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Basic stress analysis of hyperbolic regimes in plastic media

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Abstract. Nonlinear problems of two-dimensional deformation or stress in solid continua are considered where the in-plane components of stress are self-equilibrated and subject to a scalar constraint. In applications the latter is often a yield condition for plastic media. Such field equations are frequently hyperbolic, with a pair of characteristic curves through any point. The primary aim is to express the integrable differential relations along the curves in their simplest form, by an optimal choice of coordinates and variables. Special cases of this problem are now classical, but the general case has received little attention and the universal canonic structure of the relations has escaped notice.

1. *Objectives.* We are here concerned with a class of nonlinear problems in two-dimensional stress analysis. The particular type of problem is encountered under a variety of circumstances, but in essence its statement is both uniform and simple. Let x_1, x_2 be orthogonal coordinates in the mid-plane of the medium, with x_3 normal. The Cauchy stress tensor has components σ_{ij} depending only on x_1, x_2; σ_{13} and σ_{23} are zero but σ_{33} may vanish or not according to context. Then, in the absence of body-force, the medium is in equilibrium when

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0. \tag{1·1}$$

The in-plane components of stress are required also to comply with a scalar constraint, say

$$f(\sigma_{11}, \sigma_{22}, \sigma_{12}) = 0, \tag{1·2}$$

which is independent of position; it is usual to restrict function f to be convex in its arguments. In the event that this trio of equations is hyperbolic, the problem in outline is to identify the pair of characteristic directions at any point, and to formulate the associated integrable relations along the characteristic curves. (With appropriate boundary data any particular solution can thereafter be generated routinely.) When set in this manner, the problem can be treated at a superficial level by any one of several methods which are standard for partial differential equations of first order.

For instance, with brevity in view, we may proceed as follows (adopting a shortcut

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suggested in (7)). Introduce space derivatives $\partial/\partial s$, $\partial/\partial n$, locally tangential and normal respectively to a presumed characteristic curve, C say, such that

$$\frac{\partial}{\partial x_1} = \cos\phi\,\frac{\partial}{\partial s} - \sin\phi\,\frac{\partial}{\partial n}, \quad \frac{\partial}{\partial x_2} = \sin\phi\,\frac{\partial}{\partial s} + \cos\phi\,\frac{\partial}{\partial n},$$

where angle ϕ is anticlockwise from axis x_1 to tangent ds. Take the normal derivative of (1·2) and add it to a linear combination of equations (1·1) to produce

$$\lambda\cos\phi\,\frac{\partial\sigma_{11}}{\partial s} + \mu\sin\phi\,\frac{\partial\sigma_{22}}{\partial s} + (\lambda\sin\phi + \mu\cos\phi)\,\frac{\partial\sigma_{12}}{\partial s}$$

$$= (\lambda\sin\phi + \nu_{11})\,\frac{\partial\sigma_{11}}{\partial n} + (-\mu\cos\phi + \nu_{22})\,\frac{\partial\sigma_{22}}{\partial n} + (-\lambda\cos\phi + \mu\sin\phi + 2\nu_{12})\,\frac{\partial\sigma_{12}}{\partial n}, \quad (1\cdot3)$$

where λ, μ are arbitrary multipliers and

$$\nu_{ij} = \frac{\partial f}{\partial\sigma_{ij}} \quad (i,j = 1,2). \quad (1\cdot4)$$

As usual, f in (1·4) is re-arranged as a symmetric function of the equal components σ_{12} and σ_{21}, which are treated as distinct in the partial differentiation; then ν_{ij} transforms tensorially. Now annihilate the components of each normal derivative in (1·3) by choosing λ, μ such that

$$\lambda\sin\phi + \nu_{11} = 0, \quad \mu\cos\phi - \nu_{22} = 0,$$

with ϕ taken as a solution of

$$\nu_{11}\cos^2\phi + \nu_{22}\sin^2\phi + 2\nu_{12}\sin\phi\,\cos\phi = 0, \quad (1\cdot5)$$

assuming the existence of real roots. For the moment, the coordinate axes are being supposed to have no special connexion with the state of stress, so $\phi = 0$ or $\frac{1}{2}\pi$ is excluded ($\nu_{11} \neq 0$, $\nu_{22} \neq 0$). The residue of (1·3) is then a differential relation,

$$\nu_{11}\cos^2\phi\,d\sigma_{11} - \nu_{22}\sin^2\phi\,d\sigma_{22} + (\nu_{11} - \nu_{22})\sin\phi\,\cos\phi\,d\sigma_{12} = 0, \quad (1\cdot6)$$

which is integrable along C if its slope satisfies (1·5) at each point. Correspondingly, admissible values of σ_{11}, σ_{22}, σ_{12} prescribed along any segment of a characteristic necessarily comply with (1·6) and, in consequence, do not suffice with the field equations to determine their normal derivatives. These can accordingly be discontinuous across C, subject to the two independent constraints imposed by (1·1) and (1·2) over and above (1·6).

