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Bohm sheath criteria and double layers in multispecies plasmas

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Bohm sheath criteria and double layers are investigated in general multispecies plasmas containing a number of lighter, thermal species, described by Boltzmann distributions, in the presence of a number of heavier species described by cold fluid theory. A nonlinear description for one-dimensional stationary self-similar electrostatic modes is given using the Sagdeev potential. We generalize the classic Bohm sheath criteria by giving appropriate multispecies definitions of drift and ion-acoustic velocities, leading to a global supersonic condition. We also discuss plasmas with only one Boltzmann species, to show that double layers cannot be supported at all, no matter how many cold fluid species are present, and do so by analytical rather than numerical means, based on the full Sagdeev potential. Series expansions for weak double layers occurring in the literature can only claim to find double layers, erroneously as we indicate, by not confirming whether the necessary consistency relations are obeyed, and also by not checking that 'neglected' terms in the expansion overwhelm those retained.

1. Introduction

Recent developments in astrophysical multi-ion plasmas, like cometary or dusty plasmas with unusual characteristics compared to ordinary hydrogen plasmas, have rekindled interest in some basic aspects of multispecies plasma descriptions. One of these is the proper description of Bohm sheath criteria, for plasmas containing both positive and negative ions (Deutsch and Räuchle 1992; Yu *et al.* 1992; Hellberg *et al.* 1993). Another has to do with the ongoing effort to better delineate the occurrence of double layers in different plasmas, in view of their importance for the acceleration of particles, such as in auroras and other space plasmas (Raadu 1989, 1993). It turns out that time and again authors have tried and claimed to obtain double layers under conditions where none can exist, if proper care is taken. We refer to expansion techniques based on weakly nonlinear theories, which are not applied self-consistently, or when asymptotic expansions diverge because of (neglected) higher-order terms, larger than the terms retained. Examples will be given at the appropriate junctures in the paper.

Since both problems share common analytical calculations, we have striven to give a really general description of multispecies plasmas in this respect. There are

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a number of lighter, thermal species, described by Boltzmann distributions and hence effectively inertia-less. In addition, there are a number of heavier species described by cold fluid theory. In both classes the number of different species is not fixed a priori. In such plasmas we have derived the nonlinear description for one-dimensional stationary self-similar electrostatic modes by means of the Sagdeev potential. Hence we could investigate many situations, since our results are fully general.

In view of the common and repeated mistakes in the literature, however, the discussion will be limited mostly to plasmas with a single Boltzmann species, to show that double layers cannot be supported at all then, no matter how many cold fluid species are present. We hope to do so in a convincing and definitive manner, based on the full Sagdeev potential. Hence no series expansion for weak double layers can find double layers, since these cannot occur in general! The errors creep in because authors expand first to a certain (usually fourth) order in the potential, and afterwards try to fulfil conditions, which on the face of it would show the existence of double layers. Alas, this can only be claimed at the price of not looking to see whether the necessary consistency relations are obeyed, and also of not checking what happens at the next order in the expansion. The 'forgotten' terms overwhelm whatever was retained in the expansion! We hasten to add that our treatment relies on analytical proofs, and not on numerical methods.

The expressions for the Bohm sheath criteria follow as a natural by-product of our calculations, and give the most general form for the types of plasmas considered, with cold positive and negative species drifting into the sheath region and different hot Boltzmann components. Using the small-potential expansion approach of Bohm, we obtain results that generalize the classic requirement that the ion velocity exceed the ion-acoustic speed at the sheath edge (Chen 1974), by giving appropriate multispecies globalizations of these parameters over the different species. The first paper to discuss the influence of negative ions on the Bohm sheath criteria was by Deutsch and Räuchle (1992), whereas Hellberg *et al.* (1993) included heavy massive and charged dust grains.

