Mathematika 62 (2016) 79–100 © 2015 University College London
doi:10.1112/S0025579315000182

ON THE SPECTRUM OF DIOPHANTINE APPROXIMATION CONSTANTS

JOHANNES SCHLEISCHITZ

Abstract. The approximation constant $\lambda_k(\zeta)$ is defined as the supremum of $\eta \in \mathbb{R}$ such that the estimate $\max_{1\leqslant j\leqslant k} \|\zeta^j x\| \leqslant x^{-\eta}$ has infinitely many integer solutions x. Here $\|.\|$ denotes the distance to the closest integer. We establish a connection on the joint spectrum $(\lambda_1(\zeta), \lambda_2(\zeta), \ldots)$, which will lead to various improvements of known results on the individual spectrum of the approximation constants $\lambda_k(\zeta)$ as well. In particular, for given $k \geqslant 1$ and $\lambda \geqslant 1$, we construct ζ in the Cantor set with $\lambda_k(\zeta) = \lambda$. Moreover, we establish an estimate for the uniform approximation constants $\widehat{\lambda}_k(\zeta)$, which enables us to determine classical approximation constants for Liouville numbers.

§1. *Introduction and main results.*

1.1. *Definition of the constants.* We begin with the definition of the quantities $\lambda_k(\zeta)$ that we will predominately consider and their uniform versions $\widehat{\lambda}_k(\zeta)$. For a vector $\underline{\zeta} = (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^k$ we denote by $\lambda_k(\underline{\zeta})$ and $\widehat{\lambda}_k(\underline{\zeta})$, respectively, the supremum of $\eta \in \mathbb{R}$ such that the system

$$|x| \leqslant X, \qquad 0 < \max_{1\leqslant j\leqslant k} |\zeta_j x - y_j| \leqslant X^{-\eta} \tag{1}$$

has a solution $(x, y_1, y_2, \ldots, y_k) \in \mathbb{Z}^{k+1}$ for arbitrarily large X and for all $X \geqslant X_0$, respectively. Furthermore, let

$$\lambda_k(\zeta) := \lambda_k(\zeta, \zeta^2, \ldots, \zeta^k), \qquad \widehat{\lambda}_k(\zeta) := \widehat{\lambda}_k(\zeta, \zeta^2, \ldots, \zeta^k).$$

In the special case $\zeta \in \mathbb{Q}$, it is not difficult to see that due to the non-vanishing condition in (1) we have $\lambda_k(\zeta) = \widehat{\lambda}_k(\zeta) = 0$ for all $k \geqslant 1$. If ζ is algebraic of degree $d \geqslant 2$, then $\lambda_k(\zeta) = \widehat{\lambda}_k(\zeta) = \max\{1/k, 1/(d-1)\}$ is a consequence of the Schmidt Subspace theorem [22], as pointed out in [6]. Assuming otherwise that ζ is not algebraic of degree $\leqslant k$, Dirichlet's box principle implies

$$\frac{1}{k} \leqslant \widehat{\lambda}_k(\zeta) \leqslant \lambda_k(\zeta) \leqslant \infty. \tag{2}$$

In fact, (2) holds for any irrational real number ζ. The analogue result holds for $\underline{\zeta} \in \mathbb{R}^k$ for which $\{1, \zeta_1, \zeta_2, \ldots, \zeta_k\}$ is $\mathbb{Q}$-linearly independent as well. However,

Received 2 October 2014, published online 22 June 2015.
MSC (2010): 11H06 (primary), 11J13, 11J25, 11J82, 11J83 (secondary).
Supported by the Austrian Science Fund FWF grant P24828.

https://doi.org/10.1112/S0025579315000182 Published online by Cambridge University Press

the remaining results of §1.1 require the assumption $\underline{\zeta} = (\zeta, \zeta^2, \dots, \zeta^k)$, and some cannot be defined in the general case anyway.

It follows from the definitions that for any $\zeta \in \mathbb{R}$ we have the chains of inequalities

$$\cdots \leqslant \lambda_3(\zeta) \leqslant \lambda_2(\zeta) \leqslant \lambda_1(\zeta), \tag{3}$$

$$\cdots \leqslant \widehat{\lambda}_3(\zeta) \leqslant \widehat{\lambda}_2(\zeta) \leqslant \widehat{\lambda}_1(\zeta). \tag{4}$$

It is not hard to construct ζ such that the asymptotic constants $\lambda_k(\zeta)$ take the value ∞ (which for a given ζ holds for all k simultaneously or for none by [**6**, Corollary 2]), which one may deduce from Theorem 4.4 in §4.1. The uniform constants, on the other hand, can be effectively bounded. An elementary result by Khintchine [**11**] implies that, for any irrational ζ, the one-dimensional uniform approximation constant is given as

$$\widehat{\lambda}_1(\zeta) = 1. \tag{5}$$

For $k = 2$ and ζ neither rational nor quadratic irrational, we have

$$\widehat{\lambda}_2(\zeta) \leqslant \frac{\sqrt{5}-1}{2},$$

and this constant is optimal. Equality holds for so-called extremal numbers that can be explicitly constructed, see [**18**–**20**]. As usual, $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to $\alpha \in \mathbb{R}$. For any $k \geqslant 2$ and ζ not algebraic of degree $\leqslant \lceil k/2 \rceil$, the upper bound

$$\widehat{\lambda}_k(\zeta) \leqslant \frac{1}{\lceil k/2 \rceil} \tag{6}$$

is known, which is a slight refinement due to Laurent [**12**] of a slightly weaker result established in [**8**]. A refinement has been made for $k = 3$ by Roy [**21**]. The bound in (6) is not considered to be optimal for any $k \geqslant 2$.

1.2. *Problems from* [**6**] *and partial results.* We now introduce the problems stated as [**6**, Problems 1–3], upon which we will focus.

PROBLEM 1.1. Let k be a positive integer. Is the spectrum of the function λ_k equal to $[1/k, \infty]$?

PROBLEM 1.2. Let k be a positive integer and $\lambda \geqslant 1/k$. Determine the Hausdorff dimensions

$$\dim(\{\zeta \in \mathbb{R} : \lambda_k(\zeta) = \lambda\}), \qquad \dim(\{\zeta \in \mathbb{R} : \lambda_k(\zeta) \geqslant \lambda\}). \tag{7}$$

Concerning the third problem one needs to know that

$$\lambda_{kn}(\zeta) \geqslant \frac{\lambda_n(\zeta) - k + 1}{k}, \quad n \geqslant 1, k \geqslant 1, \tag{8}$$

which was proved by Bugeaud [**6**, Lemma 1].

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PROBLEM 1.3. Let $(\lambda_k)_{k\geqslant 1}$ be a sequence of non-increasing real numbers satisfying

$$\lambda_k \geqslant \frac{1}{k}, \quad k \geqslant 1,$$

and

$$\lambda_{kn} \geqslant \frac{\lambda_n - k + 1}{k}, \quad n \geqslant 1, k \geqslant 1.$$

Does there exist a real number ζ with $\lambda_k(\zeta) = \lambda_k$ for all $k \geqslant 1$?

Next we recall some notes and known partial results on the problems that can be found in [**6**] as well. We remark that in these results both sides in (7) will coincide, which is not explicitly mentioned in [**6**]. This can be inferred from the original results with a standard argument on dimension functions.

A positive answer to Problem 1.1 has been established only for $k \in \{1, 2\}$. In this case it is a consequence of the subsequent results we quote. Until now, for general k, it has been shown in [**6**, Theorem 2] that the spectrum contains the interval $[1, \infty]$. Nevertheless, we will not improve Problem 1.1. However, our methods will allow for improving [**6**, Theorem BL] on the spectrum of $\lambda_k(\zeta)$ among ζ restricted to Cantor's middle third set, see §4.

We turn to the metrical results. For $k = 1$, the following result concerning Problem 1.2 is due to Jarník [**10**].

THEOREM 1.4 (Jarník). *Let $k = 1$ and $\lambda \geqslant 1$ be a real number. Then both sides in (7) equal $2/(1 + \lambda)$.*

For $k \geqslant 2$ the following theorem is known [**3**].