Returning to (1·5) it is apparent that there are just two characteristic curves through any point and that these are such that the corresponding direct components of tensor ν_{ij} vanish (and hence the rates of extension in a medium under the normality rule of plastic flow). From this interpretation it immediately follows (for example by the Mohr circle representation) that the necessary and sufficient condition for real characteristics can be stated to be oppositely-signed principal values of ν_{ij}. Henceforward, attention is concentrated on this hyperbolic regime in some domain of stress space.

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Equations (1·5) and (1·6) are apparently not in the literature. While conceivably adequate as they stand, for numerical computation of characteristic nets, I regard this form of solution as superficial on two counts. First, it is couched in cartesian components on a non-specific background and not in quantities *intrinsic* to either the stress field or the net. Second, it fails to expose the basic inner structure of the integrable relations. It is the prime purpose of this paper to explore both these aspects; different forms of intrinsic solution will incidentally be exhibited. Surprisingly, perhaps, having regard to the long history of the problem, a unified general approach at a fundamental level has been lacking.

Before reviewing the present position, the main categories of (1·2) in solid mechanics are recalled. The function f may, for example, be a scalar invariant of stress, corresponding to isotropy of material response in some respect; then (1·2) is expressible as

$$F(\sigma_1, \sigma_2) = 0, \tag{1·7}$$

where σ_1 and σ_2 are the principal in-plane components and F depends on them symmetrically. Anisotropic response, symmetric relative to the mid-plane, is modelled by

$$F(\sigma_1, \sigma_2, \alpha) = 0, \tag{1·8}$$

where angle α is anticlockwise from axis x_1 to axis σ_1; with arbitrary dependence on α this is no less general than (1·2). Again, f may be independent of $\sigma_{11} + \sigma_{22}$, the pressure-sensitive part of the in-plane stress; in that event

$$f(\sigma_{11} - \sigma_{22}, \sigma_{12}) = 0. \tag{1·9}$$

The hyperbolic regime extends over the whole of stress space, and by (1·5) the net of characteristics is orthogonal. Correspondingly, (1·7) reduces to

$$\sigma_1 - \sigma_2 = 2k, \tag{1·10}$$

where k, the maximum shear stress, is a material constant; and (1·8) reduces to

$$\sigma_1 - \sigma_2 = 2k(\alpha), \tag{1·11}$$

where k now depends on α. As to physical interpretations of f, the most usual is of course the yield function for a classical plastic medium in a state of plane strain ($\sigma_3 \neq 0$, dependent on σ_1, σ_2 and α) or of generalized plane stress ($\sigma_3 = 0$). Beyond that, in the theory of extremal fields of stress(5), f can be any convex 'test' function, without regard to a constitutive law.

The earliest attempt on the problem appears to have been in 1903 by Kötter(8), in the context of isotropic pressure-sensitive plasticity in plane strain. This is category (1·7), though Kötter restricted F to be linear (the net is then equiangular). Integrable relations were obtained in intrinsic form, namely in terms of stress components relative to the net. The still more restricted case of pressure insensitivity, namely (1·10), was independently analysed in similar fashion by Hencky(2). An intrinsic general solution for (1·7) in the context of plane plastic strain is due to Mandel(9) with subsequent refinements by Hill((4) pp. 294–299). Plane *stress* in isotropic plasticity also comes under (1·7); some special functions F were treated in that context by Sokolovsky((11) p. 201) and Hill((4) pp. 300–303). For orthotropic plasticity in plane strain without pressure sensitivity, which is category (1·11), the characteristic integrals

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were obtained by Hill(3) for a particular $k(\alpha)$ which is representative of textured metals. The solution with any $k(\alpha)$ was given only recently by Rice(10) and simultaneously by Booker and Davis(1). Characteristic integrals were presented in an intrinsic form which is attractively concise. Unfortunately, this does not carry over to the pressure-sensitive problem (1·7) or (1·8), nor does it bring to light the inner structure mentioned previously. Booker and Davis do in fact also give (without derivation) one variant of the integrable relations for (1·8) in full generality. This could be useful in applications, especially since valid also for plane stress (not mentioned by the authors). But it still falls short of the present objectives.

