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THE DIAGONAL CYCLE EULER SYSTEM FOR $\mathrm{GL}_2 \times \mathrm{GL}_2$

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Abstract We construct an anticyclotomic Euler system for the Rankin–Selberg convolutions of two modular forms, using p-adic families of generalised Gross–Kudla–Schoen diagonal cycles. As applications of this construction, we prove new results on the Bloch–Kato conjecture in analytic ranks zero and one, and a divisibility towards an Iwasawa main conjecture.

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1. Introduction

In this paper, we study the anticyclotomic Iwasawa theory of Rankin–Selberg convolutions of two modular forms using a new Euler system arising from p-adic families of diagonal cycles. By an application of Kolyvagin's methods, our construction yields results towards the Bloch–Kato conjecture and the Iwasawa main conjecture in this setting.

1.1. Statement of the main results

Let $g \in S_l(N_g,\chi_g)$ and $h \in S_m(N_h,\chi_h)$ be newforms of weights $l \geq m \geq 2$ of the same parity and nebentypus χ_g and χ_h. Let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D<0$. Let $k>0$ be an even integer, and let ψ be a Hecke character of K of infinity type $(1-k,0)$, conductor $\mathfrak{f}$ and central character

$$\varepsilon_\psi = \bar{\chi}_g\bar{\chi}_h.$$

Fix an odd prime $p \nmid N_gN_h$, such that $(\mathfrak{f},p)=1$ and an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $E = L_{\mathfrak{P}}$ be a finite extension of $\mathbb{Q}_p$ containing the image under ι_p of the values of ψ and the Fourier coefficients of g and h. We consider the E-valued G_K-representation

$$V_{g,h}^{\psi} := V_g \otimes V_h(\psi_{\mathfrak{P}}^{-1})(1-c),$$

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where $c = (k+l+m-2)/2$, V_g and V_h are the (dual of Deligne's) p-adic Galois representations associated to g and h, respectively, and $\psi_{\mathfrak{P}}$ is a p-adic Galois character attached to ψ.

The cyclotomic Iwasawa theory of $V_g \otimes V_h$ has been extensively studied in a series of works of Lei–Loeffler–Zerbes [LLZ14, LLZ15] and Kings–Loeffler–Zerbes [KLZ17, KLZ20], among others ([BLLV19], [BL21], etc.). The key tool exploited in these works is the Euler system of Beilinson–Flach classes, a system of cohomology classes arising from certain special elements (introduced by Beilinson [Bei84], and further studied by Flach [Fla92] and Bertolini–Darmon–Rotger [BDR15a, BDR15b]) in the K_1 of products of two modular curves.

In contrast, the anticyclotomic Iwasawa theory of $V_g \otimes V_h$ (or rather of its conjugate self-dual twists, such as $V_{g,h}^{\psi}$) appears to not have been studied before. The principal contribution of this paper is the construction of an anticyclotomic Euler system for $V_{g,h}^{\psi}$. As stated in Theorem A below (which corresponds to Theorem 6.5 in the body of the paper), for general weights (k,l,m), our construction requires the additional assumptions that p splits in K and $p \nmid h_K$ (the class number of K), and that both g and h are ordinary at p, but note that for $(k,l,m) = (2,l,l)$, Theorem 4.6 contains a version of our main result without these additional hypotheses.

Theorem A. *Suppose that p splits in K and $p \nmid h_K$, and that both g and h are ordinary at p. Let $\mathcal{S}$ be the set of all squarefree products of primes split q in K and coprime to $DN_gN_h\mathfrak{f}$, and denote by $K[n]$ the maximal p-extension of K inside the ring class field of conductor n. Then there exists a family of cohomology classes*

$$\kappa_{\psi,g,h,np^r} \in H^1(K[np^r], T_{g,h}^{\psi})$$

for all $n \in \mathcal{S}$ and $r \geq 0$, where $T_{g,h}^{\psi}$ is a fixed G_K-stable lattice inside $V_{g,h}^{\psi}$, such that for all $nq \in \mathcal{S}$ with q prime, we have