Our paper runs as follows. In Sec. 2 we set up the general framework, and derive in Sec. 3 the linear electrostatic dispersion law for the plasmas we consider. In Sec. 4 the general nonlinear evolution of these modes is described via a Sagdeev potential in the frame of the nonlinear structures, and allowing in principle for arbitrary amplitudes. Section 5 then gives the Bohm sheath criteria. In Sec. 6 we look at the possibilities of double layers occurring, and do so in steps, beginning with weak double layers. We focus on the special case when only one Boltzmann species is present, in order to clearly point out where existing treatments go wrong in claiming that double layers are possible. Finally our conclusions are stated in Sec. 7, where we shall also briefly indicate what happens when finite inertia effects are retained for the hot species.

2. Basic equations

In order to outline the ideas, we consider the most general case of a multispecies plasma, where some of the lighter and hotter species can be described as isothermal and inertia-less, with Boltzmann distributions for their densities, whereas the cooler and heavier species will be described by cold fluid equations. With a running index

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i characterizing the different Boltzmann species, we have for the densities that

$$n_i = N_i \exp\left(-\frac{q_i\varphi}{\kappa T_i}\right). \tag{1}$$

Here N_i and q_i refer to the equilibrium densities and signed charges of the Boltzmann species with temperature T_i in the presence of an electrostatic potential φ, with κ being Boltzmann's constant. We shall introduce a non-dimensional potential Φ by

$$\Phi = \frac{e\varphi}{\kappa T_{\text{eff}}}, \tag{2}$$

where e is the unit charge and T_{eff} is an effective temperature as a global mean between the Boltzmann species. How this mean is specified will be addressed later.

In a similar way, the cold fluid equations for electrostatic phenomena propagating parallel to the z direction are

$$\frac{\partial n_f}{\partial t} + \frac{\partial}{\partial z}(n_f u_f) = 0, \tag{3}$$

$$\frac{\partial u_f}{\partial t} + u_f\frac{\partial u_f}{\partial z} = -\frac{q_f}{m_f}\frac{\partial \varphi}{\partial z}. \tag{4}$$

Now n_f, u_f, q_f and m_f refer to the densities, fluid velocities, signed charges and masses of the cold fluid species, of which there could be any number. The set of equations is then closed by Poisson's equation for the plasma as a whole,

$$\varepsilon_0\frac{\partial^2\varphi}{\partial z^2} + \sum_i n_i q_i + \sum_f n_f q_f = 0. \tag{5}$$

3. Linear electrostatic dispersion law

Before looking at the nonlinear evolution, we shall derive the dispersion law for plane electrostatic waves with infinitesimal amplitudes, where all quantities vary as $\exp[i(kz - \omega t)]$. After linearization of the basic equations (1) and (3)–(5), we find that

$$\sum_f \frac{\omega_{pf}^2}{(\omega - kU_f)^2} = 1 + \sum_i \frac{1}{k^2\lambda_{Di}^2}. \tag{6}$$

Here λ_{Di} is the Debye length per Boltzmann species and ω_{pf} is the plasma frequency per fluid species, both defined through their squares

$$\lambda_{Di}^2 = \frac{\varepsilon_0\kappa T_i}{N_i q_i^2}, \qquad \omega_{pf}^2 = \frac{N_f q_f^2}{\varepsilon_0 m_f}, \tag{7}$$

where N_f are the equilibrium densities of the fluid species. Their velocities U_f parallel to the direction of wave propagation are given in a frame in which the Boltzmann species are at rest, since Boltzmann distributions with their attendant inertia-less description cannot incorporate finite velocities. We introduce for further reference a global Debye length Λ_D through

$$\Lambda_D^2 = \frac{\varepsilon_0\kappa T_{\text{eff}}}{N_0 e^2}. \tag{8}$$

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We now use the fact that the effective temperature $T_{\rm eff}$ for the Boltzmann species is not yet specified to define it in such a way that

$$\frac{1}{\Lambda_D^2} = \sum_i \frac{1}{\lambda_{Di}^2}, \tag{9}$$

leading to

$$\frac{N_0}{T_{\rm eff}} = \sum_i \frac{N_i q_i^2}{e^2 T_i}. \tag{10}$$

Hence the dispersion law becomes

$$\sum_f \frac{\omega_{pf}^2}{(\omega - kU_f)^2} = 1 + \frac{1}{k^2\Lambda_D^2}. \tag{11}$$