2. *Intrinsic analysis in principal components.* In seeking to improve on the preceding treatment, a natural first step is to specify the field of stress by σ_1, σ_2, α in place of components on an arbitrary background. Then, in conjunction with (1·8), the equations of equilibrium are taken in the well-known form

$$\frac{\partial \sigma_1}{\partial t_1} + (\sigma_1 - \sigma_2)\frac{\partial \alpha}{\partial t_2} = 0, \quad \frac{\partial \sigma_2}{\partial t_2} + (\sigma_1 - \sigma_2)\frac{\partial \alpha}{\partial t_1} = 0, \tag{2·1}$$

where $\partial/\partial t_1$, $\partial/\partial t_2$ are space derivatives along the trajectories of principal stress. These equations are just specializations of (1·1) since the principal-axes components of $d\sigma_{ij}$ are

$$d\sigma_1, \quad d\sigma_2, \quad (\sigma_1 - \sigma_2)\, d\alpha. \tag{2·2}$$

Likewise, the corresponding components of ν_{ij} are proportional to

$$\frac{\partial F}{\partial \sigma_1}, \quad \frac{\partial F}{\partial \sigma_2}, \quad \frac{1}{2(\sigma_1 - \sigma_2)}\frac{\partial F}{\partial \alpha}, \tag{2·3}$$

since $\nu_{ij} d\sigma_{ij} = 0$ in any coordinates whenever $dF = 0$. Let ψ denote the anticlockwise angle from axis σ_1 to a characteristic direction; then $\psi = \phi - \alpha$ when α is reckoned from the same background as before. By the preceding technique one obtains

$$\cot\psi \frac{\partial F}{\partial \sigma_1} + \tan\psi \frac{\partial F}{\partial \sigma_2} + \frac{1}{(\sigma_1 - \sigma_2)}\frac{\partial F}{\partial \alpha} = 0, \tag{2·4}$$

along with

$$\cot\psi \frac{\partial F}{\partial \sigma_1} d\sigma_1 - \tan\psi \frac{\partial F}{\partial \sigma_2} d\sigma_2 + \left(\frac{\partial F}{\partial \sigma_1} - \frac{\partial F}{\partial \sigma_2}\right)(\sigma_1 - \sigma_2)\, d\alpha = 0. \tag{2·5}$$

Alternatively, these can be written down from (1·5) and (1·6) by choosing the local principal axes as the background; then ϕ is replaced by ψ, while $d\sigma_{ij}$ and ν_{ij} reduce to (2·2) and (2·3).

In addition to (2·5) the differential increments in σ_1, σ_2, α are governed universally, and not only along characteristics, by

$$\frac{\partial F}{\partial \sigma_1} d\sigma_1 + \frac{\partial F}{\partial \sigma_2} d\sigma_2 + \frac{\partial F}{\partial \alpha} d\alpha = 0. \tag{2·6}$$

Along each characteristic, equations (2·5) and (2·6) together determine the increment ratios uniquely. To express this concisely, it is necessary to distinguish the two solutions of (2·4). Let a typical characteristic of each family be denoted A and B, respec-

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tively, and their anticlockwise inclinations to axis σ_1 by ψ_A and ψ_B, where $\psi_A < \psi_B$ (a unique specification and explicit formulae are given later). Then the increment ratios are such that

$$\left.\begin{aligned} \tan\psi_B\, d\sigma_1 = \cot\psi_B\, d\sigma_2 = (\sigma_1-\sigma_2)\, d\alpha \quad &\text{on} \quad A; \\ \tan\psi_A\, d\sigma_1 = \cot\psi_A\, d\sigma_2 = (\sigma_1-\sigma_2)\, d\alpha \quad &\text{on} \quad B. \end{aligned}\right\} \tag{2·7}$$

These integrable relations are believed to be new, both in spirit and in substance.