THEOREM 1.5 (Budarina *et al*). *Let $k \geqslant 2$ be an integer and $\lambda \geqslant k - 1$. Then both sides in (7) equal $2/(k(1 + \lambda))$.*

Problem 1.2 for the sets in (7) was solved for $k = 2$ as well by Beresnevich *et al* [**2**, **24**]. Indeed, in the case $1/2 \leqslant \lambda < 1$ not contained in Theorem 1.5, they prove that

$$\dim(\{\zeta \in \mathbb{R} : \lambda_2(\zeta) = \lambda\}) = \dim(\{\zeta \in \mathbb{R} : \lambda_2(\zeta) \geqslant \lambda\}) = \frac{2 - \lambda}{1 + \lambda}. \tag{9}$$

It is worth noting that for any integer $k \geqslant 2$ and $\lambda \geqslant 1/k$ we have

$$\dim(\{\zeta \in \mathbb{R} : \lambda_k(\zeta) \geqslant \lambda\}) \geqslant \frac{2}{k(1 + \lambda)}. \tag{10}$$

For small values of λ and $k \geqslant 3$ a metric result due to Beresnevich can be found in [**1**]. However, (10) does not even allow for answering Problem 1.1 for $k \geqslant 3$. As a last reference concerning Problem 1.2 we want to point out a consequence of a famous result of Sprindžuk [**23**], that the set of $\zeta \in \mathbb{R}$ with $\lambda_k(\zeta) > 1/k$ has Lebesgue measure 0.

Concerning Problem 1.3, for now we solely mention that it is based on the fact that each of the assumptions is known to hold for any real number ζ. We quoted the origin of (8), the other restrictions are due to (2) and (3).

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1.3. *Outline of new results on the problems.* Our methods yield a converse to (8), provided that the constants λ_k are strictly greater than 1. This is the key to a better understanding of Problems 1.2, 1.3. We prove the following theorem.

THEOREM 1.6. *Let k be a positive integer and ζ be a real number such that $\lambda_k(\zeta) > 1$. Then*

$$\lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k}.$$

Moreover, for $1 \leqslant j \leqslant k$ we have $\widehat{\lambda}_j(\zeta) = 1/j$.

Remark 1.7. Note that the restriction $\lambda_k > 1$ is necessary. [**6**, Theorem 4.3] shows that there exists ζ such that $\lambda_k(\zeta) = 1$ for all $k \geqslant 1$. [**6**, Theorem 4.4] shows that for $1 \leqslant \lambda \leqslant 3$ there exists ζ with $\lambda_1(\zeta) = \lambda$ and $\lambda_2(\zeta) = 1$. This shows that, in general, $\lambda_1(\zeta)$ is not determined by $\lambda_2(\zeta)$ as in Theorem 1.6.

We now present several corollaries to Theorem 1.6. Concerning Problem 1.2, we deduce from Theorem 1.6 the assertion of Theorem 1.5 for a larger class of parameters λ.

COROLLARY 1.8. *Let $k \geqslant 2$ be an integer and λ be a parameter. For $\lambda > 1$, both sides in (7) equal $2/(k(1+\lambda))$. For $\lambda = 1$, both dimensions in (7) coincide and are at least $1/k$.*

Proof. For $\lambda > 1$, Theorem 1.6 implies the set identity

$$\{\zeta \in \mathbb{R} : \lambda_k(\zeta) = \lambda\} = \{\zeta \in \mathbb{R} : \lambda_1(\zeta) = k\lambda + k - 1\}, \tag{11}$$

and similarly for the right-hand sides in (7). In particular the corresponding Hausdorff dimensions coincide. The dimension of the right-hand side in (11) (respectively for $\geqslant$) can be determined with Theorem 1.4, which gives just $2/(k(1+\lambda))$.

For $\lambda = 1$, note that in view of (8), the equality $\lambda_1(\zeta) = 2k - 1$ implies $\lambda_k(\zeta) \geqslant 1$. However, $\lambda_k(\zeta) > 1$ is impossible due to Theorem 1.6. Thus

$$\{\zeta \in \mathbb{R} : \lambda_k(\zeta) = 1\} \supset \{\zeta \in \mathbb{R} : \lambda_1(\zeta) = 2k - 1\},$$

and the lower bound $1/k$ follows from Theorem 1.4 again. It remains to be shown that both sides in (7) coincide for $\lambda = 1$ as well. The standard argument on dimension functions mentioned subsequent to Problem 1.3 and the results in [**10**] show that actually the $1/k$-dimensional Hausdorff measure of the set $\{\zeta : \lambda_1(\zeta) > 2k-1\}$ is 0. On the other hand, $\{\zeta : \lambda_k(\zeta) > 1\} = \{\zeta : \lambda_1(\zeta) > 2k-1\}$ by virtue of Theorem 1.6. The claim follows. □

Note that (9) suggests that we cannot expect Corollary 1.8 to hold for $\lambda < 1$. We turn towards Problem 1.3. First we discuss the case of Liouville numbers, i.e. real numbers with $\lambda_1(\zeta) = \infty$, separately. In §1.1 it was pointed out that in our setting $\zeta \in \mathbb{Q}$ satisfies $\lambda_1(\zeta) = 0$, such that Liouville numbers are irrational. They are in fact transcendental by Liouville's theorem (or Roth's theorem, see §5.2). Recall from §1.1 that for Liouville numbers, in fact, $\lambda_k(\zeta) = \infty$ for

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all $k \geqslant 1$ by virtue of (8), and, on the other hand, if $\zeta \in \mathbb{Q}$ then $\lambda_k(\zeta) = 0$ for $k \geqslant 1$. Thus we need consider only the case of irrational ζ with $\lambda_1(\zeta) < \infty$. The following Corollary 1.9 to Theorem 1.6 shows that in this case all $\lambda_\cdot(\zeta) > 1$ are determined by $\lambda_1(\zeta)$ as well, and there can only be a finite number of them. This readily implies a negative answer to Problem 1.3.

COROLLARY 1.9. *Let ζ be an irrational real number with $\lambda_1(\zeta) < \infty$ and define $k_0 := \lceil (\lambda_1(\zeta) + 1)/2 \rceil$. Then*

$$\lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k}, \quad 1 \leqslant k \leqslant k_0 - 1, \tag{12}$$

$$\max\left\{\frac{\lambda_1(\zeta) - k + 1}{k}, \frac{1}{k}\right\} \leqslant \lambda_k(\zeta) \leqslant 1, \quad k \geqslant k_0. \tag{13}$$

In particular, there are only finitely many indices k with $\lambda_k(\zeta) > 1$. Moreover, the answer to Problem 1.3 is no.

Proof. Note that, for all integers $k < k_0$, by construction $(\lambda_1(\zeta) - k + 1)/k > 1$ holds. By (8) for $n = 1$, we have $\lambda_k(\zeta) > 1$ such that we can apply Theorem 1.6 to obtain the reverse inequalities in (12) for all $k < k_0$. The left inequalities in (13) are due to (8) and (2). (Observe that (2) holds for irrational algebraic numbers too, as pointed out in §1.1.) The right-hand-side inequality follows again from the definition of k_0 and Theorem 1.6.

For the last assertion, put $\lambda_1 = \lambda_2 = 1.1$ and $\lambda_k = 1/k$ for $k \geqslant 3$. One readily sees that the conditions of Problem 1.3 are satisfied, but the choice contradicts Theorem 1.6. □

In view of Remark 1.7, the finiteness assertion is wrong if we relax the assumption to $\lambda_k(\zeta) \geqslant 1$. A rearrangement of Corollary 1.9 allows for determining $\lambda_m(\zeta)$ from $\lambda_n(\zeta)$ for arbitrary indices m, n, provided that both values are strictly larger than one. In particular, this implies equality in (8) for $n > 1$ as well. This is part of the following corollary.

COROLLARY 1.10. *Let $1 \leqslant n \leqslant m$ be integers and $\zeta \in \mathbb{R}$. If $\lambda_n(\zeta) > 1$, then we have the inequality*

$$\lambda_m(\zeta) \geqslant \frac{n\lambda_n(\zeta) + n - m}{m}.$$

If, moreover, $\lambda_m(\zeta) > 1$, then

$$\lambda_m(\zeta) = \frac{n\lambda_n(\zeta) + n - m}{m}.$$

In particular, choosing $m = kn$, we obtain equality in (8).

Proof. Assuming $\lambda_n(\zeta) > 1$, Theorem 1.6 implies $(\lambda_1(\zeta) - n + 1)/n = \lambda_n(\zeta)$. On the other hand, (8) yields $\lambda_m(\zeta) \geqslant (\lambda_1(\zeta) - m + 1)/m$. The assertion follows from basic rearrangements.

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Assuming $\lambda_m(\zeta) > 1$, since $n \leqslant m$ implies $\lambda_n(\zeta) \geqslant \lambda_m(\zeta) > 1$, we may apply Theorem 1.6 to both indices. This yields $(\lambda_1(\zeta) - n + 1)/n = \lambda_n(\zeta)$ and $(\lambda_1(\zeta) - m + 1)/m = \lambda_m(\zeta)$. It is easy to deduce the identity. □

The first assertion of Corollary 1.10 motivates the question of whether the assumption $\lambda_n(\zeta) > 1$ is necessary or not. If not, we would obtain an ultimate generalization of (8).

PROBLEM 1.11. Let $m \geqslant n \geqslant 1$ be integers. Does the estimate

$$\lambda_m(\zeta) \geqslant \frac{n\lambda_n(\zeta) + n - m}{m} \tag{14}$$

hold for any $\zeta \in \mathbb{R}$?

