SUMMABILITY AND BOUNDED SEQUENCES

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It is our purpose in this paper to present concise proofs of four theorems which take up the greater part of a paper due to Brudno (1). The proofs of Theorems 1 and 2 have appeared already ((2), (3)), but we present them again so that comparisons can be made with the other two.

We shall consider the partial sums $\{u_k\}$ of the series

$$1+0+\ldots+0-1+0+\ldots+0+\tfrac{1}{2}+\tfrac{1}{2}+0+\ldots+0-\tfrac{1}{2}-\tfrac{1}{2}$$
$$+0+\ldots+0+\ldots+\frac{1}{q}+\ldots+\frac{1}{q}+0+\ldots+0-\frac{1}{q}-\ldots-\frac{1}{q}+\ldots,$$

where $1/q$ appears in the series $2q$ times, the first q times with the plus sign, followed by $g(q)$ zeros; the second q times with the minus sign, followed by $h(q)$ zeros, $h(q)$ and $g(q)$ being integer valued functions of q. It is evident that $0 \leqslant u_k \leqslant 1$ for all k and if $\zeta_k = |u_{k+1}-u_k|$ then $\lim_{k\to\infty} \zeta_k = 0$.

In this paper we shall be concerned throughout with the bounded sequences summed by regular sequence to sequence transformations. If $A = (a_{mn})$ is a regular method, there is a sequence $\{m_\nu\}$ and two non-decreasing functions $\lambda(m)$ and $\mu(m)$ such that

$$\sum_{n=1}^{\lambda(m)} |a_{mn}| < \epsilon_m, \qquad \sum_{n=\mu(m)}^{\infty} |a_{mn}| < \epsilon_m,$$

where $\lambda(m_\nu) = \mu(m_{\nu-1})$ and $\lim_{m\to\infty} \epsilon_m = 0$. Clearly, if s_n is bounded and $A-\lim s_n = 0$, then if

$$\delta_m = \left| \sum_{\lambda(m)}^{\mu(m)} a_{mn} s_n \right|,$$

$\lim_{m\to\infty} \delta_m = 0$. We shall also write $M = \sup_m \Sigma |a_{mn}|$ and $|s_n| \leqslant H$ for all n. We first prove a lemma.

LEMMA. *Let $\{s_n\}$ be a bounded sequence and $A-\lim s_n = 0$. Then, for arbitrary $g(q)$ and $h(q)$, $A-\lim s'_n = 0$, where $s'_n = u_k s_n$, $\lambda(m_k) \leqslant n < \mu(m_k)$.*

Proof. If $t_m = \Sigma a_{mn} s_n$ and $m_k \leqslant m < m_{k+1}$,

$$|t_m| \leqslant \sum_{n=1}^{\lambda(m)-1} |a_{mn}|\,|s'_n| + |u_k| \left| \sum_{n=\lambda(m)}^{\mu(m)} a_{mn} s_n \right| + |u_{k+1}-u_k| \sum_{n=\mu(m_k)}^{\mu(m)} |a_{mn}|\,|s_n|$$
$$+ \sum_{n=\mu(m)+1}^{\infty} |a_{mn}|\,|s'_n|$$
$$\leqslant H\epsilon_m + \delta_m + HM\zeta_k + H\epsilon_m,$$

so that $A-\lim s'_n = 0$.

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THEOREM 1. *If two regular matrix methods, $A = (a_{mn})$ and $B = (b_{mn})$ sum a bounded sequence $\{s_n\}$ to different sums, then there exists a bounded sequence which is summed by A, but not by B.*

Proof. We can choose $\{s_n\}$ so that $A - \lim s_n = 0$ and $B - \lim s_n = 1$. The non-decreasing functions $\lambda_A(m)$, $\mu_A(m)$, $\lambda_B(m)$ and $\mu_B(m)$ can be found for the matrices A and B as in the lemma. The non-decreasing functions $\lambda(m)$ and $\mu(m)$,

$$\lambda(m) \leqslant \min(\lambda_A(m), \lambda_B(m)) \quad \text{and} \quad \mu(m) \geqslant \max(\mu_A(m), \mu_B(m)),$$

can then be used for both matrices A and B as in the lemma. A sequence $\{m_\nu\}$ can be chosen so that $\lambda(m_\nu) \geqslant \mu(m_{\nu-1})$, and then $\mu(m)$ increased, if necessary, so that $\lambda(m_\nu) = \mu(m_{\nu-1})$. Let $g(q) = h(q) = 0$, $q = 0, 1, 2, \ldots$; we know from our lemma that $A - \lim u_k s_n = 0$. However, if $t_m = \Sigma b_{mn} s'_n$, $\lim t_{m_{p2}} = 1$ and $\lim t_{m_{p(p+1)}} = 0$ so that $\{s'_n\}$ is not B-summable. This proves our theorem.