For brevity they will be proved here by simple verification. First, (2·7) on A or B satisfies (2·6) since the left side there is rendered proportional to that of (2·4), with ψ_A or ψ_B in place of ψ. Second, on both A and B (2·7) makes the left side of (2·5) proportional to

$$\cot\psi_A \cot\psi_B \frac{\partial F}{\partial \sigma_1} - \tan\psi_A \tan\psi_B \frac{\partial F}{\partial \sigma_2} + \frac{\partial F}{\partial \sigma_1} - \frac{\partial F}{\partial \sigma_2}.$$

This is zero since $\partial F/\partial\sigma_1 = \tan\psi_A \tan\psi_B\, \partial F/\partial\sigma_2$, by considering the product of the roots of (2·4) regarded as a quadratic.

Not all four relations in (2·7) are needed to compute a characteristic net, since there are only three primary unknowns. To clarify this redundancy, consider a typical Cauchy boundary-value problem. Suppose that infinitesimal segments, OA and OB, of intersecting characteristics are given in the physical plane, along with compatible values of σ_1, σ_2, α at 0. Their increments at A (and similarly at B) are determined by the relations along OA in conjunction with the given $\phi_A = \alpha + \psi_A$. Let AC and BC be the (unknown) characteristic segments that complete the infinitesimal quadrangular cell, i.e. the domain of influence of the Cauchy data. To compute σ_1, σ_2, α at C we can use finite-difference replacements of the two integrable relations along AC together with one of those along BC. The remaining relation along BC will then be satisfied automatically within the computing error. This is because $F(\sigma_1, \sigma_2, \alpha)$ is, in effect, held constant along OA, OB, AC, and so its values at B and C are automatically equal. Thereafter, to determine the actual position of C, we can compute the roots ψ at C from σ_1, σ_2, α there; next, the background slopes $\phi = \alpha + \psi$ of the characteristics at C; lastly, x_1 and x_2 at C via the mean slopes of AC and BC. Naturally, the entire programme can be refined as to accuracy in various ways; this rudimentary outline is purely to explain the function and status of the four relations (2·7).

By splitting the stress into its hydrostatic and deviatoric parts, we may rewrite (2·7) as

$$\left.\begin{aligned} \sin(2\psi_B)\, d\{\tfrac{1}{2}(\sigma_1+\sigma_2)\} = \tan(2\psi_B)\, d\{\tfrac{1}{2}(\sigma_1-\sigma_2)\} = (\sigma_1-\sigma_2)\, d\alpha \quad &\text{on} \quad A; \\ \sin(2\psi_A)\, d\{\tfrac{1}{2}(\sigma_1+\sigma_2)\} = \tan(2\psi_A)\, d\{\tfrac{1}{2}(\sigma_1-\sigma_2)\} = (\sigma_1-\sigma_2)\, d\alpha \quad &\text{on} \quad B. \end{aligned}\right\} \tag{2·8}$$

Coming closer to the usual practice in classical plasticity, we may suppose (1·8) converted to

$$\sigma_1 - \sigma_2 = 2k(p, \alpha), \tag{2·9}$$

where $p = -\frac{1}{2}(\sigma_1+\sigma_2)$ is the so-called hydrostatic pressure. Correspondingly, (2·4) is transformed to

$$\frac{1}{2k}\frac{\partial k}{\partial \alpha}\sin(2\psi) - \cos(2\psi) = \frac{\partial k}{\partial p}, \tag{2·10}$$

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and the roots ψ_A, ψ_B are obtained as functions of p, α. This being done, the equations for dk in (2·8) can be dropped, leaving just

$$\left.\begin{aligned} \sin(2\psi_B)\,dp + 2k\,d\alpha = 0 \quad \text{on} \quad A, \\ \sin(2\psi_A)\,dp + 2k\,d\alpha = 0 \quad \text{on} \quad B, \end{aligned}\right\} \tag{2·11}$$

to be integrated in p and α, along with (2·9) as it stands. This is the standpoint of Booker and Davis (1), who state relations amounting to (2·11), though without explicit reference to ψ; their preferred derivation is unfortunately not indicated.

When the medium is insensitive to the hydrostatic pressure, we have

$$\cot(2\psi) = \frac{1}{2k}\frac{dk}{d\alpha}$$

by (2·10); consequently $\psi_B - \psi_A = \frac{1}{2}\pi$ and the net is orthogonal. Suppose $k(\alpha)$ represented as a locus generated by polar coordinates $(k, 2\alpha)$. According to the preceding equation, $2\psi_B$ is the anticlockwise angle from the radial direction to the tangential direction. Therefore, if l is the arc-length measured in the same sense,

$$\sin(2\psi_B)\,dl = 2k\,d\alpha.$$

Consequently, for category (1·11), the integrals of (2·4) simplify to

$$\left.\begin{aligned} p + l &= \text{constant on } A, \\ p - l &= \text{constant on } B, \end{aligned}\right\} \tag{2·12}$$

as shown by Rice (10) and by Booker and Davis (1). The classical integrals of Kötter (8) and Hencky (2) for category (1·10) are recovered with $l = 2k\alpha$, the locus being a circle.

