \mathrm{amp}A$. Then $\sup \mathrm{R}\Gamma_{\bar{\mathfrak{m}}}X \geq \sup X$ by [28, Remark 2.3 and Theorem 2.15] and $\inf \mathrm{R}\Gamma_{\bar{\mathfrak{m}}}X \leq \inf X$ by [28, Propositions 3.3 and 3.5]. So we have $\mathrm{amp}\mathrm{R}\Gamma_{\bar{\mathfrak{m}}}X \geq \mathrm{amp}X$, as desired.

Lemma 4.9. *The subcategory $\mathcal{R}(A)$ is thick and it is the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ that contains the objects from $\mathcal{A}$.*

Proof. Obviously, $\mathcal{R}(A)$ is closed under shifts and isomorphisms. Assume that $X \in \mathcal{R}(A)$ and $X^{'}$ is a direct summand of X. There exits two split triangles $X' \xrightarrow{u} X \xrightarrow{v} X'' \xrightarrow{0} X'[1]$ and $X'' \xrightarrow{p} X \xrightarrow{q} X' \xrightarrow{0} X''[1]$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. For simplicity, we write $((-,A),A)$ for $\mathrm{RHom}_A(\mathrm{RHom}_A(-,A),A)$. Then we have the following commutative diagrams of exact triangles:

$$\begin{array}{ccccccc} X' & \xrightarrow{u} & X & \xrightarrow{v} & X'' & \xrightarrow{0} & X'[1] \\ \downarrow{\scriptstyle \delta^{X'}} & & \downarrow{\scriptstyle \delta^{X}} & & \downarrow{\scriptstyle \delta^{X''}} & & \downarrow{\scriptstyle \delta^{X'}[1]} \\ ((X',A),A) & \longrightarrow & ((X,A),A) & \longrightarrow & ((X'',A),A) & \xrightarrow{0} & ((X',A),A)[1], \end{array} \tag{4.1}$$

$$\begin{array}{ccccccc} X'' & \xrightarrow{p} & X & \xrightarrow{q} & X' & \xrightarrow{0} & X''[1] \\ \downarrow{\scriptstyle \delta^{X''}} & & \downarrow{\scriptstyle \delta^{X}} & & \downarrow{\scriptstyle \delta^{X'}} & & \downarrow{\scriptstyle \delta^{X''}[1]} \\ ((X'',A),A) & \longrightarrow & ((X,A),A) & \longrightarrow & ((X',A),A) & \xrightarrow{0} & ((X'',A),A)[1]. \end{array} \tag{4.2}$$

Since u is a split monomorphism and δ^X is an isomorphism, it is straightforward to check that $\delta^{X'}$ is a split monomorphism by the commutative diagram (4.1). On the other hand, it follows from the commutativity of the diagram (4.2) that $\delta^{X'}q$ is an epimorphism, which implies that $\delta^{X'}$ is also an epimorphism. Thus, $\delta^{X'}$ is an isomorphism and $X^{'}$ belongs to $\mathcal{R}(A)$, and therefore $\mathcal{R}(A)$ is a thick subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ by Proposition 3.1.

Recall that $\langle \mathcal{A} \rangle$ is the smallest thick triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ that contains the objects from $\mathcal{A}$. Since $\mathcal{A} \subseteq \mathcal{R}(A)$, it follows that $\langle \mathcal{A} \rangle \subseteq \mathcal{R}(A)$. For the reverse containment, we assume that X is a non-zero object in $\mathcal{R}(A)$. Note that $\mathcal{R}(A)$ is precisely the subcategory of DG-modules with finite G-dimension. If $\mathrm{amp}X \geq \mathrm{amp}A$, applying Theorem 1.2(1), there exists a natural number m such that X belongs to $\mathcal{G}_0[-\sup X] * \mathcal{P}[-\sup X + 1] * \cdots * \mathcal{P}[-\sup X + m]$. By the fact that $\mathcal{P} \subseteq \mathcal{G}_0 \subseteq \mathcal{A}$, we deduce that X belongs to $\langle \mathcal{A} \rangle$. If $\mathrm{amp}X < \mathrm{amp}A$, applying Theorem 1.2(2), then $X \oplus X[\mathrm{amp}A - \mathrm{amp}X]$ belongs to $\mathcal{P}[-\sup X] * \cdots * \mathcal{P}[-\sup X + n - 1] * \mathcal{G}_0[-\sup X + n]$ for some n. Similarly, since $\langle \mathcal{A} \rangle$ is thick, one has $X \in \langle \mathcal{A} \rangle$. Altogether, we have $\mathcal{R}(A) = \langle \mathcal{A} \rangle$. □