Dropping the assumption $m \geqslant n$, counterexamples are provided by taking $m = 1, n \geqslant 2$ and any ζ with $\lambda_1(\zeta) < 2$, noting $n\lambda_n(\zeta) \geqslant 1$ by (2). Note also that (8) is trivial in the case of $\lambda_n \leqslant 1$, whereas (14) is generally not. We check that the smallest pair (m, n) that leads to a non-trivial case of (14) is $m = 4, n = 3$. We can assume $\lambda_n(\zeta) \leqslant 1$, otherwise Corollary 1.10 applies. Now, indeed, in view of $\lambda_n(\zeta) \leqslant 1$, for $m = 3, n = 2$ the estimate (14) is trivially implied by (2). For $m \geqslant 5, n = 2$, it turns out to be trivial anyway, and all other pairs (m, n) with either $m \leqslant 4$ or $n \leqslant 3$ or both yield a special case of (8). For $m = 4, n = 3$, if $\lambda_3(\zeta) \in (2/3, 1]$, then (14) would lead to some bound $\lambda_4(\zeta) > \eta > 1/4$, where $1/4$ is the trivial lower bound from (2).

Similar to the proof of Lemma 2.3, which is the main tool for the proof of Theorem 1.6, we will derive the following result in §3.

THEOREM 1.12. *Let k be a positive integer and ζ be a real number. Then*

$$\widehat{\lambda}_k(\zeta) \leqslant \max\left\{\frac{1}{k}, \frac{1}{\lambda_1(\zeta)}\right\}.$$

The assertion of Theorem 1.12, for ζ not algebraic of degree $\leqslant \lceil k/2 \rceil$, is of interest only in the case of $\lambda_1(\zeta) > \lceil k/2 \rceil$ due to (6). Also note that Theorem 1.12 improves the assertion on $\widehat{\lambda}_j$, $1 \leqslant j \leqslant k$ from Theorem 1.6, since $\lambda_k(\zeta) > 1$ is a stronger assumption than $\lambda_k(\zeta) \geqslant k$ by Theorem 1.6. See also Theorem 5.1 in §5.2.

The remainder of the paper is organized as follows. In §2 we gather preparatory results for the proofs of Theorems 1.6 and 1.12, which will be carried out in §3. In §4 we study the simultaneous rational approximation properties of numbers in special fractal sets, such as Cantor's middle third set, as already indicated in §1.2. Finally, in §5 we discuss the consequences of our results for the well-studied related constants $w_k(\zeta)$, $\widehat{w}_k(\zeta)$ dealing with polynomial approximation.

§2. *Preparatory results.* In this section we gather technical lemmas for the proofs of Theorems 1.6 and 1.12. The lemmas will be applied in the proof of

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Theorem 4.4 in §4 as well. First we need the rather elementary Lemma 2.1 concerning one-dimensional Diophantine approximation. It can be proved using elementary facts of continued fraction expansions. In the sequel we will denote by $\|\alpha\|$ the distance of $\alpha \in \mathbb{R}$ to the nearest integer as usual.

LEMMA 2.1. *Let $\zeta \in \mathbb{R}$. Suppose that for a positive integer x we have the estimate*

$$\|\zeta x\| < \tfrac{1}{2}x^{-1}. \tag{15}$$

Then there exist positive integers x_0, y_0, M_0 such that $x = M_0x_0$, $(x_0, y_0) = 1$ and

$$|\zeta x_0 - y_0| = \|\zeta x_0\| = \min_{1\leqslant v\leqslant x} \|\zeta v\|. \tag{16}$$

Moreover, we have the identity

$$\|\zeta x\| = M_0\|\zeta x_0\|. \tag{17}$$

The integers x_0, y_0, M_0 are uniquely determined by the fact that y_0/x_0 is the convergent (in lowest terms) of the continued fraction expansion of ζ with the largest denominator not exceeding x, and $M_0 = x/x_0$.

Proof. Let y be the integer for which $|\zeta x - y| < (1/2)x^{-1}$ holds. It is a well-known result on continued fractions [**16**, Satz 11] that y/x must be a convergent of ζ. Let M_0 be the greatest common divisor of x, y, such that $(x, y) = (M_0x_0, M_0y_0)$ for coprime integers x_0, y_0 and y_0/x_0 a convergent of ζ in lowest terms. Then (17) follows from

$$\|\zeta x\| = |\zeta x - y| = |\zeta M_0x_0 - M_0y_0| = M_0|\zeta x_0 - y_0| = M_0\|\zeta x_0\|. \tag{18}$$

Notice that in particular $x_0 \leqslant x$ and

$$|\zeta x_0 - y_0| \leqslant |\zeta x - y| < (1/2)x^{-1}. \tag{19}$$

Next we check (16). Assume (16) is false. Then there exist integers x_1, y_1 with $x_0 \neq x_1$, $x_1 \leqslant x$ such that

$$|\zeta x_1 - y_1| < |\zeta x_0 - y_0| \leqslant |\zeta x - y| < (1/2)x^{-1}. \tag{20}$$

By an argument very similar to (18) we can reduce the proof to the case that (x_0, y_0), (x_1, y_1) are linearly independent. A combination of (19), (20) and the conditions $x_0 \leqslant x$, $x_1 \leqslant x$ and $x_0 \neq x_1$ imply

$$\left|\frac{y_1}{x_1}-\frac{y_0}{x_0}\right| \leqslant \left|\zeta-\frac{y_1}{x_1}\right|+\left|\zeta-\frac{y_0}{x_0}\right| \leqslant \frac{1}{2xx_1}+\frac{1}{2xx_0} < \frac{1}{2x_0x_1}+\frac{1}{2x_1x_0} = \frac{1}{x_0x_1}. \tag{21}$$

On the other hand, since $y_0/x_0 \neq y_1/x_1$, we have

$$\left|\frac{y_1}{x_1} - \frac{y_0}{x_0}\right| = \left|\frac{x_0y_1 - x_1y_0}{x_0x_1}\right| \geqslant \frac{1}{x_0x_1}. \tag{22}$$

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The combination of (21) and (22) contradicts the hypothesis and hence (16) holds. It remains to be shown that x_0 must be the *largest* convergent denominator not greater than x. Otherwise, if y'/x' is another convergent (in lowest terms) with $x_0 < x' \leqslant x$, then by the *monotonic* convergence of $|q_n\alpha - p_n|$ to 0 for the convergents p_n/q_n of $\alpha \in \mathbb{R}$, see [**16**, Satz 16], the estimate (20) would be satisfied for $x_1 := x'$, $y_1 := y'$, which we just falsified. □

Definition 2.2. We call a positive integer x a *best approximation for* ζ if it satisfies

$$\|\zeta x\| = \min_{1 \leqslant v \leqslant x} \|\zeta v\|.$$

The numbers x_0 from Lemma 2.1 are obviously best approximations. We used in the proof of Lemma 2.1 that the denominators of convergents in lowest terms are best approximations. It actually follows from the law of best approximation [**16**, Satz 16] that the converse is true as well. Moreover, it is shown in [**16**, Satz 14] that at least one of two consecutive convergents satisfies (15). It is further well known that (15), with the factor $1/2$ replaced by 1, is valid for any convergent, see [**16**, Satz 10].

The most technical ingredient in the proofs of Theorems 1.6, 1.12 is the following Lemma 2.3.

LEMMA 2.3. *Let k be a positive integer and ζ be a real number.*

Then there exists a constant $C = C(k, \zeta) > 0$ such that for any integer $x > 0$ the estimate

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| < C \cdot x^{-1} \tag{23}$$

implies $y/x = y_0/x_0$ for integers $(x_0, y_0) = 1$ and x_0^k divides x, where y denotes the closest integer to ζx. A suitable choice for C is given by $C = C_0 := (1/2) \cdot k^{-1}(1 + |\zeta|)^{1-k}$.

Moreover, y_0^j/x_0^j is a convergent of the continued fraction expansion of ζ^j for $1 \leqslant j \leqslant k$. Furthermore, if (23) holds for some pair $(x, C) = (Nx_0^k, C)$ with an integer $N \geqslant 1$ and $C \leqslant C_0$, then

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| = N \cdot \max_{1 \leqslant j \leqslant k} \|\zeta^j x_0^k\|. \tag{24}$$

In particular, (23) holds for any pair $(x', C) = (Mx_0^k, C)$ with $1 \leqslant M \leqslant N$, and the minimum of the left-hand side among those x' is obtained for $x' = x_0^k$.

Proof. Suppose (23) holds for some x and $C = C_0$. Denote by y the closest integer to ζx and let y_0/x_0 be the fraction y/x in lowest terms.