We can also introduce a general plasma frequency Ω_p defined through

$$\Omega_p^2 = \frac{N_0 e^2}{\varepsilon_0 m_{\rm ref}}, \tag{12}$$

and take this as the sum over all fluid species,

$$\Omega_p^2 = \sum_f \omega_{pf}^2. \tag{13}$$

This defines a reference mass through

$$\frac{N_0}{m_{\rm ref}} = \sum_f \frac{N_f q_f^2}{e^2 m_f}, \tag{14}$$

Boltzmann species being treated as inertia-less and hence effectively massless. In the absence of equilibrium drifts, the dispersion law can be solved for ω, to find that

$$\omega^2 = \frac{k^2\Lambda_D^2\Omega_p^2}{1 + k^2\Lambda_D^2} = \frac{k^2 C_s^2}{1 + k^2\Lambda_D^2}. \tag{15}$$

We see that for long wavelengths (small k) the linear phase velocity obeys

$$\frac{\omega}{k} = \Omega_p\Lambda_D = C_s = \left(\frac{\kappa T_{\rm eff}}{m_{\rm ref}}\right)^{1/2}. \tag{16}$$

Here and in (15), C_s is the proper definition, in the plasma mixture under consideration, of a typical sound speed, with which we shall later non-dimensionalize all velocities.

In the above definitions there is still some ambiguity, since we have not fixed the reference density N_0, for which several obvious choices are possible. If there is only one Boltzmann species then its density is the logical choice, since this is the species participating in the shielding. The implication then is that the effective temperature is the temperature of the only Boltzmann species. On the other hand, if there is only one fluid species, its density could serve as the standard, with its plasma frequency through (13) as the total plasma frequency and its mass as the reference mass. Of course, N_0 can only be fixed once, with evident repercussions either on the reference mass of the fluid species or on the effective temperature of the Boltzmann species.

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4. Nonlinear modes

We now consider one-dimensional stationary self-similar nonlinear phenomena, i.e. phenomena travelling in the z direction with a velocity v, so that all quantities will depend upon z and t through $\eta = z - vt$. The fluid equations (3) and (4) can then be integrated exactly with the one-sided boundary conditions

$$\varphi, \quad \frac{d\varphi}{d\eta} \to 0, \quad n_f \to N_f, \quad u_f \to U_f \quad \text{as} \quad \eta \to +\infty. \tag{17}$$

After elimination of the velocities we find expressions for the fluid densities

$$n_f = \frac{N_f}{\left[1 - \dfrac{2q_f\varphi}{m_f(v-U_f)^2}\right]^{1/2}} = \frac{N_f}{\left[1 - \dfrac{2Z_f\Phi}{\mu_f(M-M_f)^2}\right]^{1/2}}. \tag{18}$$

In the last expression dimensionless variables have been introduced, i.e. signed charge numbers $Z_f = q_f/e$, mass ratios $\mu_f = m_f/m_{\text{ref}}$, Mach numbers $M = v/C_s$ and $M_f = U_f/C_s$ and the potential $\Phi = e\varphi/\kappa T_{\text{eff}}$. With the dimensionless equilibrium densities $\alpha_{f,i} = N_{f,i}/N_0$ and inverse temperature ratios $\beta_i = T_{\text{eff}}/T_i$, we write Poisson's equation as

$$\frac{d^2\Phi}{d\xi^2} + \sum_i Z_i\alpha_i \exp\left(-Z_i\beta_i\Phi\right) + \sum_f \frac{Z_f\alpha_f}{\left[1 - \dfrac{2Z_f\Phi}{\mu_f(M-M_f)^2}\right]^{1/2}} = 0, \tag{19}$$

with ξ the dimensionless travelling-frame coordinate. This equation can be integrated in the traditional way to give

$$\frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + V(\Phi, M) = 0, \tag{20}$$

with the Sagdeev or classical potential $V(\Phi, M)$ given by

$$V(\Phi, M) = \sum_i \frac{\alpha_i}{\beta_i}\left[1 - \exp\left(-Z_i\beta_i\Phi\right)\right] + \sum_f \alpha_f\mu_f(M-M_f)^2\left\{1 - \left[1 - \frac{2Z_f\Phi}{\mu_f(M-M_f)^2}\right]^{1/2}\right\}. \tag{21}$$