Let $\mathcal{T}$ be a triangulated category. Recall that a subcategory $\omega \subseteq \mathcal{T}$ is called *presilting* if $\mathrm{Hom}_{\mathcal{T}}(M, M'[i]) = 0$ for all $M, M' \in \omega$ and $i > 0$. Let $\mathcal{E}$ be an arbitrary category. A full additive subcategory $\mathcal{M} \subseteq \mathcal{E}$ is called *contravariantly finite*, if every object X in $\mathcal{E}$ admits a right $\mathcal{M}$-approximation, i.e. there exists an object $M \in \mathcal{M}$ and a morphism $f : M \to X$, such that the induced map $\mathrm{Hom}_{\mathcal{E}}(N, M) \to \mathrm{Hom}_{\mathcal{E}}(N, X)$ is surjective for all $N \in \mathcal{M}$. Dually, we define the notion of a *covariantly finite* subcategory. We say that M is *functorially finite* if it is both covariantly and contravariantly finite.

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Lemma 4.10. *The following statement hold.*

(1) $\mathcal{A}$ *is closed under extensions and direct summands and finite direct sums in* $\mathcal{R}(A)$.
(2) *The subcategory* $\mathcal{P} = \operatorname{add} A$ *is presilting and functorially finite in* $\mathcal{R}(A)$.

Proof. (1) Since $\mathcal{A} = A[<0]^{\perp_f} \cap {}^{\perp_f}A[>0]$, one can check that all objects in $\mathcal{A}$ is closed under extensions and direct summands and finite direct sums in $\mathcal{R}(A)$.

(2) The subcategory $\mathcal{P}$ is presilting follows from the fact that A is a non-positive DG-ring. Note that for any $N \in \mathcal{R}(A)$, the set of the generators of $\operatorname{Hom}_{\mathrm{D}(A)}(A, N) \cong \mathrm{H}^0(N)$ is finite. If $f_1, \cdots, f_s$ are generators of $\operatorname{Hom}_{\mathrm{D}(A)}(A, N)$, then the canonical homomorphism $f : A^{(s)} \to N$ given by $(f_1, \cdots, f_s)$ is a right $\mathcal{P}$-approximation. On the other hand, since $\operatorname{RHom}_A(X, A) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ for any $X \in \mathcal{R}(A)$, the set $\operatorname{Hom}_{\mathrm{D}(A)}(X, A)$ also has finite generators. So the left $\mathcal{P}$-approximation can be constructed by a similar way. Therefore, $\mathcal{P}$ is functorially finite in $\mathcal{R}(A)$. □

Following [30], we denote by $[\mathcal{D}]$ the ideal of morphisms in $\mathcal{C}$ which factor through objects in $\mathcal{D}$. The stable category of $\mathcal{C}$ by $\mathcal{D}$ is the quotient category $\mathcal{C}/[\mathcal{D}]$, for which objects are the same as $\mathcal{C}$, but morphisms are morphisms in $\mathcal{C}$ modulo those in $[\mathcal{D}]$.

The following results are analogues of well-known properties of Cohen-Macaulay modules.