Assumption (23) for $j = 1$ leads to

$$\left|\frac{y_0}{x_0} - \zeta\right| = \left|\frac{y}{x} - \zeta\right| < C_0 x^{-2}.$$

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Since $C_0 < 1/2 < 1$, we have $|y_0/x_0 - \zeta| \leqslant 1$ and thus $|y_0/x_0| \leqslant 1 + |\zeta|$. The combination of these facts yields for $1 \leqslant j \leqslant k$ the estimate

$$\left|\frac{y_0^j}{x_0^j} - \zeta^j\right| = \left|\frac{y_0}{x_0} - \zeta\right| \cdot \left|\left(\frac{y_0}{x_0}\right)^{j-1} + \cdots + \zeta^{j-1}\right| < C_0 x^{-2} \cdot k(1+|\zeta|)^{k-1} = \frac{1}{2}x^{-2}. \tag{25}$$

Suppose $x_0^k \nmid x$. Then, since $x_0 | x$, the integer x has a representation in base x_0 as

$$x = b_1 x_0 + b_2 x_0^2 + \cdots + b_{k-1} x_0^{k-1} + b_k x_0^k + \cdots + b_l x_0^l,$$

where at least one of $\{b_1, b_2, \ldots, b_{k-1}\}$ is not zero. Put $u = i + 1 \in \{2, 3, \ldots, k\}$ with i the smallest index such that $b_i \neq 0$. By construction, for all $j \neq i$ we have $b_j x_0^j (y_0^u / x_0^u) \in \mathbb{Z}$. Hence, using $(x_0, y_0) = 1$ and $b_i \neq 0$, we have the estimate

$$\left\| x \frac{y_0^u}{x_0^u} \right\| = \left\| b_i x_0^i \frac{y_0^u}{x_0^u} \right\| = \left\| \frac{b_i y_0^u}{x_0} \right\| \geqslant x_0^{-1}. \tag{26}$$

On the other hand, the estimate (25) for $j = u$ implies

$$\left| x \left(\zeta^u - \frac{y_0^u}{x_0^u} \right) \right| \leqslant \frac{1}{2} x^{-1} \leqslant \frac{1}{2} \cdot x_0^{-1}. \tag{27}$$

The combination of (26) and (27) and triangular inequality imply

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \geqslant \|\zeta^u x\| > \tfrac{1}{2} x_0^{-1} \geqslant \tfrac{1}{2} x^{-1},$$

contradicting (23) since $C_0 < 1/2$. Hence, $x_0^k | x$.

From $x_0^k | x$ we infer $x_0^k \leqslant x$, and (25) yields

$$\left|\frac{y_0^j}{x_0^j} - \zeta^j\right| < \frac{1}{2}x^{-2} \leqslant \frac{1}{2}x_0^{-2k} = \frac{1}{2}x_0^{2j-2k} \cdot (x_0^j)^{-2} \leqslant \frac{1}{2} \cdot (x_0^j)^{-2}, \quad 1 \leqslant j \leqslant k.$$

Since clearly $(x_0^j, y_0^j) = 1$ for any $1 \leqslant j \leqslant k$, Lemma 2.1 implies that y_0^j / x_0^j is indeed a convergent of ζ^j for every $1 \leqslant j \leqslant k$. Finally, we show (24). Let $x = N x_0^k$. It follows from (25) that $y_j := N x_0^{k-j} y_0^j$ is the closest integer to $\zeta^j x$, for $1 \leqslant j \leqslant k$. Thus, we obtain

$$\|\zeta^j x\| = |\zeta^j x - y_j| = N \cdot |\zeta^j x_0^k - x_0^{k-j} y_0^j| = N \cdot \|\zeta^j x_0^k\|, \quad 1 \leqslant j \leqslant k.$$

Hence the relation holds for the maximum as well. □

Remark 2.4. Analyzing the estimates in (25), the constant C_0 can be improved if we additionally assume x to be sufficiently large. One finds that Lemma 2.3 actually holds with $C_0 = (1/2) \cdot L_k(\zeta)^{-1} - \epsilon$ for arbitrary small $\epsilon > 0$, where $L_k(\zeta) := \max_{1 \leqslant j \leqslant k} (j|\zeta|^{j-1})$, for all $x \geqslant x_0(\epsilon)$. Moreover, in the case of $|\zeta| < 1/2$, the maximum in $L_k(\zeta)$ is obtained for $j = 1$ and we may put $\epsilon = 0$, which yields $C_0 = 1/2$.

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Remark 2.5. It is not hard to see Lemma 2.3 would be wrong with the right-hand side of (23) replaced by $(1/2)\cdot x_0^{-1}$ for any ζ with $\widehat{\lambda}_k(\zeta) > 1/k$ and in particular for $k = 2$ and the extremal numbers ζ mentioned in §1.1. The proof of the false stronger version fails since (27) is no longer correct.

§3. *Proof of Theorems 1.6 and 1.12.* First we prove Theorem 1.12 with a method very similar to the proof of Lemma 2.3. It might be possible to deduce Theorem 1.12 directly from this lemma; however, due to technical difficulties, we prefer to prove it directly.

Proof of Theorem 1.12. Consider k, ζ fixed. The assertion is trivial for $\zeta \in \mathbb{Q}$ and in the case of $\lambda_1(\zeta) = 1$, so we may assume ζ is irrational and $\lambda_1(\zeta) > 1$.

Let $1 < T < \lambda_1(\zeta)$ be arbitrary. By definition of the quantity $\lambda_1(\zeta)$, and since $\zeta \notin \mathbb{Q}$, there exist arbitrarily large coprime x_0, y_0 with the property that

$$\left|\zeta - \frac{y_0}{x_0}\right| \leqslant x_0^{-T-1}.$$

For sufficiently large x_0 and a constant $D_0 = D_0(k, \zeta)$, similarly as in (25) we deduce

$$\left|\zeta^j - \frac{y_0^j}{x_0^j}\right| < D_0 x_0^{-T-1}, \quad 1 \leqslant j \leqslant k. \tag{28}$$

We distinguish the cases $\lambda_1(\zeta) > k$ and $\lambda_1(\zeta) \leqslant k$.

Case 1. Suppose $\lambda_1(\zeta) > k$. Then we may assume $T > k$ as well. Let $X := x_0^k/2$. Write $1 \leqslant x \leqslant X$ in base x_0 as

$$x = b_0 + b_1 x_0 + b_2 x_0^2 + \cdots + b_{k-1} x_0^{k-1} + b_k x_0^k + \cdots + b_l x_0^l,$$

and put $u = i + 1 \in \{1, 2, \ldots, k\}$ with i the smallest index such that $b_i \neq 0$. Since x_0, y_0 are coprime and $b_i \neq 0$, we have

$$\left\| x \frac{y_0^u}{x_0^u} \right\| = \left\| b_i x_0^{u-1} \frac{y_0^u}{x_0^u} \right\| = \left\| \frac{b_i y_0^u}{x_0} \right\| \geqslant \frac{1}{x_0}. \tag{29}$$

Moreover, (28) yields for $1 \leqslant x \leqslant X$ the upper bounds

$$\left| x\left(\zeta^u - \frac{y_0^u}{x_0^u}\right) \right| \leqslant \frac{X}{2}\left|\zeta^u - \frac{y_0^u}{x_0^u}\right| \leqslant \frac{D_0}{2} x_0^{k-T-1}. \tag{30}$$

Since $T > k$, the right-hand side is smaller than $(1/2)x_0^{-1}$ for large x_0, so combining (29), (30) with triangular inequality yields for $1 \leqslant x \leqslant X$ the estimate

$$M_x(\zeta) := \max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \geqslant \|\zeta^u x\| \geqslant \tfrac{1}{2} x_0^{-1}.$$

Using the definition of $\widehat{\lambda}_k$, we conclude

$$\widehat{\lambda}_k(\zeta) \leqslant \liminf_{X \to \infty} \max_{1 \leqslant x \leqslant X} -\frac{\log M_x(\zeta)}{\log X} \leqslant \frac{1 + (\log 2/\log x_0)}{k - (\log 2/\log x_0)},$$

and with $X \to \infty$, or equivalently $x_0 \to \infty$, indeed, $\widehat{\lambda}_k(\zeta) \leqslant 1/k$.