$$\mathrm{cor}_{K[nqp^r]/K[np^r]}(\kappa_{\psi,g,h,nqp^r}) = \begin{cases} P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\,\kappa_{\psi,g,h,np^r} & \text{if } q \neq p, \\ \kappa_{\psi,g,h,np^r} & \text{if } q = p, \end{cases}$$

where $\mathfrak{q}$ is any of the primes of K above q, and $P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | (V_{g,h}^{\psi})^{\vee}(1))$.

The construction of this Euler system, which is taken up in the first part of the paper, is based on the diagonal classes studied by Darmon–Rotger [DR14, DR17, DR22] and Bertolini–Seveso–Venerucci [BSV22], extending earlier constructions due to Gross–Kudla [GK92] and Gross–Schoen [GS95]. Roughly speaking, our classes κ_{ψ,g,h,np^r} are suitable modifications of diagonal classes for the triples $(\tilde{\theta}_{\psi,np^r}, g, h)$, where $\tilde{\theta}_{\psi,np^r}$ is a certain deformation of the theta series associated to ψ, and the main difficulty in the proof of Theorem A is in establishing the Euler system norm relations.

The main results in the second part of the paper are the proof of new cases of the Bloch–Kato conjecture for $V_{g,h}^{\psi}$ in analytic rank zero and a divisibility towards the Iwasawa main conjecture for $V_{g,h}^{\psi}$. These are obtained by applying Kolyvagin's methods (in the form recently developed by Jetchev–Nekovář–Skinner [JNS] in the anticyclotomic setting) to

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our Euler system. In the results that follow, we use 'big image' to refer to Hypothesis (HS) in Section 8.1, for which sufficient conditions are given in Section 8.2.

Theorem B. *Suppose that:*

(a) *g and h are ordinary at p, non-Eisenstein and p-distinguished,*

(b) *p splits in K,*

(c) *p does not divide the class number of K,*

(d) *$V_{g,h}^{\psi}$ has big image.*

Let

$$\kappa_{\psi,g,h} := \kappa_{\psi,g,h,1}.$$

If $l-m<k<l+m$, then the following implication holds:

$$\kappa_{\psi,g,h} \neq 0 \quad \Longrightarrow \quad \dim_E \mathrm{Sel}(K,V_{g,h}^{\psi}) = 1,$$

where $\mathrm{Sel}(K,V_{g,h}^{\psi}) \subset H^1(G_K,V_{g,h}^{\psi})$ is the Bloch–Kato Selmer group.

Remark.

(1) For $k=l=m=2$, together with the Gross–Zagier formula for diagonal cycles by Yuan–Zhang–Zhang [YZZ], Theorem B supports the Bloch–Kato conjecture for $V_{g,h}^{\psi}$ in analytic rank one, reducing it to the expected injectivity of the p-adic étale Abel–Jacobi map.

(2) Still in the case $k=l=m=2$, combined with the p-adic Gross–Zagier formula for diagonal cycles in forthcoming work of Hsieh–Yamana [HY], Theorem B establishes some cases of Perrin-Riou's p-adic Beilinson conjecture in analytic rank one.

(3) In general, by the main result of [DR14], the nonvanishing of $\kappa_{\psi,g,h}$ also follows from the nonvanishing of a special value of the triple product p-adic L-function $\mathscr{L}_p(\mathbf{f},g,h)$ introduced below.

In analytic rank zero, we get unconditional applications to the Bloch–Kato conjecture. Let $f=\theta_{\psi}\in S_k(N_{\psi},\varepsilon_{\psi})$ be the theta series associated to ψ, let $\varepsilon_{\ell}(V_{g,h}^{\psi})$ be the epsilon factor of the Weil–Deligne representation associated to the restriction of $V_f\otimes V_g\otimes V_h(1-c)$ to $G_{\mathbb{Q}_\ell}$, and put $N=\mathrm{lcm}(N_{\psi},N_g,N_h)$.