Equation (20) represents the energy integral for a classical particle of unit mass moving with velocity $d\Phi/d\xi$ in a potential $V(\Phi, M)$, M appearing as a parameter. Clearly, charge neutrality in equilibrium demands that

$$\sum_i Z_i\alpha_i + \sum_f Z_f\alpha_f = 0, \tag{22}$$

and where necessary, likewise for current neutrality

$$\sum_f Z_f\alpha_f M_f = 0. \tag{23}$$

5. Bohm sheath criteria

Before discussing the Bohm sheath criteria themselves, we shall for further reference expand the Sagdeev potential up to fifth order in Φ, and find from (20) and (21)

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that

$$\frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + A\Phi^2 + B\Phi^3 + C\Phi^4 + D\Phi^5 = 0, \tag{24}$$

with the coefficients given by

$$A = -\frac{1}{2}\sum_i \alpha_i\beta_i Z_i^2 + \frac{1}{2}\sum_f \frac{\alpha_f Z_f^2}{\mu_f(M-M_f)^2}, \tag{25}$$

$$B = \frac{1}{6}\sum_i \alpha_i\beta_i^2 Z_i^3 + \frac{1}{2}\sum_f \frac{\alpha_f Z_f^3}{\mu_f^2(M-M_f)^4}, \tag{26}$$

$$C = -\frac{1}{24}\sum_i \alpha_i\beta_i^3 Z_i^4 + \frac{5}{8}\sum_f \frac{\alpha_f Z_f^4}{\mu_f^3(M-M_f)^6}, \tag{27}$$

$$D = \frac{1}{120}\sum_i \alpha_i\beta_i^4 Z_i^5 + \frac{7}{8}\sum_f \frac{\alpha_f Z_f^5}{\mu_f^4(M-M_f)^8}. \tag{28}$$

For the Bohm sheath we only need results up to order Φ^2, and for exponential decay in the sheath region we need A to be negative in a stationary frame (with $v = 0$ or $M = 0$), which leads to

$$\sum_i \alpha_i\beta_i Z_i^2 > \sum_f \frac{\alpha_f Z_f^2}{\mu_f M_f^2}, \tag{29}$$

or, in more transparent notation,

$$\sum_i \frac{1}{\lambda_{Di}^2} > \sum_f \frac{\omega_{pf}^2}{U_f^2}. \tag{30}$$

This supersonic condition is the most general fluid form of the Bohm sheath criterion for cold drifting species and hot Boltzmann components. It includes various special cases, like the discussion given by Hellberg *et al.* (1993) for a plasma with Boltzmann electrons, isothermal or fluid ions and fluid charged dust grains, according to the different regimes. That in itself extended results of Deutsch and Räuchle (1992) to several ionic species. Our expressions are also equivalent to the form given by Raadu and Rasmussen (1988) in connection with the existence of double layers, obtained from kinetic theory.

6. Double layers

For double layers to exist, the Sagdeev potential $V(\Phi, M)$ is required to possess a double root in $\Phi = \Phi_m$, besides the double root in $\Phi = 0$ it already has, owing to the choice of integration constants. Hence the potential Φ changes from 0 to Φ_m and in between $V(\Phi, M)$ has to be negative to guarantee real solutions. In the language of classical mechanics, this ensures that a particle with coordinate Φ transits just once from 0 to Φ_m or vice versa.