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Case 2. Now suppose $\lambda_1(\zeta) \leqslant k$. Define $X := (1/2)D_0^{-1}x_0^T$, and again write $x \leqslant X$ in base x_0 and define $i, u, b_.$ as in case 1. We have $0 \leqslant i \leqslant \lfloor T \rfloor \leqslant T$, such that from $T < k$ we infer that $1 \leqslant u \leqslant k$. We have (29) precisely as in case 1, such that

$$\left| x\left(\zeta^u - \frac{y_0^u}{x_0^u}\right)\right| \leqslant X\left|\zeta^u - \frac{y_0^u}{x_0^u}\right| \leqslant \frac{1}{2D_0}x_0^T \cdot D_0 x_0^{-T-1} = \frac{1}{2}x_0^{-1}. \tag{31}$$

So, combining (29), (31) and triangular inequality yields

$$M_x(\zeta) := \max_{1\leqslant j\leqslant k} \|\zeta^j x\| \geqslant \|\zeta^u x\| \geqslant \tfrac{1}{2}x_0^{-1}.$$

Again we conclude

$$\widehat{\lambda}_k(\zeta) \leqslant \liminf_{X\to\infty} \max_{1\leqslant x\leqslant X} -\frac{\log M_x(\zeta)}{\log X} \leqslant \frac{1 + (\log 2/\log x_0)}{T - (\log D_0 + \log 2/\log x_0)}.$$

As we may choose T arbitrarily close to $\lambda_1(\zeta)$, $\widehat{\lambda}_k(\zeta) \leqslant 1/\lambda_1(\zeta)$ follows again with $X \to \infty$ or, equivalently, $x_0 \to \infty$. □

Next we prove Theorem 1.6 using Lemmas 2.3 and 2.1.

Proof of Theorem 1.6. In view of (8), for the assertion on $\lambda_k(\zeta)$ we only have to show that provided $\lambda_k(\zeta) > 1$ holds we have

$$\lambda_k(\zeta) \leqslant \frac{\lambda_1(\zeta) - k + 1}{k}. \tag{32}$$

The definition of the quantity $\lambda_k(\zeta)$ implies that, for any fixed $1 < T < \lambda_k(\zeta)$, the inequality

$$\max_{1\leqslant j\leqslant k} \|\zeta^j x\| \leqslant x^{-T} \tag{33}$$

has arbitrarily large integer solutions x. One checks that for any $\tau > 0$ and sufficiently large $x > \hat{x}(\tau, T) := \tau^{1/(1-T)}$ we have $x^{-T} < \tau x^{-1}$. Choosing $\tau \leqslant C_0$ with $C_0 < 1/2$ from Lemma 2.3, condition (33) ensures we may apply both Lemmas 2.3 and 2.1 for $x \geqslant \hat{x}$, with coinciding pairs x_0, y_0 such that y_0/x_0 is the reduced fraction y/x. Further, let M_0 be as in Lemma 2.1. Writing $M_0 = x_0^\eta$, by Lemma 2.3 we infer $\eta \geqslant k - 1$. Moreover, define T_0 implicitly by $x_0^{-T_0} = |\zeta x_0 - y_0|$, i.e.

$$T_0 = -\frac{\log|\zeta x_0 - y_0|}{\log x_0}.$$

The derived properties yield

$$\begin{aligned} T \leqslant -\frac{\log\|\zeta x\|}{\log x} &= -\frac{\log(M_0|\zeta x_0 - y_0|)}{\log(M_0 x_0)} \\ &\leqslant \frac{T_0 - \eta}{1+\eta} \leqslant \frac{T_0 - (k-1)}{1 + (k-1)} = \frac{T_0 - k + 1}{k}. \end{aligned}$$

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Since this is true for arbitrarily large values of x (and thus x_0) and we may choose T arbitrarily close to $\lambda_k(\zeta)$, the definition of T_0 implies (32).

Since $\lambda_1(\zeta) = k\lambda_k(\zeta) + k - 1 > 2k - 1 \geqslant k$, the assertion on $\widehat{\lambda}_j(\zeta)$ follows from Theorem 1.12. □

We actually proved something stronger than Theorem 1.6. We point out the more general results that evolved from the proof as a corollary.

COROLLARY 3.1. *Let $k \geqslant 2$ be an integer and ζ be a real number. For any fixed $T > 1$, there exists $\hat{x} = \hat{x}(T, \zeta)$ such that the estimate*

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \leqslant x^{-T}$$

for an integer $x \geqslant \hat{x}$ implies the existence of x_0, y_0, M_0 as in Lemma 2.1 with the properties

$$x \geqslant x_0^k, \qquad M_0 \geqslant x_0^{k-1}, \qquad |\zeta x_0 - y_0| \leqslant x_0^{-kT-k+1}. \tag{34}$$

Similarly, if for $C_0 = C_0(k, \zeta)$ from Lemma 2.3 the inequality

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| < C_0 \cdot x^{-1}$$

has an integer solution $x > 0$, then (34) holds with $T = 1$.

For direct consequences of Corollary 3.1, see §5.

§4. *Diophantine approximation in fractal sets.*

4.1. *Definitions and results.* The middle-third Cantor set $\mathscr{C}$ is defined as the real numbers a in $[0, 1]$ that can be written in the form

$$a = c_1 3^{-1} + c_2 3^{-2} + \cdots, \quad c_i \in \{0, 2\}.$$

The spectrum of the quantity $\lambda_k(\zeta)$ with the restriction that ζ belongs to the Cantor set has been studied. For $k = 1$, the question is solved by the following constructive result [**5**, Theorem 2]. We use slightly different notation than in [**5**] for correlation with our upcoming results.

THEOREM 4.1 (Bugeaud). *Let $\tau \in [1, \infty)$ and $\alpha > 0$. Any number*

$$\zeta = 2 \sum_{n \geqslant 1} 3^{-\lceil \alpha(1+\tau)^n \rceil} \tag{35}$$

belongs to $\mathscr{C}$ and satisfies $\lambda_1(\zeta) = \tau$. In particular, the spectrum of λ_1 restricted to $\mathscr{C}$ equals $[1, \infty]$.

Indeed, the case $\lambda_1(\zeta) = \infty$ not explicitly mentioned in [**5**, Theorem 2] is obtained similarly by a sequence with hyper-exponential growth in the exponent, such that we can include the value ∞ in Theorem 4.1.

The best current result concerning the spectrum of λ_k within $\mathscr{C}$ for $k \geqslant 2$, which is [**6**, Theorem BL], originates in ζ as in (35) incorporating [**4**, Theorem 7.7].

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THEOREM 4.2 (Bugeaud and Laurent). *Let $k \geqslant 2$ be an integer. The spectrum of λ_k among ζ in $\mathscr{C}$ contains the interval* $[(1+\sqrt{4k^2+1})/(2k), \infty]$.

Again, as conjectured for real numbers ζ in Problem 1.1, there is reason to believe that the spectrum actually equals $[1/k, \infty]$. The following immediate consequence of Theorem 1.6 yields an improvement of Theorem 4.2.

THEOREM 4.3. *Let $k \geqslant 2$ be an integer and $\alpha > 0, \rho > 0$. Then $\zeta = 2\sum_{n\geqslant 1} 3^{-\lceil \alpha(k(1+\rho))^n \rceil}$ belongs to the Cantor set. If $\rho \in (0, 1]$, then*

$$\max\left\{\rho, \frac{1}{k}\right\} \leqslant \lambda_k(\zeta) \leqslant 1. \tag{36}$$

If $\rho \in (1, \infty]$, we have equality

$$\lambda_k(\zeta) = \rho. \tag{37}$$

Proof. By Theorem 4.1 with $\tau := k\rho + k - 1 > 1$, we have $\lambda_1(\zeta) = k\rho + k - 1$. If $\rho > 1$, then $k < k_0 := \lceil (\lambda_1(\zeta) + 1)/2 \rceil$ such that with Corollary 1.9, we obtain (37). If $\rho \in (0, 1]$, then $k \geqslant k_0$ and the assertion (36) again follows from Corollary 1.9. □

Theorem 4.3 obviously yields the improvement of Theorem 4.2 so that the spectrum of λ_k contains $[1, \infty]$. However, we want to prove the more general statement, Theorem 4.4. It is more flexible in the choice of suitable ζ and extends Theorem 4.3 to expansions in an arbitrary base.

THEOREM 4.4. *Let $k \geqslant 2, b \geqslant 2$ be integers and $\rho \in (0, \infty]$. Let $(a_n)_{n\geqslant 1}$ be a strictly increasing sequence of positive integers with the property*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = k(\rho + 1). \tag{38}$$

Let

$$\zeta = \sum_{n\geqslant 1} b^{-a_n}. \tag{39}$$

If $\rho \in (0, 1)$, then

$$\max\left\{\frac{1}{k}, \rho\right\} \leqslant \lambda_k(\zeta) \leqslant 1. \tag{40}$$

If $\rho \in [1, \infty]$, we have equality

$$\lambda_k(\zeta) = \rho. \tag{41}$$

Moreover, if $\rho \geqslant 1/k$, then $\widehat{\lambda}_k(\zeta) = 1/k$.