Theorem C. *Let the hypotheses be as in Theorem B, and assume, in addition, that*

- *$\varepsilon_{\ell}(V_{g,h}^{\psi})=+1$ for all primes $\ell \mid N$,*
- *$\gcd(N_{\psi},N_g,N_h)$ is squarefree.*

If $k\geq l+m$, then

$$L(V_{g,h}^{\psi},0)\neq 0 \quad \Longrightarrow \quad \mathrm{Sel}(K,V_{g,h}^{\psi})=0,$$

and hence, the Bloch–Kato conjecture for $V_{g,h}^{\psi}$ holds in analytic rank zero.

Remark. Here, $L(V_{g,h}^{\psi},s)$ is the triple product L-function introduced by Garrett, Piatetski–Shapiro and Rallis, which satisfies a functional equation relating its values at s

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and $-s$. When $k \geq l+m$, the local root number condition in Theorem C implies that the sign in this functional equation is $+1$, and so the central L-values $L(V_{g,h}^{\psi},0)$ are expected to be generically nonzero.

A third application is to the anticyclotomic Iwasawa main conjectures for Rankin–Selberg convolutions. Let $(\mathbf{f},\mathbf{g},\mathbf{h})$ be a triple of p-adic Hida families. In [Hsi21], Hsieh has constructed a square root triple product p-adic L-function $\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ whose square interpolates the central values of the triple product L-function attached to the classical specialisations of $(\mathbf{f},\mathbf{g},\mathbf{h})$ to weights (k_1,k_2,k_3) with $k_1 \geq k_2+k_3$. Letting $\mathbf{g}$ and $\mathbf{h}$ be the Hida families passing through the ordinary p-stabilisations of g and h, respectively, we obtain an element

$$\mathscr{L}_p(\mathbf{f},g,h) \in \Lambda_{\mathbf{f}}$$

interpolating a square root of the above central L-values for the specialisations of $\mathbf{f}$ to weights $k \geq l+m$, where $\Lambda_{\mathbf{f}}$ is the finite flat extension of $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ generated by the coefficients of $\mathbf{f}$. Greenberg's generalisation of the Iwasawa main conjectures [Gre94] predicts that $\mathscr{L}_p(\mathbf{f},g,h)^2$ generates the $\Lambda_{\mathbf{f}}$-characteristic ideal of a certain torsion Selmer group $X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$. We also show that our classes $\kappa_{\psi,g,h,n}$ are universal norms in the p-direction, therefore giving rise, in particular, to an Iwasawa cohomology class

$$\kappa_{\psi,g,h,\infty} \in H^1_{\mathrm{Iw}}(K_\infty, T_{g,h}^{\psi})$$

for the anticyclotomic $\mathbb{Z}_p$-extension K_∞/K. The class $\kappa_{\psi,g,h,\infty}$ is associated with the triple $(\mathbf{f},g,h)$, where $\mathbf{f} = \mathbf{f}_\psi$ is a CM Hida family attached to ψ for which $\Lambda_{\mathbf{f}} \cong \Lambda_{\mathrm{ac}}$, the anticyclotomic Iwasawa algebra. Assuming the nontriviality of $\kappa_{\psi,g,h,\infty}$, we can prove the following result towards the Iwasawa main conjecture for $\mathscr{L}_p(\mathbf{f},g,h)^2$.