3. *Integrable relations: inner structure.* We proceed to the main task, which is to uncover the essence of the integrable relations.

Guided by the analysis in principal components, we express (1·2) in differential form,

$$\nu_{11}\,d\sigma_{11} + \nu_{22}\,d\sigma_{22} + 2\nu_{12}\,d\sigma_{12} = 0, \tag{3·1}$$

with the notation (1·4). The aim is to solve (1·6) with (3·1) for the increment ratios. A direct attack leads to complicated formulae, whose optimal reduction via (1·5) is far from obvious without foreknowledge. Instead (following my own route in the actual investigation) we have recourse to (2·7), which suggests

$$\left.\begin{aligned} \tan\phi_B\,d\sigma_{11} = \cot\phi_B\,d\sigma_{22} = d\sigma_{12} \quad \text{on} \quad A; \\ \tan\phi_A\,d\sigma_{11} = \cot\phi_A\,d\sigma_{22} = d\sigma_{12} \quad \text{on} \quad B. \end{aligned}\right\} \tag{3·2}$$

Verification is immediate. For A and B in turn, substitution of the ratios in (3·1) gives (1·5) with ϕ replaced by ϕ_A and ϕ_B, respectively. Likewise, by substitution in (1·6),

$$\nu_{11}(\cot\phi_A \cot\phi_B + 1) = \nu_{22}(\tan\phi_A \tan\phi_B + 1),$$

in both cases, which is compatible with $\tan\phi_A \tan\phi_B = \nu_{11}/\nu_{22}$ from (1·5).

Suppose, now, that the coordinates are oriented specifically in relation to the characteristic net. Let x_1, x_2 be respectively tangential and normal to curve A at some

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point O. Then $\phi_A = 0$ there while $\phi_B = \psi_B - \psi_A$, the angle between the characteristic directions at O. Relations (3·2) become

$$\left.\begin{aligned} \tan(\psi_B - \psi_A)\, d\sigma_{11} = \cot(\psi_B - \psi_A)\, d\sigma_{22} = d\sigma_{12} \quad \text{on} \quad A, \\ d\sigma_{22} = 0 = d\sigma_{12} \quad \text{on} \quad B. \end{aligned}\right\} \tag{3·3}$$

A corresponding pair is obtained by orienting x_1, x_2 tangential and normal to curve B, so that $\phi_B = 0$, $\phi_A = \psi_A - \psi_B$. These pairs can be combined selectively as

$$\left.\begin{aligned} d\sigma^B_{22} = 0 = d\sigma^B_{12} \quad \text{on} \quad A, \\ d\sigma^A_{22} = 0 = d\sigma^A_{12} \quad \text{on} \quad B, \end{aligned}\right\} \tag{3·4}$$

where the superscripts A, B emphasize that the cartesian increments are based on the tangent and normal at O to curves A, B respectively. The relations on A in the pair (3·3) have been omitted in favour of those on A in (3·4); similarly, the relations on B in the corresponding pair have been omitted in favour of those on B in (3·3). (These equivalences are readily checked by the elementary formulae for transforming cartesian components of stress and their increments.)

The integrable relations are considered to be in the sought-for canonical form. They may be epitomised in words: *the increment of in-plane stress on either characteristic is uniaxial in the direction of the other*.

It is worth making the connexion between (3·2) and (3·4) absolutely explicit. With the non-specific coordinates Cauchy's basic formula in stress analysis gives

$$(-\sin\phi_B\, d\sigma_{11} + \cos\phi_B\, d\sigma_{12}, \quad \cos\phi_B\, d\sigma_{22} - \sin\phi_B\, d\sigma_{12}),$$

as the x_1, x_2 components of the $d\sigma_{ij}$ traction acting on an arc-element with unit normal $(-\sin\phi_B, \cos\phi_B)$. According to (3·2) these components vanish for the A-increments of stress. Consequently, so do the components $(d\sigma^B_{12}, d\sigma^B_{22})$ of the $d\sigma_{ij}$ traction with the specific coordinates in (3·4). Alternatively, by tensor transformation, there are the direct connexions

$$\begin{aligned} d\sigma^B_{22} &= \sin^2\phi_B\, d\sigma_{11} + \cos^2\phi_B\, d\sigma_{22} - 2\sin\phi_B\cos\phi_B\, d\sigma_{12}, \\ d\sigma^B_{12} &= \sin\phi_B\cos\phi_B(d\sigma_{22} - d\sigma_{11}) + (\cos^2\phi_B - \sin^2\phi_B)\, d\sigma_{12}. \end{aligned}$$