For two regular sequence to sequence transformations, let $B \supset A$ mean that B sums all bounded A-summable sequences, and at least one bounded sequence that is not A-summable.

THEOREM 2. *If $B \supset A$, then there exists a C such that $B \supset C \supset A$.*

Proof. Let the sequence which is B- but not A-summable be denoted by $\{s'_n\}$. Let $s''_n = u_k s'_n$, $\lambda(m_k) \leqslant n < \mu(m_k)$. If $t'_m = \Sigma a_{mn} s'_n$, we can assume $\limsup t'_m > 0$ and we can choose $g(q)$ and $h(q)$ so that $\limsup t''_m = \limsup t'_m$, since by a choice of $g(q)$, $\{u_k\}$ can contain as many successive 1's as we may wish. Also, by choosing $h(q)$, we can have $\{u_k\}$ contain as many 0's as we wish and so define $h(q)$ such that a subsequence $\{m'_r\}$ exists such that $\lim t''_{m'_r} = 0$ and $\limsup t'_{m'_r} = \limsup t'_m$, $\liminf t'_{m'_r} = \liminf t'_m$. If we choose these rows of A to form a new matrix A', A' sums $\{s''_n\}$ to zero but does not sum $\{s'_n\}$. We now intertwine the rows of A' and B and call the new matrix C. The matrix C sums all sequences that are A'- and B-summable, and in particular all bounded sequences that are A-summable. But $\limsup t''_m = \limsup t'_m$, $\liminf t''_m = 0$ and C sums $\{s''_n\}$ which is not A-summable. This means $C \supset A$. The matrix B sums $\{s'_n\}$ which is not C-summable and since a C-summable sequence must be B-summable $B \supset C$. Thus we have $B \supset C \supset A$.

If B sums all bounded A-summable sequences we shall write $B \supseteq A$. We shall see that for our next theorem, we shall enlarge our technique somewhat.

THEOREM 3. *Let the sequence of matrices $\{A^k\}$, $A^k = (a^k_{mn})$, be such that $A^1 \subseteq A^2 \subseteq A^3 \subseteq \ldots \subseteq A^k \subseteq \ldots$ and suppose that the matrix A sums every bounded sequence that is summable by A^k for some k. Then either $A \supset A^k$ for every $k = 1, 2, 3, \ldots$ or A is b-equivalent to A^k for some k.*

Proof. Without loss of generality we can assume $A^1 \subset A^2 \subset \ldots \subset A^k \subset \ldots$ for otherwise we can remove the redundant matrices. If only a finite set of matrices remain, it is clear that A can be taken b-equivalent to A^k with the largest k. From Theorem 1, if $B \supset A$, then A and B are consistent; the bounded sequences summed by both matrices are summed to the same sum by both. If an infinite set of matrices remain we have bounded sequences $\{s^1_n\}, \ldots, \{s^k_n\}, \ldots$ such that $\{s^k_n\}$ is not A^k-summable, but is A^n-summable to zero for all $n > k$. We can assume that $|s^k_n| \leqslant 1$ for all k and all n. Also we

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choose $\{s_n^k\}$ so that $\limsup \Sigma a_{mn}^k s_n^k = U^k = \limsup |\Sigma a_{mn}^k s_n^k|$ and $U^k \leqslant 1$ in every case. We write $M_k = \sup_m \Sigma |a_{mn}^k|$. We now choose a positive sequence $\{\alpha_k\}$, $\alpha_1 = 1$, such that for all k

$$3M_k\alpha_{k+1} \leqslant U^k\alpha_k, \quad \ldots, \quad 3^r M_k \alpha_{k+r} \leqslant U^k\alpha_k, \quad \ldots.$$