Theorem D. *Let* $\mathbf{f} = \mathbf{f}_\psi$*, and suppose that:*

(a) *g and h are ordinary at p, non-Eisenstein and p-distinguished,*

(b) *p splits in K,*

(c) *p does not divide the class number of K,*

(d) *$V_{g,h}^{\psi}$ has big image,*

(e) *$\varepsilon_\ell(V_{g,h}^{\psi}) = +1$ for all primes $\ell \mid N$,*

(f) *$\gcd(N_\psi, N_g, N_h)$ is squarefree.*

If $\kappa_{\psi,g,h,\infty}$ is not Λ_{ac}-torsion, then the module $X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$ is Λ_{ac}-torsion, and

$$\mathrm{Char}_{\Lambda_{\mathrm{ac}}}(X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})) \supset (\mathscr{L}_p(\mathbf{f},g,h)^2)$$

in $\Lambda_{\mathrm{ac}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Remark. The classes $\kappa_{\psi,g,h,n}$ may be viewed as a counterpart in the study of the arithmetic of $V_{g,h}^{\psi}$ to systems of Heegner points and Heegner cycles for individual modular forms. It would be interesting to see whether the methods of Cornut–Vatsal can be extended to establish the nontriviality of $\kappa_{\psi,g,h,\infty}$.

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Remark. The 'big image' hypothesis on $V_{g,h}^{\psi}$ excludes some cases of arithmetic interest; notably, the case in which $h = g^*$ is the dual of g (assuming ψ has trivial central character) is excluded from our applications in this paper. We study this case in [ACR22], where, building on (a suitable projection of) the classes $\kappa_{\psi,g,g^*,n}$ constructed in this paper, we obtain a new anticyclotomic Euler system for twists of the three-dimensional G_K-representation $\mathrm{ad}^0(V_g)$, with applications to the Bloch–Kato conjecture in rank zero and the Iwasawa main conjecture in this setting.

Remark. As already noted, the anticyclotomic Euler system classes constructed in this paper arise from diagonal classes attached to triples (f,g,h) of modular forms in which f varies over certain CM forms by K. A modification of this construction with g and h varying among certain CM forms for the same imaginary quadratic field K gives rise to a new anticyclotomic Euler system for twists of $V_f|_{G_K}$. This construction, and its arithmetic applications, is studied in [Do22, CD23].

Part 1. The diagonal cycle Euler system

2. Preliminaries

In this section, we begin by discussing our conventions regarding modular curves and Hecke operators, for which we shall largely follow [Kat04, Section 2] and [BSV22, Section 2].

2.1. Modular curves

Given integers $M \geq 1$, $N \geq 1$, $m \geq 1$ and $n \geq 1$ with $M + N \geq 5$, we denote by $Y(M(m),N(n))$ the affine modular curve over $\mathbb{Z}[1/MNmn]$ representing the functor taking a $\mathbb{Z}[1/MNmn]$-scheme S to the set of isomorphism classes of 5-tuples (E,P,Q,C,D), where:

- E is an elliptic curve over S,
- P is an S-point of E of order M,
- Q is an S-point of E of order N,
- C is a cyclic order-Mm subgroup of E defined over S and containing P,
- D is a cyclic order-Nn subgroup of E defined over S and containing Q,

and such that C and D have trivial intersection. If either $m = 1$ or $n = 1$, we omit it from the notation, and we will often write $Y_1(N)$ for $Y(1,N)$.

We will denote by

$$E(M(m),N(n)) \to Y(M(m),N(n))$$

the universal elliptic curve over $Y(M(m),N(n))$.

Define the modular group

$$\Gamma(M(m),N(n)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv 1\,(M), b \equiv 0\,(Mm), c \equiv 0\,(Nn), d \equiv 1\,(N) \right\}.$$

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Then, letting $\mathcal{H}$ be the Poincaré upper half-plane, we have the complex uniformisation

$$Y(M(m),N(n))(\mathbb{C}) \cong (\mathbb{Z}/M\mathbb{Z})^\times \times \Gamma(M(m),N(n))\backslash\mathcal{H}, \tag{2.1}$$

with a pair (a,τ) on the right-hand side corresponding to the isomorphism class of the 5-tuple $(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau, a\tau/M, 1/N, \langle\tau/Mm\rangle, \langle 1/Nn\rangle)$.