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6.1. Weak double layers

For weak double layers one traditionally supposes that the Sagdeev potential can be expanded up to order Φ^4, and that the resulting energy integral

$$\frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + A\Phi^2 + B\Phi^3 + C\Phi^4 = 0 \tag{31}$$

can be written in the double-layer form

$$\frac{1}{2}\left(\frac{d\Phi}{d\xi}\right)^2 + C\Phi^2(\Phi - \Phi_m)^2 = 0 \qquad (C < 0). \tag{32}$$

Identification of both expansions yields

$$\Phi_m = -\frac{B}{2C}, \qquad \Phi_m^2 = \frac{A}{C}, \tag{33}$$

and the typical solution of (32) is

$$\Phi = -\frac{B}{4C}\{1 - \tanh[(\tfrac{1}{2}|A|)^{1/2}\xi]\} \qquad (A < 0). \tag{34}$$

Two important remarks have to be borne in mind, which are sometimes not adequately taken care of. One is that (33) implies a compatibility condition,

$$B^2 = 4AC, \tag{35}$$

which must be fulfilled. The other is that the terms that were left out of the expansion be small indeed (Baboolal *et al.* 1991; Hellberg *et al.* 1992; Verheest 1993). That is,

$$\left|\frac{D\Phi_m^5}{C\Phi_m^4}\right| = \left|\frac{D\Phi_m}{C}\right| \ll 1, \tag{36}$$

or in other words,

$$\left|\frac{BD}{2C^2}\right| = \left|\frac{2AD}{BC}\right| \ll 1 \tag{37}$$

must hold. Mace and Hellberg (1993*c*) have shown that one can, for example, easily obtain mKdV-type electron-acoustic double layers if self-consistency is ignored, but not if the calculation is carried out properly.

6.2. Weak double layers and single Boltzmann species

We shall first show that, in the above sense of mathematical consistency, weak double layers cannot exist in a plasma with a single Boltzmann and a single cold fluid species. In such a plasma we can put $\beta_i = \mu_f = 1$ and assume that $M_f = 0$. We find from the charge neutrality conditions that

$$\alpha_f Z_f = -\alpha_i Z_i, \tag{38}$$

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and hence the coefficients of (31) become

$$\left.\begin{aligned} A &= -\frac{\alpha_i Z_i^2}{2}\left(1+\frac{Z_f}{M^2 Z_i}\right), \\ B &= \frac{\alpha_i Z_i^3}{6}\left(1-\frac{3Z_f^2}{M^4 Z_i^2}\right), \\ C &= -\frac{\alpha_i Z_i^4}{24}\left(1+\frac{15Z_f^3}{M^6 Z_i^3}\right), \\ D &= \frac{\alpha_i Z_i^5}{120}\left(1-\frac{105Z_f^4}{M^8 Z_i^4}\right). \end{aligned}\right\} \tag{39}$$

Using the shorthand $a = -Z_f/M^2 Z_i$, a positive quantity because of the opposing charges, allows us to recast the compatibility condition (35) in the form of a polynomial of degree 4 in a:

$$36a^4 - 45a^3 + 6a^2 - 3a + 2 = 0. \tag{40}$$

We have to find a real root a, such that A and C are negative, requiring that

$$a < 1, \qquad 15a^3 < 1. \tag{41}$$

There is only one such root, $a = 0.38$, the others being either complex-conjugate or larger than 1. One can than easily check that for this value the first neglected term, $O(\Phi^5)$, leads to

$$\left|\frac{BD}{2C^2}\right| = 9.2 \gg 1 \tag{42}$$

contradicting the requirement (37)! That shows that we cannot neglect the other terms in the expansion of the Sagdeev potential in considering double layers, and thus that small-amplitude double layers do not exist in such a simple plasma.