Now consider $$s_n' = \sum_{k=1}^{\infty} \alpha_k s_n^k.$$

Since for all k, $M_k \geqslant 1$, $$|s_n'| \leqslant \sum_{k=1}^{\infty} 3^{-(k-1)}$$

so that $\{s_n'\}$ is bounded. In the first place, we show that $\{s_n'\}$ is A-summable to zero. If k is a given positive integer, we can find an N such that for all $m \geqslant N$

$$|\Sigma a_{mn} s_n^r| \leqslant k^{-1}.3^{-k} \quad (r \leqslant k),$$

$$|\Sigma a_{mn} s_n'| \leqslant \left|\Sigma a_{mn}\left[\sum_1^k \alpha_r s_n^r\right]\right| + \left|\Sigma a_m \left[\sum_{k+1}^{\infty} \alpha_r s_n^r\right]\right|$$

$$\leqslant 3^{-k} + \Sigma |a_{mn}| \left[\sum_{k+1}^{\infty} |\alpha_r|\right] \leqslant 3^{-k} + \Sigma |a_{mn}| \sum_{k+1}^{\infty} 3^{-(k-1)}$$

$$\leqslant 3^{-k}(1 + \tfrac{3}{2}M).$$

Clearly, $\{s_n'\}$ is A-summable to zero. Moreover,

$$\limsup \Sigma a_{mn}^k s_n' \geqslant \limsup \Sigma a_{mn}^k \alpha_k s_n^k - \limsup \left|\Sigma a_{mn}^k \left[\sum_1^{k-1} \alpha_r s_n^r\right]\right|$$

$$- \limsup \left|\Sigma a_{mn}^k \left[\sum_{k+1}^{\infty} \alpha_r s_n^r\right]\right|.$$

Now, $$\limsup \left|\Sigma a_{mn}^k \left[\sum_1^{k-1} \alpha_r s_n^r\right]\right| = 0, \quad \limsup \Sigma a_{mn}^k \alpha_k s_n^k = \alpha_k U^k$$

and $$\limsup \left|\Sigma a_{mn}^k \left[\sum_{k+1}^{\infty} \alpha_r s_n^r\right]\right| \leqslant \limsup \Sigma |a_{mn}^k| \left[\sum_{k+1}^{\infty} |\alpha_r|\,|s_n^r|\right]$$

$$\leqslant \limsup \frac{\alpha_k U^k}{M_k} \Sigma |a_{mn}^k| \left[\sum_1^{\infty} 3^{-r}\right] \leqslant \tfrac{1}{2}\alpha_k U^k.$$

Hence $$\limsup \Sigma a_{mn}^k s_n' \geqslant \alpha_k U^k - \tfrac{1}{2}\alpha_k U^k = \tfrac{1}{2}\alpha_k U^k.$$

It is clear that $\{s_n'\}$ cannot be summed to the same sum by A and A^k. Since $A \supseteq A^k$ we conclude from Theorem 1 that $\{s_n'\}$ is A but not A^k-summable, $k = 1, 2, 3, \ldots$.

THEOREM 4. *If $A^1 \supseteq A^2 \supseteq \ldots \supseteq A^k \supseteq \ldots$ and each A^k sums at least one bounded sequence, then there is a bounded sequence that is summable A^k for every k.*

Proof. Let $A_m^k = \Sigma a_{mn}^k s_n$ and consider the matrix $A = (a_{mn})$, $A_m = \Sigma a_{mn} s_n$, defined by

$$A_m = \left(1 - \frac{1}{r}\right) A_v^1 + \frac{1}{r} A_v^r,$$

where $m = \frac{1}{2}v(v+1) + r$, $1 \leqslant r \leqslant v+1$. We first show this matrix sums a given bounded sequence if and only if it is summed by A^k for all k. We can assume that $\Sigma |a_{mn}^r| \leqslant M$ for all r, by (4). Let a sequence $\{s_n'\}$, $|s_n'| \leqslant H$ be summed to zero by all A^k, we can assume

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this because of Theorem 1. If $\epsilon > 0$ is given and $MH/r < \epsilon, r \geqslant i$, then there is an N such that for all $v \geqslant N$, $|A_v^k| \leqslant \epsilon$, $0 \leqslant k \leqslant i$. We then have for $v \geqslant N$, $r \leqslant i$,

$$|A_m| \leqslant \left|1-\frac{1}{r}\right||A_v^1|+\frac{1}{r}|A_v^r| \leqslant \epsilon;$$

for $v \geqslant N$, $r > i$,

$$|A_m| \leqslant \left|1-\frac{1}{r}\right||A_v^1|+\frac{1}{r}|A_v^r| \leqslant \left(1-\frac{1}{r}\right)\epsilon+\frac{MH}{r} < 2\epsilon$$

and $A-\lim s_n' = 0$. On the other hand, if a sequence is not summable by each matrix, either $A_v^1 = A\frac{1}{2}v(v+1)$ diverges or A_v^1 converges and A_v^k diverges for some k. In either case the sequence is not A-summable. This proves our assertion.