Note the similarity to the constructions of Theorem 4.1, where the analogue result was established for $k = 1$ and the sequence $a_n = \lceil \alpha(k(1+\rho))^n \rceil$ for $\alpha > 0$. Roughly speaking, the additional factor k in the quotient a_{n+1}/a_n allows

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for generalizing the one-dimensional result. However, the methods of the proofs of Theorems 4.1 and 4.4 are much different. The approach in this paper is rather connected to the one in [**13**], where a slightly weaker result than Theorem 4.1 was established.

We encourage the reader to compare the following corollary to Theorem 4.4 with [**5**, Theorem 1], which we will not state, where a more general result in the special case $k = 1$ was established.

COROLLARY 4.5. *Let $k \geqslant 2$, $b \geqslant 2$ be integers and $\mathscr{A} \subset \{0, 1, \dots, b-1\}$ of cardinality $|\mathscr{A}| \geqslant 2$. The spectrum of the approximation constant $\lambda_k(\zeta)$, restricted to $\zeta \in (0, 1)$, whose expansion in base b have all digits in $\mathscr{A}$, contains $[1, \infty]$. In particular, for any $\epsilon > 0$ there exists a set $\mathscr{B}$ of Hausdorff dimension less than ϵ such that the spectrum of λ_k within $\mathscr{B}$ contains $[1, \infty]$.*

For the proof of Theorem 4.4 we will need an estimate for the concrete numbers ζ in (39). This will be established in Lemma 4.7. For its proof we apply a proposition connected to Lemma 2.1.

PROPOSITION 4.6. *Let $\zeta \in \mathbb{R}$. Then for no parameter $Q > 0$ the system*

$$|m| \leqslant Q, \qquad |\zeta m - n| < \frac{1}{2Q} \tag{42}$$

has two linearly independent solutions $(m, n) \in \mathbb{Z}^2$.

Proof. We may assume $Q \geqslant 1$, otherwise there do not exist two linearly independent vectors anyway. Hence, (42) implies $m \neq 0$, so $m > 0$ is no restriction (for else if $m < 0$ consider $(-m, -n)$). For fixed Q, say $(m_1, n_1) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ is a solution to (42) with largest m_1 among all such solutions. We have to show that any vector $(m_2, n_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ linearly independent of (m_1, n_1) with $m_2 \leqslant m_1 \leqslant Q$ satisfies $|\zeta m_2 - n_2| \geqslant (1/2)Q^{-1}$. We infer

$$\left|\zeta - \frac{n_1}{m_1}\right| < \frac{1}{2m_1 Q} \leqslant \frac{1}{2m_1 m_2}$$

from (42). Thus, the linear independence condition implies

$$\left|\zeta - \frac{n_2}{m_2}\right| \geqslant \left|\frac{n_1}{m_1} - \frac{n_2}{m_2}\right| - \left|\zeta - \frac{n_1}{m_1}\right| \geqslant \frac{1}{m_1 m_2} - \frac{1}{2m_1 m_2} = \frac{1}{2m_1 m_2} \geqslant \frac{1}{2m_2 Q}.$$

Multiplying with m_2 yields the assertion. □

An alternative proof of Proposition 4.6 is obtained by regarding it as a special case, in the plane, of Minkowski's second lattice point theorem [**15**] on convex bodies.

LEMMA 4.7. *Let $k \geqslant 2, b \geqslant 2$ be integers, $\rho > 0$ and ζ be as in (39) for some sequence $(a_n)_{n \geqslant 1}$ as in (38). Then for $(x, y) \in \mathbb{Z}^2$ with sufficiently large x, the estimate*

$$|\zeta x - y| \leqslant x^{-k/(k-1)} \tag{43}$$

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implies (x, y) *an integral multiple of some*

$$\underline{x}_n := (x_n, y_n) := \left(b^{a_n}, \sum_{i \leqslant n} b^{a_n - a_i}\right). \tag{44}$$

Proof. First note that for any fixed n, the entries x_n, y_n of the vectors $\underline{x}_n = (x_n, y_n)$ are coprime, since x_n consists of prime factors dividing b and $y_n \equiv 1 \bmod b$. Hence any vector $(x, y) \in \mathbb{Z}^2$ which is no integral multiple of $\underline{x}_n$ is actually linearly independent of it.

Assume the lemma is false. Then by the above observation, there exist arbitrarily large (x, y) for which (43) holds and which are linearly independent of all $\underline{x}_n$. Let $\delta > 0$ not be too large; in particular, $\delta = 1$ will be a proper choice if we assume x (or n) is sufficiently large. Say n is the index with $b^{a_n} \leqslant x < b^{a_{n+1}}$.

First, suppose $x \leqslant b^{a_{n+1} - (1+\delta)a_n}$. Put $Q = x$. If n or equivalently x is sufficiently large, then by assumption (43) we have

$$-\frac{\log|\zeta x - y|}{\log Q} = -\frac{\log|\zeta x - y|}{\log x} \geqslant \frac{k}{k-1} > 1.$$

On the other hand, the estimate

$$\|b^{a_n}\zeta\| = \sum_{i \geqslant n+1} b^{a_n - a_i} \leqslant 2 \cdot b^{a_n - a_{n+1}} \tag{45}$$

implies

$$-\frac{\log|\zeta x_n - y_n|}{\log Q} = -\frac{\log\|b^{a_n}\zeta\|}{\log x} \geqslant \frac{a_{n+1} - a_n - (\log 2/\log b)}{a_{n+1} - (1+\delta)a_n} > 1. \tag{46}$$

Hence, for some fixed $\epsilon > 0$ and arbitrarily large n, the system

$$|M| \leqslant Q, \qquad |\zeta M - N| \leqslant Q^{-1-\epsilon} \tag{47}$$

has two linearly independent integral solutions $(M, N) = (x, y), (M, N) = (x_n, y_n)$. Since the above holds for all $n \geqslant 1$ and we may assume $Q = x > 2^{1/\epsilon}$, we infer a contradiction to Proposition 4.6.

In the remaining case $b^{a_{n+1} - (1+\delta)a_n} \leqslant x < b^{a_{n+1}}$, put $Q = b^{a_{n+1}}$. For sufficiently large n, clearly (45) for n replaced by $n+1$ shows that (x_{n+1}, y_{n+1}) satisfies (47) for some $\epsilon > 0$ (actually any $\epsilon \geqslant k(\rho+1) - 1 \geqslant k - 1 \geqslant 1$). On the other hand, (43) yields

$$-\frac{\log|\zeta x - y|}{\log Q} = -\frac{\log|\zeta x - y|}{\log x} \cdot \frac{\log x}{\log Q} \geqslant \frac{k}{k-1} \cdot \frac{a_{n+1} - (1+\delta)a_n}{a_{n+1}}. \tag{48}$$

For $\rho > 0$, we have

$$\lim_{n \to \infty, \delta \to 0} \frac{a_{n+1} - (1+\delta)a_n}{a_{n+1}} = \frac{k(\rho+1) - 1}{k(\rho+1)} > \frac{k-1}{k}.$$

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Hence, the right-hand side in (48) is strictly greater than 1 and consequently the left is too. Thus, for some $\epsilon > 0$, the system (47) has linearly independent integral solutions $(x, y), (x_{n+1}, y_{n+1})$ again, contradicting Proposition 4.6 for large n (respectively Q). □

Remark 4.8. The continued fraction expansion of numbers ζ as in Theorem 4.4 can be explicitly established using some variant of the Folding Lemma, see [**5**] or [**14**]. This should allow for proving the assertion of Lemma 4.7 even for slightly larger exponents than $-k/(k-1)$ in (43). It is reasonable that even the optimal value in (43) in the dependence of ρ for which the claim of Lemma 4.7 holds can be determined. It is possible to show, though, that Lemma 4.7 does not apply with exponent $-1-\epsilon$ for some $\epsilon > 0$ (otherwise the proof of Theorem 4.4 could be simplified). However, improvements of this kind are not necessary for our purposes. In fact, we only need the much weaker bound $-2k+1$ instead of $-k/(k-1)$ for the proof of Theorem 4.4.

Remark 4.9. In fact, Lemma 4.7 is still true by essentially the same proof if we relax the assumption (38) to the weaker condition $\liminf_{n\to\infty} a_{n+1}/a_n \geqslant k(\rho+1)$.

4.2. *Proof of Theorem 4.4.* We now prove Theorem 4.4 by using Lemmas 2.1 and 4.7 and Corollary 3.1. Lower bounds for $\lambda_k(\zeta)$ in Theorem 4.4 will be rather straightforward to derive by looking at integers of the form $x = b^{ka_n}$ for large n, whereas the proof of the more interesting upper bounds is slightly technical. We sketch the outline of the proof of the upper bounds. We distinguish between integers x with the property that $\|\zeta x_0\| < x_0^{-2k+1}$ for x_0 the largest best approximation $\leqslant x$, see Definition 2.2, and those for which this inequality is wrong. Lemmas 2.1 and 4.7 allow for an easy classification of the values x belonging to the first class to which Lemmas 2.3 and 2.1 can be effectively applied to obtain upper bounds. For the remaining class of integers x, the negated formulation of Corollary 3.1 immediately yields the upper bound 1.