6.3. Double layers of arbitrary amplitude

The result of the preceding subsection is even more generally valid for double layers of any strength. We write the Sagdeev potential (21) for one Boltzmann and one fluid species with the parameters taken as simple as possible:

$$\begin{aligned} V(\Phi, M) &= 1 - \exp\left(-Z_i\Phi\right) - \frac{Z_i M^2}{Z_f}\left[1 - \left(1 - \frac{2Z_f\Phi}{M^2}\right)^{1/2}\right] \\ &= 1 - \exp\left(-\Psi\right) + \frac{1}{a}[1 - (1 + 2a\Psi)^{1/2}]. \end{aligned} \tag{43}$$

Here $\Psi = Z_i\Phi$ could have either sign, and a was defined before. The double-layer conditions are that both V and its first derivative V' with respect to Ψ vanish, ensuring the existence of double roots, and that the second derivative V'' be negative in these roots. The conditions on the first derivative mean that

$$V' = \exp\left(-\Psi\right) - (1 + 2a\Psi)^{-1/2} = 0. \tag{44}$$

Hence we can replace the square root everywhere by $\exp\Psi$ and find from $V = 0$ that

$$\exp\Psi = 1 \qquad \text{or} \qquad \exp\Psi = a. \tag{45}$$

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At this stage there would seem to be two possibilities, one giving indeed the value $\Psi = 0$, the second one corresponding to $\Psi_m = \ln a$. However, we still need from (44) that

$$\exp \Psi_m = a = (1 + 2a \ln a)^{1/2}. \tag{46}$$

The latter equality admits only the root $a = 1$, bringing us back to the case where $\Phi = 0$. Another way of proving this is by reasoning on (46), recast in the form (M. A. Raadu 1995 personal communication)

$$\sinh \Psi_m = \Psi_m. \tag{47}$$

In other words, there is no second double root for the Sagdeev potential. That proves beyond doubt that double layers cannot be supported by only one Boltzmann and one cold fluid species, irrespective of amplitude. Harin and Strasser (1995) came to the same conclusion, although their mathematical proof did not involve the Sagdeev potential but was based on an equivalent description of travelling electrostatic structures.

6.4. Double Boltzmann distributions

For two Boltzmann species of the same sign and under the right conditions, there are enough papers to show that weak and strong double layers are possible, even if one has to be careful with the limits of validity in the case of weak double layers, consistent with the expansion of the Sagdeev potential up to fourth order. From a vast number of papers, we only quote some that adequately discuss the inherent limitations: Baboolal (1988), Baboolal *et al.* (1988, 1990*a*,*b*) and Verheest (1993). Recently, attention has also been drawn to the influence of the inertia of the thermal species on the existence of double layers, showing that these tend to restrict the parameter domain in rather stringent ways (Mace and Hellberg 1993*a*,*b*) compared with Boltzmann treatments. These aspects, however, are beyond the scope of our discussion here. Nevertheless, as pointed out by Mace and Hellberg (1993*a*,*b*) for dust-acoustic double layers, the Boltzmann limit is not essential to whether or not double layers are obtained. What is important to note is that double layers can only be supported by at least two thermal species providing the necessary pressures. Having only one species exert a pressure is not enough, irrespective of whether finite-mass effects are retained or, for simplicity, neglected so that a Boltzmann description can be used. A corollary, of course, is that the fluid species are really cold.

6.5. Single Boltzmann but multiple cold fluid species

Hence it only remains to prove that one Boltzmann species cannot support double layers, no matter how we fiddle around with the cold fluid species. This would corroborate results of the Durban group (Baboolal *et al.* 1991), based, however, on numerical computations, concerning earlier work by Verheest (1989) and other authors who forgot to check the consistency relations, like most recently Sah (1995) when dealing with weak double layers. The proof, nevertheless, is easier to give for the general case of arbitrary amplitudes.