We next show by a construction that the matrix we have defined sums at least one bounded sequence. We have a set of sequences $\{s_n^k\}$, such that $\{s_n^k\}$ is a bounded sequence summable A^j, $1 \leqslant j \leqslant k$, to zero, and we can assume $|s_n^k| \leqslant 1$ for all k and n. We select $\lambda(m)$ and $\mu(m)$ and a sequence $\{m_\nu\}$ so that $\lambda(m)$ and $\mu(m)$ satisfy the same conditions for A as in the lemma. Since the $\lambda(m)$, $\mu(m)$ are defined directly for A, they also apply to A^1, and by the linear relation, to A^k with ϵ_m replaced by $\epsilon_m(k)$ where $\lim_{m\to\infty} \epsilon_m(k) = 0$ for each k. Without loss of generality we can assume $\lambda(m)-\lambda(m-1) \leqslant 1$ and $\lambda(m)-\lambda(m-1) = \mu(m)-\mu(m-1) = 0$ whenever $m = \frac{1}{2}v(v+1)+r$, $2 \leqslant r \leqslant v+1$. Finally, if $\lambda(m)$ increases even more slowly and $\mu(m)$ more rapidly then they can be kept constant over longer stretches, so we can select $\{m_\nu\}$ so that not only $\lambda(m_\nu) = \mu(m_{\nu-1})$, $m_\nu = \frac{1}{2}v_\nu(v_\nu+1)$ but also so that

$$\left|\sum_{\lambda(m)}^{\mu(m)} a_{mn}^r s_n^p\right| < \frac{1}{p} \quad (r \leqslant p,\ m > m_{p(p-1)}).$$

We take $g(q) = h(q) = 0$ for all q. We define our sequence

$$\{s_n'\} \text{ by } s_n' = u_\nu s_n^p, \quad \lambda(m_{p(p-1)}) \leqslant \lambda(m_\nu) \leqslant n < \mu(m_\nu) < \mu(m_{p(p+1)}).$$

We assert that $\{s_n'\}$ is A-summable to zero. Let $\Sigma|a_{mn}| = M'$ and also $p < v_{\nu+1}$. If

$$m = \tfrac{1}{2}v(v+1)+r \quad (1 \leqslant r \leqslant p) \quad v_\nu \leqslant v < v_{\nu+1} \leqslant v_{p(p+1)} \quad (\nu > p(p-1)),$$

$$|A_m| \leqslant \left|\sum_1^{\lambda(m)-1} a_{mn}s_n'\right| + u_\nu\left|\sum_{\lambda(m)}^{\mu(m)}\left[\left(1-\frac{1}{r}\right)a_{mn}^1+\frac{1}{r}a_{mn}^r\right]s_n^p\right|$$

$$+|u_\nu-u_{\nu+1}|\left|\sum_{\mu(m_\nu)}^{\mu(m)} a_{mn}s_n^p\right|+\left|\sum_{\mu(m)+1}^{\infty} a_{mn}s_n'\right| \leqslant \epsilon_m+u_\nu\frac{1}{p}+\zeta_\nu M'+\epsilon_m.$$

If $r > p$

$$\left|\Sigma\left[\left(1-\frac{1}{r}\right)a_{mn}^1+\frac{1}{r}a_{mn}^r\right]s_n^p\right| \leqslant \left|\Sigma\left(1-\frac{1}{r}\right)a_{mn}^1 s_n^p\right|+\left|\Sigma\frac{1}{r}a_{mn}^r s_n^p\right|$$

$$\leqslant 2\epsilon_m(1)+\frac{1}{p}+\frac{M}{r} \leqslant 2\epsilon_m(1)+(1+M)\frac{1}{p}.$$

If $\nu = p(p+1)$, where we change from $\{s_n^p\}$ to $\{s_n^{p+1}\}$, a similar analysis follows and so our assertion is true.

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