Proof of Theorem 4.4. We consider $k \geqslant 2$ and $\rho > 0$ fixed, and a corresponding sequence $(a_n)_{n\geqslant 1}$ and ζ as in (39) is constructed via the sequence. Note that

$$\|b^{a_n}\zeta\| = \sum_{i\geqslant n+1} b^{a_n-a_i} \leqslant 2\cdot b^{a_n-a_{n+1}}. \tag{49}$$

We first prove the assertion on $\widehat{\lambda}_k$. Assuming $\rho \geqslant 1/k$, for any $\delta > 0$ and sufficiently large $n \geqslant \widehat{n}(\delta)$ we have

$$a_{n+1} \geqslant (k+k\rho-\delta)a_n \geqslant (k+1-\delta)a_n.$$

If we choose integers x of the form b^{a_n}, the estimate (49) and $\delta \to 0$ imply

$$\lambda_1(\zeta) \geqslant \limsup_{n\geqslant 1} -\frac{\log(2\cdot b^{a_n-a_{n+1}})}{\log b^{a_n}} \geqslant \limsup_{n\geqslant 1} \frac{(k+1)a_n - a_n}{a_n} = k.$$

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For any such real number ζ, the assertion follows directly from Theorem 1.12.

To prove (40) and (41), we show

$$\max\left\{\frac{1}{k}, \rho\right\} \leqslant \lambda_k(\zeta) \leqslant \max\{1, \rho\}. \tag{50}$$

We start with the left inequality. We only have to show $\lambda_k(\zeta) \geqslant \rho$; the other inequality $\lambda_k(\zeta) \geqslant 1/k$ is trivial by (2). It suffices to consider integers x of the form $x = b^{ka_n}$. Write $\zeta = S_n + \epsilon_n$ with

$$S_n = \sum_{i=1}^{n} b^{-a_i}, \qquad \epsilon_n = \sum_{i=n+1}^{\infty} b^{-a_i}.$$

Since $S_n < 1$, $\epsilon_n < 1$ and the binomial coefficients are bounded above by $k!$, we have that

$$\zeta^j = \sum_{i=0}^{j} \binom{j}{i} S_n^i \epsilon_n^{j-i} = S_n^j + O(\epsilon_n), \quad 1 \leqslant j \leqslant k,$$

as $n \to \infty$, with the implied constant depending on k only. The crucial point now is that xS_n^j is an integer for $1 \leqslant j \leqslant k$ by construction. Thus for some constant $C_0 > 0$ independent of n and $C_1 = 2C_0$, we have

$$\|x\zeta^j\| \leqslant C_0 \sum_{i=n+1}^{\infty} b^{-a_i} x = C_0 \cdot b^{ka_n} \sum_{i=n+1}^{\infty} b^{-a_i} \leqslant C_1 \cdot b^{ka_n - a_{n+1}}, \quad 1 \leqslant j \leqslant k. \tag{51}$$

The condition (38) implies, for any $\nu > 0$ and sufficiently large $n \geqslant \hat{n}(\nu)$,

$$ka_n - a_{n+1} \leqslant (-k\rho + \nu)a_n.$$

For sufficiently large n (or equivalently x), combination with (51) yields

$$\max_{1 \leqslant j \leqslant k} \|x\zeta^j\| \leqslant C_1 \cdot b^{(-k\rho+\nu)a_n} = C_1 \cdot x^{-\rho+\nu/k} \leqslant x^{-\rho+2\nu/k}.$$

Since ν can be taken arbitrarily small we indeed obtain $\lambda_k(\zeta) \geqslant \rho$ for any fixed $\rho > 0$.

We are left to prove the right-hand side of (50), which we do indirectly. Suppose there exist $\rho > 0$ and ζ as in Theorem 4.4 such that $\lambda_k(\zeta) > \max\{1, \rho\}$. Then for $\epsilon = (\lambda_k(\zeta) + 1)/2 - 1 > 0$ the inequality

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \leqslant x^{-1-\epsilon}$$

has arbitrarily large solutions x. Consequently, Corollary 3.1 applies. It yields that $x = M_0 x_0$, for some best approximation x_0 such that $|\zeta x_0 - y_0| < x_0^{-2k+1}$ for some y_0 with $(x_0, y_0) = 1$, and that $M_0 \geqslant x_0^{k-1}$. Note that $k/(k-1) < 2k-1$ for $k \geqslant 2$. Recall from the proof of Lemma 4.7 that for any fixed n, the entries

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of the vectors $\underline{x}_n = (x_n, y_n)$ defined in (44) are coprime too. Thus Lemma 4.7 shows that for large x_0, the inequality $|\zeta x_0 - y_0| < x_0^{-2k+1}$ can be satisfied only if $(x_0, y_0) = \underline{x}_n$ for some n. Hence we can write $x_0 = b^{a_n}$, and consequently $M_0 \geqslant b^{(k-1)a_n}$ and $x \geqslant b^{ka_n}$. Observe that

$$\|b^{a_n}\zeta\| = \sum_{i \geqslant n+1} b^{a_n - a_i} \geqslant b^{a_n - a_{n+1}}.$$

Hence, (17) of Lemma 2.1 yields

$$\|\zeta x\| = M_0 |\zeta x_n - y_n| \geqslant b^{(k-1)a_n} \cdot b^{a_n - a_{n+1}} = b^{ka_n - a_{n+1}}. \tag{52}$$

By the assumption (38) on the sequence $(a_n)_{n \geqslant 1}$, we have, for any $\eta > 0$ and sufficiently large $n \geqslant \widehat{n}(\eta)$ (or equivalently x large enough),

$$ka_n - a_{n+1} \geqslant (-k\rho - \eta)a_n.$$

Together with (52) we infer

$$\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \geqslant \|\zeta x\| \geqslant b^{(-k\rho - \eta)a_n} \geqslant x^{-\rho - \eta/k}.$$

Thus, the approximation constant $\lambda_k(\zeta)$ restricted to pairs (x, y) linearly dependent on some $\underline{x}_n$ is bounded above by $\rho + \eta/k$. As we may choose η arbitrarily small, this contradicts the assumption $\lambda_k(\zeta) > \max\{1, \rho\}$. □

Remark 4.10. In fact, we proved that, for any $\epsilon > 0$ and sufficiently large $n \geqslant \hat{n}(\epsilon)$, any integer $x \in [b^{a_n}, b^{a_{n+1}})$ not divisible by b^{a_n} satisfies $\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \geqslant x^{-1-\epsilon}$. Using the argument of case $T = 1$ in Corollary 3.1 within the proof instead of the $T > 1$ case, this can be sharpened to $\max_{1 \leqslant j \leqslant k} \|\zeta^j x\| \geqslant C_0 x^{-1}$ with C_0 from Lemma 2.3.

Eventually, we state two obvious conjectures concerning generalizations of Theorem 4.4. Both would imply a positive answer to Problem 1.1.

CONJECTURE 4.11 (Weak). *Let $k \geqslant 2$, $b \geqslant 2$ be integers and let $\rho \geqslant 1/k$. Let $(a_n)_{n \geqslant 1}$ be a strictly increasing sequence of positive integers with the property (38) and ζ as in (39). Then $\lambda_k(\zeta) = \rho$.*

CONJECTURE 4.12 (Strong). *Let $k \geqslant 2$, $b \geqslant 2$ be integers and let $\rho > 0$. Let $(a_n)_{n \geqslant 1}$ be a strictly increasing sequence of positive integers with the property (38) and ζ as in (39). Then $\lambda_k(\zeta) = \max\{\rho, 1/k\}$.*

The crucial point why the methods in the proof of Theorem 4.4 do not allow for establishing better upper bounds for $\lambda_k(\zeta)$ in the context of Theorem 4.4 in the case of $\rho < 1$ is that no extension of Lemma 2.3 to this case seems available.

https://doi.org/10.1112/S0025579315000182 Published online by Cambridge University Press

§5. *Consequences of Theorems 1.6, 1.12 for the dual constants.*

5.1. *Definition of the dual problem.* We conclude with applications to the dual constants $w_k(\zeta)$, $\widehat{w}_k(\zeta)$. For $\underline{\zeta} = (\zeta_1, \dots, \zeta_k) \in \mathbb{R}^k$, the quantities $w_k(\underline{\zeta})$ and $\widehat{w}_k(\underline{\zeta})$ are respectively defined as the supremum of real numbers ν such that

$$\max_{0 \leqslant j \leqslant k} |x_j| \leqslant X, \qquad 0 < |x_0 + \zeta_1 x_1 + \cdots + \zeta_k x_k| \leqslant X^{-\nu},$$

has a solution $(x_0, x_1, \dots, x_k) \in \mathbb{Z}^{k+1}$ for arbitrarily large X and for all $X \geqslant X_0$, respectively. Further, let

$$w_k(\zeta) := w_k(\zeta, \zeta^2, \dots, \zeta^k), \qquad \widehat{w}_k(\zeta) := \widehat{w}_k(\zeta, \zeta^2, \dots, \zeta^k).$$