With one Boltzmann species, we normalize density and temperature in terms of that species, such that $\alpha_i = \beta_i = 1$, and introduce the abbreviations

$$a_f = -\frac{Z_f}{\mu_f (M - M_f)^2 Z_i}, \qquad b_f = -\frac{\alpha_f Z_f}{Z_i}. \tag{48}$$

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Clearly, owing to charge neutrality, we have

$$\sum_f b_f = 1. \tag{49}$$

We remark that a_f and b_f have the same sign for a given species, and the preceding condition shows that not all b_f (and a_f) can be negative. In the variables defined in (48), the Sagdeev potential (21) becomes

$$V = 1 - \exp(-\Psi) + \sum_f \frac{b_f}{a_f}[1 - (1 + 2a_f\Psi)^{1/2}]. \tag{50}$$

The double-layer conditions are that both V and

$$V' = \exp(-\Psi) - \sum_f \frac{b_f}{(1 + 2a_f\Psi)^{1/2}} \tag{51}$$

vanish at 0, which is obvious, and at another point Ψ_m, and that at both points the second derivative

$$V'' = -\exp(-\Psi) + \sum_f \frac{a_f b_f}{[(1 + 2a_f\Psi)^{1/2}]^3} \tag{52}$$

has the same (negative) sign. Hence, in order to have any non-trivial solutions at all, we require at the outset that

$$V''(0) = \sum_f a_f b_f - 1 < 0, \tag{53}$$

yielding conditions on the different a_f and b_f, which we do not, however, need further in explicit form. Suppose that V has a positive double root at Ψ_1, apart from the one at 0. That is,

$$V(0) = V(\Psi_1) = 0, \qquad \text{with} \quad 0 < \Psi_1. \tag{54}$$

A limiting case of Rolle's theorem (originally formulated for distinct roots) then tells us that the derivative has zeros in between the roots of the function itself (as well as at the double roots themselves, of course); in other words,

$$V'(0) = V'(\Psi_2) = V'(\Psi_1) = 0, \qquad \text{with} \quad 0 < \Psi_2 < \Psi_1. \tag{55}$$

Repeating the argument for the second derivative, we get

$$V''(\Psi_3) = V''(\Psi_4) = 0, \qquad \text{with} \quad 0 < \Psi_3 < \Psi_2 \ \text{ and } \ \Psi_2 < \Psi_4 < \Psi_1, \tag{56}$$

with strict inequalities. In other words, the second derivative would need to have two distinct positive roots. Analogous results follow for a negative double root. Translated to the behaviour of V, the conclusion is that V is concave at zero (with $V(0) = 0$), becomes convex after a while, and then the convex stretch would have to be followed by another concave part. In order to see whether such a scenario is possible, we rewrite (52) as

$$V'' = \left\{ (\exp\ \Psi) \sum_f \frac{a_f b_f}{[(1 + 2a_f\Psi)^{1/2}]^3} - 1 \right\} \frac{1}{\exp\ \Psi}. \tag{57}$$

For the discussion, we have to distinguish between the cases when all a_f (and

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also then all b_f) are positive and those when some are negative. The case with all a_f positive means that the cold fluid species all have the opposite sign to the Boltzmann species, for instance a plasma with Boltzmann electrons and multiple cold fluid positive ions. In that case the a_f obey

$$0 < a_f \leqslant a_{\max}, \tag{58}$$

and V is defined in the open Ψ-interval $]-1/2a_{\max}, +\infty[$. The expression inside the curly brackets in (57) eventually goes to $+\infty$ for positive Ψ, since the exponential overpowers every other expression. Moreover, the functions inside the curly brackets are sufficiently monotonic that the bracket can only change sign once for positive Ψ. That means that V'' is negative at $\Psi = 0$, becomes positive for some positive Ψ and then decays to zero while remaining positive. Hence V'' has one root in the positive domain, and V could have at most a single positive root outside 0. This can be seen by applying Rolle's theorem in its original formulation for successive distinct roots. However, $\lim_{\Psi\to+\infty} V \to -\infty$, and, together with $V(0) < 0$, this implies no or two positive roots. The combination of both arguments indicates that V cannot have a root at all in the domain where $\Psi > 0$. The case for negative Ψ will be discussed together with that for some negative a_f, since these are qualitatively similar.