Both expressions vanish when the increment ratios on A satisfy (3·2), and conversely. Naturally the link between (2·7) and (3·4) is precisely similar.

Further insight can be gained by deriving (3·4) *ab initio* as it stands, rather than by (3·2). With the specific coordinates relative to curve A, we combine the field equations linearly:

$$\lambda\left(\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2}\right) + \mu\left(\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2}\right) = \nu_{11}\frac{\partial\sigma_{11}}{\partial x_2} + \nu_{22}\frac{\partial\sigma_{22}}{\partial x_2} + 2\nu_{12}\frac{\partial\sigma_{12}}{\partial x_2},$$

where ν_{ij} is defined by (1·4). The terms in $\partial/\partial x_2$, the space derivative normal to A, are annihilated by choosing

$$\nu_{11} = 0, \quad \lambda = 2\nu_{12}, \quad \mu = \nu_{22}. \tag{3·5}$$

The first of these accords with the necessary and sufficient condition enunciated earlier for a curve to be characteristic. The residue from the linear combination is

$$\lambda\frac{\partial\sigma_{11}}{\partial x_1} + \mu\frac{\partial\sigma_{12}}{\partial x_1} = 0. \tag{3·6}$$

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We also have $\partial f/\partial x_1 = 0$, which is

$$\lambda \frac{\partial \sigma_{12}}{\partial x_1} + \mu \frac{\partial \sigma_{22}}{\partial x_1} = 0 \tag{3·7}$$

by (3·5). Together (3·6) and (3·7) can be written as

$$\frac{\lambda}{\mu} d\sigma_{11}^A = \frac{\mu}{\lambda} d\sigma_{22}^A = -d\sigma_{12}^A \quad \text{on} \quad A. \tag{3·8}$$

Now (3·6) and (3·7) are convertible to

$$\left(\mu \frac{\partial}{\partial x_1} - \lambda \frac{\partial}{\partial x_2}\right) \sigma_{12} = 0, \quad \left(\mu \frac{\partial}{\partial x_1} - \lambda \frac{\partial}{\partial x_2}\right) \sigma_{22} = 0,$$

by virtue of the equations of equilibrium. So

$$d\sigma_{22}^A = 0 = d\sigma_{12}^A \quad \text{on} \quad B, \tag{3·9}$$

where B is inclined at $\tan^{-1}(-\lambda/\mu)$ to A. Relations (3·8) and (3·9) are identical with (3·3), which completes the derivation. This line of argument has its origins in my treatment of category (1·7) in ((4) pp. 294–299).

4. *Intrinsic analysis in characteristic components.* In the isotropic problem, category (1·7), it is advantageous to express the integrable relations in stress components intrinsic to the families of characteristic curves ((9) and (4) pp. 294–299). In the general case, however, this version is attended by extra complexity. Nevertheless, we give the formulae for future evaluation.

Components of stress intrinsic to a curve are defined as follows. At a generic point the tangent and normal are taken as axes of reference. The direct component along the tangent is denoted by P; the direct component along the normal by R; and the shear component by Q. Subscripts A and B are attached to distinguish the families of characteristics. With regard to Q we adopt the sign convention in ((4) pp. 294–299), such that Q_A and Q_B at the particular point O coincide with σ_{12}^A and $-\sigma_{12}^B$ defined there previously. (In the isotropic case the characteristics are equally inclined to the principal axes. Therefore, at a generic point, the intrinsic components on A and B have equal values P, Q, R and subscripts are superfluous.) To link with the analysis in principal components, we note that

$$\left.\begin{aligned} P_A &= \sigma_1 \cos^2\psi_A + \sigma_2 \sin^2\psi_A, & P_B &= \sigma_1 \cos^2\psi_B + \sigma_2 \sin^2\psi_B, \\ Q_A &= (\sigma_2 - \sigma_1) \sin\psi_A \cos\psi_A, & Q_B &= (\sigma_1 - \sigma_2) \sin\psi_B \cos\psi_B, \\ R_A &= \sigma_1 \sin^2\psi_A + \sigma_2 \cos^2\psi_A, & R_B &= \sigma_1 \sin^2\psi_B + \sigma_2 \cos^2\psi_B, \end{aligned}\right\} \tag{4·1}$$

where ψ_A, ψ_B are functions of σ_1, σ_2, α determinable from (2·4). It is reiterated that $\psi_A < \psi_B$, both being reckoned anticlockwise from axis σ_1.