If $r \geq 1$ is an integer, there is an isomorphism of $\mathbb{Z}[1/MNmnr]$-schemes

$$\varphi_r : Y(M(m),N(nr)) \xrightarrow{\simeq} Y(M(mr),N(n))$$

defined in terms of moduli by

$$(E,P,Q,C,D) \mapsto (E',P',Q',C',D'),$$

where $E' = E/NnD$, P' is the image of P in E', Q' is the image of $r^{-1}(Q) \cap D$ in E', C' is the image of $r^{-1}(C)$ in E' and D' is the image of D in E'. Under the complex uniformisations (2.1), the isomorphism φ_r sends $(a,\tau) \mapsto (a, r\cdot\tau)$. If

$$\varphi_r^*(E(M(mr),N(n))) \to Y(M(m),N(nr))$$

denotes the base change of $E(M(mr),N(n)) \to Y(M(mr),N(n))$ under φ_r, there is a natural degree-r isogeny

$$\lambda_r : E(M(m),N(nr)) \to \varphi_r^*(E(M(mr),N(n))).$$

2.2. Degeneracy maps

With the same notations as above, we have natural degeneracy maps

$$Y(M(m),Nr(n)) \xrightarrow{\mu_r} Y(M(m),N(nr)) \xrightarrow{\nu_r} Y(M(m),N(n)),$$
$$Y(Mr(m),N(n)) \xrightarrow{\check{\mu}_r} Y(M(mr),N(n)) \xrightarrow{\check{\nu}_r} Y(M(m),N(n)),$$

forgetting the extra level structure, for example

$$\mu_r(E,P,Q,C,D) = (E,P,r\cdot Q,C,D),$$
$$\nu_r(E,P,Q,C,D) = (E,P,Q,C,rD).$$

We also define degeneracy maps

$$\begin{aligned} \pi_1 &: Y(M(m),Nrs(nt)) \to Y(M(m),N(ns)), \\ \pi_2 &: Y(M(m),Nrs(nt)) \to Y(M(m),N(ns)), \end{aligned} \tag{2.2}$$

acting on the moduli space by

$$\pi_1(E,P,Q,C,D) = (E,P,rs\cdot Q,C,rtD),$$
$$\pi_2(E,P,Q,C,D) = (E',P',Q',C',D'),$$

where $E' = E/NnsD$, P' is the image of P in E', Q' is the image of $t^{-1}(s\cdot Q) \cap D$ in E', C' is the image of C in E' and D' is the image of D in E'. Under the complex uniformisations in (2.1), the maps π_1 and π_2 correspond to the identity and to multiplication by rt, respectively, on $\mathcal{H}$. It is straightforward to check that the maps π_1 and π_2 are given by the compositions

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$$Y(M(m),Nrs(nt)) \xrightarrow{\mu_{rs}} Y(M(m),N(nrst)) \xrightarrow{\nu_{rt}} Y(M(m),N(ns)),$$
$$Y(M(m),Nrs(nt)) \xrightarrow{\mu_{rs}} Y(M(m),N(nrst)) \xrightarrow{\varphi_{rt}} Y(M(mrt),N(ns)) \xrightarrow{\check{\nu}_{rt}} Y(M(m),N(ns)),$$

respectively.

2.3. Relative Tate modules

Fix a prime p. Let S be a $\mathbb{Z}[1/MNmnp]$-scheme, and let

$$v : E(M(m),N(n))_S \to Y(M(m),N(n))_S$$

be the structural morphism. For every $\mathbb{Z}[1/MNmnp]$-scheme X, denote by $A = A_X$ either the locally constant constructible sheaf $\mathbb{Z}/p^t(j)$ or the locally constant p-adic sheaf $\mathbb{Z}_p(j)$ on X_{et}, for fixed $t \geq 1$ and $j \in \mathbb{Z}$. Set

$$\mathscr{T}_{M(m),N(n)}(A) = R^1 v_* \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} A \quad \text{and} \quad \mathscr{T}^*_{M(m),N(n)}(A) = \mathrm{Hom}(\mathscr{T}_{M(m),N(n)}(A), A).$$

In particular, in the case $A = \mathbb{Z}_p$, this gives the relative Tate module of the universal elliptic curve and its dual, respectively; in this case, we will often drop A from the notation.