Dirichlet's Theorem yields for ζ not algebraic of degree $\leqslant k$ the estimates

$$k \leqslant \widehat{w}_k(\zeta) \leqslant w_k(\zeta) \leqslant \infty. \tag{53}$$

Moreover, it is not hard to check that (3), (4) extend to

$$\begin{aligned} &\cdots \leqslant \lambda_3(\zeta) \leqslant \lambda_2(\zeta) \leqslant \lambda_1(\zeta) = w_1(\zeta) \leqslant w_2(\zeta) \leqslant w_3(\zeta) \leqslant \cdots, \\ &\cdots \leqslant \widehat{\lambda}_3(\zeta) \leqslant \widehat{\lambda}_2(\zeta) \leqslant \widehat{\lambda}_1(\zeta) = \widehat{w}_1(\zeta) \leqslant \widehat{w}_2(\zeta) \leqslant \widehat{w}_3(\zeta) \leqslant \cdots. \end{aligned} \tag{54}$$

Khintchine's transference principle [**11**] allows for a connection between the constants λ_k and the constants w_k. Indeed, the original version is shown to be equivalent to

$$\frac{w_k(\zeta)}{(k-1)w_k(\zeta) + k} \leqslant \lambda_k(\zeta) \leqslant \frac{w_k(\zeta) - k + 1}{k} \tag{55}$$

for all $\zeta \in \mathbb{R}$ in [**4**]. The analogue for the uniform constants $\widehat{w}_k, \widehat{\lambda}_k$ holds as well [**7**], in fact a refined version due to German [**9**] can be written as

$$\frac{\widehat{w}_k(\zeta) - 1}{(k-1)\widehat{w}_k(\zeta)} \leqslant \widehat{\lambda}_k(\zeta) \leqslant \frac{\widehat{w}_k(\zeta) - k + 1}{(k-1)\widehat{w}_k(\zeta) - k(k-2)}. \tag{56}$$

We should mention that (55) and (56) are valid in the much more general context of real vectors $\underline{\zeta} \in \mathbb{R}^k$ linearly independent over $\mathbb{Q}$ together with $\{1\}$. Consequently, the relations

$$\lambda_k(\zeta) = \frac{1}{k} \iff w_k(\zeta) = k, \qquad \widehat{\lambda}_k(\zeta) = \frac{1}{k} \iff \widehat{w}_k(\zeta) = k \tag{57}$$

hold for any positive integer k and any real number ζ. Furthermore,

$$\widehat{w}_k(\zeta) \leqslant 2k - 1 \tag{58}$$

for ζ not algebraic of degree $\leqslant k$ is a consequence of [**8**, Theorem 2b].

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5.2. *Results for the dual problem.* We say in advance that in the proofs in this section, we will assume $k \geqslant 2$, since for $k = 1$ the assertions follow from (2), (5) and (53) if they are not trivial. It will be convenient to use Roth's theorem [**17**] in some places to exclude the case that the numbers ζ involved are algebraic, in order to apply (58). It asserts that for irrational algebraic ζ we have $\lambda_1(\zeta) = 1$. Note also that Corollary 3.1 yields the equivalence of the inequalities $\lambda_k(\zeta) > 1$ and $w_1(\zeta) = \lambda_1(\zeta) > 2k - 1$.

First we want to point out a consequence of Theorem 1.12. It allows for improving the bound in (58) provided that $\lambda_1(\zeta)$ is sufficiently large.

THEOREM 5.1. *Let k be a positive integer and ζ an irrational real number. In case of $w_1(\zeta) \geqslant k$, which is in particular true if $\lambda_k(\zeta) > 1$, the equalities*

$$\widehat{w}_1(\zeta) = 1, \qquad \widehat{w}_2(\zeta) = 2, \quad \ldots, \quad \widehat{w}_k(\zeta) = k \tag{59}$$

hold. In the case of $k - 1 < w_1(\zeta) < k$, we have the inequalities

$$k \leqslant \widehat{w}_k(\zeta) \leqslant \min\left\{\frac{w_1(\zeta)}{w_1(\zeta) - k + 1}, 2k - 1\right\}. \tag{60}$$

Proof. Note that $w_1(\zeta) = \lambda_1(\zeta)$ by (54). Thus, Theorem 1.12 implies $\widehat{\lambda}_j(\zeta) = 1/j$ for $1 \leqslant j \leqslant k$, and (59) follows from (57). Concerning the non-trivial right-hand-side inequality in (60), note that, by assumption and Roth's theorem, ζ is transcendental, such that the upper bound $2k - 1$ is obtained from (58). The remaining upper bound in (60) follows from (56) and Theorem 1.12 via

$$\frac{\widehat{w}_k(\zeta) - 1}{(k-1)\widehat{w}_k(\zeta)} \leqslant \widehat{\lambda}_k(\zeta) \leqslant \max\left\{\frac{1}{k}, \frac{1}{\lambda_1(\zeta)}\right\} = \max\left\{\frac{1}{k}, \frac{1}{w_1(\zeta)}\right\} = \frac{1}{w_1(\zeta)}$$

by elementary rearrangements. □

One readily sees (60) is an improvement to (58) in the case of $w_1(\zeta) > k - 1/2$.

In combination with certain results established before, Theorem 5.1 allows for finally determining *all* the classical approximation constants introduced in §§1.1 and 5.1 for Liouville numbers.

COROLLARY 5.2. *Let ζ be a Liouville number, i.e. a real number which satisfies $\lambda_1(\zeta) = \infty$. Then for any $k \geqslant 1$ we have*

$$\lambda_k(\zeta) = \infty, \qquad \widehat{\lambda}_k(\zeta) = \frac{1}{k}, \qquad w_k(\zeta) = \infty, \qquad \widehat{w}_k(\zeta) = k.$$

Proof. The assertion on $\widehat{w}_k(\zeta)$ follows from (59). The claim on $\lambda_k(\zeta)$ was established in [**6**, Corollary 2], as carried out before Corollary 1.9. The assertion on $\widehat{\lambda}_k(\zeta)$ follows either from Theorem 1.12 or (57), and the assertion on $w_k(\zeta)$ is due to (54). □

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Recall $\lambda_k(\zeta) = \infty$ for any Liouville number ζ and all $k \geqslant 1$, as already mentioned in §1.1. In particular, for any $k \geqslant 1$ and any parameter $\theta > 0$, the estimate $\max_{1\leqslant j\leqslant k} \|\zeta^j x\| < \theta x^{-1}$ has arbitrarily large integer solutions x. Together with the case $T = 1$ in Corollary 3.1, we derive a new criterion for a number to be a Liouville number.

THEOREM 5.3. *An irrational real number ζ is a Liouville number if and only if, for any positive integer k, the estimate*

$$\max_{1\leqslant j\leqslant k} \|\zeta^j x\| < C_0(k, \zeta) \cdot x^{-1},$$

with C_0 defined in Lemma 2.3, has an integer solution $x = x(k, \zeta) > 0$.

Finally, in the case $\lambda_k(\zeta) > 1$, Theorem 1.6 implies the right-hand-side inequality in Khintchine's transference principle (55) and a criterion for equality.

THEOREM 5.4. *Let k be a positive integer and ζ be a real number with $\lambda_k(\zeta) > 1$, or, equivalently, $w_1(\zeta) = \lambda_1(\zeta) > 2k - 1$. We have equality in the right-hand-side inequality of (55) if and only if*

$$w_1(\zeta) = w_2(\zeta) = \cdots = w_k(\zeta).$$

Proof. In view of (54) and Theorem 1.6, for $\lambda_k(\zeta) > 1$,

$$\lambda_k(\zeta) = \frac{\lambda_1(\zeta) - k + 1}{k} = \frac{w_1(\zeta) - k + 1}{k} \leqslant \frac{w_k(\zeta) - k + 1}{k}.$$

Clearly the given equivalence is valid. □

The "only if" statement is the contribution of Theorem 1.6; the "if" part can be inferred from (8) and (55) without any restriction on $\lambda_k(\zeta)$ as already implicitly carried out in the proof of [**6**, Theorem 2].

Acknowledgements. The author warmly thanks Yann Bugeaud and the anonymous referee for remarks that helped improve the original version, in particular the presentation.

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Johannes Schleischitz,
Institute of Mathematics,
University of Natural Resources
and Life Sciences, Vienna,
Austria
E-mail: johannes.schleischitz@boku.ac.at

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