When some a_f (and the corresponding b_f) are negative, we have cold species of both signs present. Then

$$-|a_{\min}| \leqslant a_f \leqslant a_{\max}, \tag{59}$$

and V is defined in the open Ψ-interval $]-1/2a_{\max}, +1/2|a_{\min}|[$. The expression inside the curly brackets in (57) now goes to $+\infty$ at both ends of the interval in which V is defined, since the exponential remains finite, but one of the denominators goes to zero. Since V'' is negative for $\Psi = 0$, it becomes positive for some values to the left and right of the origin, in such a way that this can only happen once for negative Ψ and once for positive Ψ. At the same time,

$$\lim_{\Psi\to-1/2a_{\max}} V' \to -\infty, \qquad \lim_{\Psi\to1/2|a_{\min}|} V' \to +\infty, \tag{60}$$

because of the different signs of some of the a_f and b_f. Again translating this to the behaviour of V, the latter starts out convex with an infinitely negative slope, turns concave in the neighbourhood of zero, and ends up convex with an infinitely positive slope. It can have *at most* one strictly negative root and one strictly positive root, besides the double root at zero. That definitely excludes any double-layer behaviour, but could admit soliton solutions under the right initial conditions.

Returning now to the case of all a_f positive, and combining all the information gathered so far, we find that V could have *at most* a single negative root, apart from the double root at $\Psi = 0$. As pointed out already, this case corresponds to cold fluid species having opposite signs to the Boltzmann species. Given that $\Psi = Z_i\Phi$, if the Boltzmann species consists of electrons then negative Ψ corresponds to positive Φ, and we recover the usual compressive ion-acoustic solitons, possibly in the presence of different species of positive ions. In the case of dust-acoustic modes in the absence of free electrons and with constant, negative dust charges, the Boltzmann species are the protons and we find rarefactive solitons (Rao *et al.* 1990; Verheest 1992).

When the a_f include some negative quantities, we typically need both positive and negative cold fluid ions, and thus both rarefactive and compressive solitons are possible. This could be the case both for ion-acoustic solitons, if the Boltzmann

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species is electrons and some negative ions are present (Watanabe 1984), and for dust-acoustic solitons without free electrons but with charged dust of either sign. Other examples are possible.

7. Conclusions

We have given a general description of multispecies plasmas, in which there are a number of lighter, thermal species, described by Boltzmann distributions, in addition to a number of heavier species described by cold fluid theory, without a priori fixing of the number of species. For such plasmas, we have briefly discussed the linear characteristics of electrostatic modes, with the appropriate extensions of concepts like ion-acoustic velocities. After that, we derived the nonlinear description by means of the Sagdeev potential, giving results that are fully general.

Using the small-potential expansion approach of Bohm, we have obtained criteria that generalize the classic requirement that the ion velocity exceed the ion-acoustic speed at the sheath edge, by giving appropriate multispecies expressions for the global supersonic condition. Thus we recover as special cases recent results that have appeared in the literature, mainly in connection with negative-ion or dusty plasmas.

We have also focused on the possible occurrence of one-dimensional, stationary double layers in different types of plasmas, and reviewed in particular those results based on weakly nonlinear descriptions. It turns out that these are sometimes not applied self-consistently. Specifically for plasmas with one Boltzmann species, we have given a fully analytic proof that double layers of whatever amplitude cannot exist, no matter how many cold fluid species are present. Errors creep in when authors expand to a certain order in the potential, and afterwards neglect to see whether 'forgotten' terms are small indeed. It turns out that the omitted terms overwhelm whatever was retained in the expansion!

Finally, we remark that we have left aside all finite inertia effects for the thermal species. Recent work tends to limit the existence range of double layers compared with Boltzmann treatments, sounding another warning of due caution. From these recent papers, we learn that in order to make double layers possible, one needs at least two thermal species exerting a pressure, irrespective of whether one describes these thermal species by Boltzmann distributions or retains finite-mass effects, although the existence ranges will be affected. One thermal species is not enough to maintain double layers – that is, if the other species can really be described by cold fluids.

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