Theorem 4.11. (1) *The stable category* $\underline{\mathcal{A}} := \mathcal{A}/[\mathcal{P}]$ *is a triangulated category;*

(2) *The composition* $\mathcal{A} \hookrightarrow \mathcal{R}(A) \to \mathcal{R}(A)/\langle\mathcal{P}\rangle$ *induces a triangle equivalence*

$$H : \quad \underline{\mathcal{A}} \simeq \mathcal{R}(A)/\langle\mathcal{P}\rangle.$$

(3) $\mathcal{R}(A) = \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ *if and only if* A *is a local Gorenstein DG-ring.*

Proof. (1) By Lemma 4.10(2), we know that $\mathcal{P}$ is presilting and functorially finite in $\mathcal{R}(A)$. Then the result follows from the proof of [30, Corollary 2.7] and [30, Proposition 2.4(3)].

(2) Since $\mathcal{P} \subseteq \mathcal{R}(A)$ and $\mathcal{R}(A)$ is a triangulated full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, we have $\langle\mathcal{P}\rangle \subseteq \mathcal{R}(A)$. Thanks to [30, Corollary 2.7] or [17, Theorem 1.1], by Lemma 4.10, there is a triangle equivalence between $\underline{\mathcal{A}}$ and the singularity category $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$.

(3) This follows from Theorem 1.2(3). □

Let G be the obvious composite functor $\mathcal{A} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$. Since $G(\mathcal{P}) = 0$, there is a unique factorization of G through the canonical projection $S : \mathcal{A} \to \underline{\mathcal{A}}$. We denote by F the factorization from $\underline{\mathcal{A}}$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$. Then $G = F \circ S$. In fact, F is the composite functor $\underline{\mathcal{A}} \xrightarrow{H} \mathcal{R}(A)/\langle\mathcal{P}\rangle \hookrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$. Let A be a local Gorenstein DG-ring. By Theorem 4.11(3), we obtain a triangle equivalence: $F : \underline{\mathcal{A}} \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$.

We are now ready to prove Corollary 1.6.

Proof of Corollary 1.6. By Theorem 4.11, it suffices to show that if the functor $F : \underline{\mathcal{A}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$ is dense, then A is a local Gorenstein DG-ring. Assume that $F : \underline{\mathcal{A}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)/\langle\mathcal{P}\rangle$ is dense, and let M be any DG-module in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. It follows that

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$M \cong F(G)$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ for some $G \in \mathcal{A}$. Let $s \setminus f : M \xrightarrow{f} Z \xleftarrow{s} G$ be an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ with $\mathrm{cn}(s) \in \langle \mathcal{P} \rangle$, then $\mathrm{cn}(f) \in \langle \mathcal{P} \rangle$. Consider the triangle $G \xrightarrow{s} Z \longrightarrow \mathrm{cn}(s) \rightsquigarrow$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. We see that G and $\mathrm{cn}(s)$ lie in $\mathcal{R}(A)$, so $Z \in \mathcal{R}(A)$. It follows from $Z \in \mathcal{R}(A)$ and $\mathrm{cn}(f) \in \mathcal{R}(A)$ that $M \in \mathcal{R}(A)$. Therefore the G-dimension of M is finite, and so A is a local Gorenstein DG-ring by Theorem 1.2(3). □

Remark 4.12. If A is a local Gorenstein ring, then Corollary 1.6 gives the classical Buchweitz-Happle Theorem and its inverse. As we stated in the Introduction, Jin [19] proved that if A is a proper Gorenstein DG-algebra, then there is a triangle equivalence between the stable category of Cohen-Macaulay DG-modules (see Remark 4.8) and the singularity category $\mathrm{D}^{\mathrm{b}}(A)/\langle \mathcal{P} \rangle$. This generalizes Buchweitz-Happle Theorem to the setting of proper Gorenstein DG-algebras (see [19, Theorem 2.4]). In fact, over commutative local noetherian DG-algebras over a field k, condition (ii) of the proper Gorenstein DG-algebra is equivalent to $\mathrm{injdim}_A A < \infty$. In this case, a Gorenstein ring in our setting is a proper Gorenstein DG-algebra defined in [19]. Therefore, over a commutative local noetherian non-positive DG algebra over k, Corollary 1.6 shows that the inverse of Buchweitz-Happel Theorem is also true.

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