The desired integrable relations follow at once from the canonical forms. We have

$$\left.\begin{aligned} d\sigma_{22}^B &= dR_B - 2Q_B d\phi_B, & d\sigma_{12}^B &= -dQ_B + (P_B - R_B) d\phi_B, \\ d\sigma_{22}^A &= dR_A + 2Q_A d\phi_A, & d\sigma_{12}^A &= dQ_A + (P_A - R_A) d\phi_A, \end{aligned}\right\} \tag{4·2}$$

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by elementary resolution (as in the Jaumann derivative with rotating axes). Whence, from (3·4),

$$\left.\begin{aligned} dR_B/2Q_B = dQ_B/(P_B - R_B) = d\phi_B \quad \text{on} \quad A; \\ dR_A/2Q_A = dQ_A/(P_A - R_A) = -d\phi_A \quad \text{on} \quad B. \end{aligned}\right\} \tag{4.3}$$

Equivalence with (2·7) may be verified via (4·1) and $\phi_A - \psi_A = \phi_B - \psi_B = \alpha$.

In relations (4·3) the equation

$$dR/2Q = dQ/(P - R) \tag{4·4}$$

is common to both A and B. Now, in the isotropic problem, P and R are expressible as unique functions of Q alone on all characteristics; so dR/dQ in (4·4) enters standard theory ((9) and (4)) pp. 294–299) as a true derivative, not merely the ratio of differentials. In the Mohr plane this derivative is the slope of the envelope of stress circles subject to (1·7); in other words, each stress circle touches the envelope at points corresponding to the two characteristics. This geometric representation does not survive in the general case. To gain some other insight into (4·4), imagine that P, Q, R are constant along a curve. At a generic point consider the traction on a material line-element that is always parallel to the tangent at a fixed point O. Relative to axes along the tangent and normal at O, the components of this traction are Q, R at O but, as in (4·2), they are $Q + (P - R)\delta\theta$, $R + 2Q\delta\theta$ at a neighbouring point where the curve has turned through $\delta\theta$. The effect of the curvature is therefore to augment the traction at O by a vector increment inclined at $\tan^{-1}\{2Q/(P-R)\}$ to the tangent, independently of $\delta\theta$. To balance this increment as required by (3·4), it is necessary that Q and R vary along the curve in such a manner that $\delta R/\delta Q$ has the same inclination. This is the content of (4·4).

5. *Stress dependence of characteristic inclinations.* To avoid undue detail in some of the preceding formulae, the explicit solution of (2·4) has been deferred until now.

We recall that, according to (1·5), the normal components of tensor ν_{ij} are zero in the characteristic directions, which are therefore equally inclined to the principal axes of ν_{ij}. By (2·3) the latter are oriented anticlockwise to axis σ_1 at angles β and $\frac{1}{2}\pi + \beta$, where

$$\tan(2\beta) = \frac{\partial F/\partial\alpha}{(\sigma_1 - \sigma_2)(\partial F/\partial\sigma_1 - \partial F/\partial\sigma_2)}, \quad -\tfrac{1}{4}\pi < \beta < \tfrac{1}{4}\pi. \tag{5·1}$$

Then, in (2·7) and (2·8), we can set

$$\psi_A = \beta - \gamma, \quad \psi_B = \beta + \gamma, \quad 0 < \gamma < \tfrac{1}{2}\pi, \tag{5·2}$$

where

$$\left(\frac{\partial F}{\partial\sigma_2} - \frac{\partial F}{\partial\sigma_1}\right)\cos(2\gamma) = \left(\frac{\partial F}{\partial\sigma_1} + \frac{\partial F}{\partial\sigma_2}\right)\cos(2\beta). \tag{5·3}$$

This last is evident from the circle representation of tensor ν_{ij}, via

$$(\nu_1 - \nu_2)\cos(2\beta) = \frac{\partial F}{\partial\sigma_1} - \frac{\partial F}{\partial\sigma_2}, \quad (\nu_2 - \nu_1)\cos(2\gamma) = \frac{\partial F}{\partial\sigma_1} + \frac{\partial F}{\partial\sigma_2},$$

where ν_1 and ν_2 are its principal values. Note that, since the direct component along a characteristic is zero, the direct component perpendicular to it can be evaluated as $\nu_{11} + \nu_{22}$ with any coordinates.

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The solution (5·2) can be checked, or alternatively derived, by recasting (2·4) as

$$\left(\frac{\partial F}{\partial \sigma_1}-\frac{\partial F}{\partial \sigma_2}\right)\cos(2\psi)+\frac{1}{(\sigma_1-\sigma_2)}\frac{\partial F}{\partial \alpha}\sin(2\psi)+\left(\frac{\partial F}{\partial \sigma_1}+\frac{\partial F}{\partial \sigma_2}\right)=0.$$

With the definitions (5·1) and (5·3) this becomes $\cos 2(\psi-\beta)=\cos(2\gamma)$, leading to (5·2).

In the event that a particular constraint (1·8) can be re-arranged conveniently in the form (2·9), we can shorten (5·1) and (5·3) to

$$\sin(2\beta)\Big/\frac{1}{2k}\frac{\partial k}{\partial \alpha}=-\cos(2\beta)=\cos(2\gamma)\Big/\frac{\partial k}{\partial p}, \tag{5·4}$$

which with (5·2) is the solution of (2·10). Auxiliary angles resembling β and γ in (5·4) appear in the integrable relations of Booker and Davis(1); a definite convention for selecting the angles is lacking, however, and no motivation is offered for their introduction.

The angle between the characteristic directions is 2γ, anticlockwise from A to B, and may in principle range between 0 and π. At these limits the directions coincide, so the local state ceases to be hyperbolic and becomes parabolic. In the pressure-insensitive category (1·11), $\partial F/\partial\sigma_1+\partial F/\partial\sigma_2\equiv 0$; then $\gamma\equiv\frac{1}{4}\pi$ which means that the characteristic net is orthogonal, as mentioned in connexion with (2·12). When no real γ is defined by (5·3), the problem is elliptic and outside our terms of reference.

A geometric interpretation for β was remarked in (1). It is constructed by imagining that the surface (1·8) is represented in cylindrical coordinates such that $\frac{1}{2}(\sigma_1+\sigma_2)$ is axial while $\frac{1}{2}(\sigma_1-\sigma_2)$ and 2α are polar (the principal values are conventionally numbered so that $\sigma_1>\sigma_2$). At constant pressure a transverse section of the surface is a plane locus such that

$$d\{\tfrac{1}{2}(\sigma_1-\sigma_2)\}=\frac{-\partial F/\partial\alpha}{(\partial F/\partial\sigma_1-\partial F/\partial\sigma_2)}d\alpha=-\tfrac{1}{2}(\sigma_1-\sigma_2)\tan(2\beta)\,d(2\alpha)$$

by (2·6) and (5·1), or directly from (5·4). This states that 2β, reckoned in the same sense as 2α, is the angle from the extended radius to the outward normal. The interpretation is no surprise, since β is defined as the angle between the axes of tensors σ_{ij} and ν_{ij}, and their respective deviators are represented by vectors along the radius and the normal, according to (2·3).

The two tensors are coaxial when $\partial F/\partial\alpha\equiv 0$ (in-plane isotropy) or when the medium is anisotropic but nevertheless $\partial F/\partial\alpha=0$ for particular states of stress. Some general observations on that possibility can be found in (6) for the full three-dimensional context. A medium with orthotropic symmetry is the important case in applications. With x_1, x_2 as the in-plane axes of orthotropy, and α anticlockwise from x_1 to σ_1, function F is then such that

$$F(\sigma_1,\sigma_2,\alpha)\equiv F(\sigma_1,\sigma_2,-\alpha)\equiv F(\sigma_2,\sigma_1,\tfrac{1}{2}\pi-\alpha)\equiv F(\sigma_2,\sigma_1,\tfrac{1}{2}\pi+\alpha).$$

When α is 0 or $\frac{1}{2}\pi$, $\partial F/\partial\alpha$ vanishes trivially for any ratio σ_1/σ_2. The interesting coaxialities are those that do not coincide with the axes of orthotropy. However, their occurrence is critically dependent on the particular medium and its associated function F, and that is not at issue here.

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