From the proper base change theorem, both $\mathscr{T}_{M(m),N(n)}(A)$ and $\mathscr{T}^*_{M(m),N(n)}(A)$ are locally constant p-adic sheaves on $Y(M(m),N(n))_S$ of formation compatible with base changes along morphisms of $\mathbb{Z}[1/MNmnp]$-schemes $S' \to S$.

For every integer $r \geq 0$, define

$$\mathscr{L}_{M(m),N(n),r}(A) = \mathrm{Tsym}^r_A \mathscr{T}_{M(m),N(n)}(A), \quad \mathscr{S}_{M(m),N(n),r}(A) = \mathrm{Symm}^r_A \mathscr{T}^*_{M(m),N(n)}(A),$$

where, for any finite free module M over a profinite $\mathbb{Z}_p$-algebra R, one denotes by $\mathrm{Tsym}^r_R M$ the R-submodule of symmetric tensors in $M^{\otimes r}$ and by $\mathrm{Symm}^r_R M$ the maximal symmetric quotient of $M^{\otimes r}$.

When the level of the modular curve $Y(M(m),N(n))_S$ is clear, we may use the simplified notations

$$\mathscr{L}_r(A) = \mathscr{L}_{M(m),N(n),r}(A), \quad \mathscr{L}_r = \mathscr{L}_r(\mathbb{Z}_p), \quad \mathscr{S}_r(A) = \mathscr{S}_{M(m),N(n),r}(A), \quad \mathscr{S}_r = \mathscr{S}_r(\mathbb{Z}_p).$$

2.4. Hecke operators

Let $\mathscr{F}^r_{M(m),N(n)}$ denote either $\mathscr{L}_{M(m),N(n),r}(A)$ or $\mathscr{S}_{M(m),N(n),r}(A)$, and let q be a rational prime. Then there are natural isomorphisms of sheaves

$$\nu_q^*(\mathscr{F}^r_{M(m),N(n)}) \cong \mathscr{F}^r_{M(m),N(nq)} \quad \text{and} \quad \check{\nu}_q^*(\mathscr{F}^r_{M(m),N(n)}) \cong \mathscr{F}^r_{M(mq),N(n)}, \tag{2.3}$$

and therefore pullback morphisms

$$H^i_{\mathrm{et}}(Y(M(m),N(n))_S, \mathscr{F}^r_{M(m),N(n)}) \xrightarrow{\nu_q^*} H^i_{\mathrm{et}}(Y(M(m),N(nq))_S, \mathscr{F}^r_{M(m),N(nq)}),$$
$$H^i_{\mathrm{et}}(Y(M(m),N(n))_S, \mathscr{F}^r_{M(m),N(n)}) \xrightarrow{\check{\nu}_q^*} H^i_{\mathrm{et}}(Y(M(mq),N(n))_S, \mathscr{F}^r_{M(mq),N(n)}),$$

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and traces

$$\begin{aligned} H^i_{\mathrm{et}}(Y(M(m),N(nq))_S,\mathscr{F}^r_{M(m),N(nq)}) &\xrightarrow{\nu_{q*}} H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)}),\\ H^i_{\mathrm{et}}(Y(M(mq),N(n))_S,\mathscr{F}^r_{M(mq),N(n)}) &\xrightarrow{\check{\nu}_{q*}} H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)}). \end{aligned} \tag{2.4}$$

Also, the isogeny λ_q induces morphisms of sheaves

$$\lambda_{q*} : \mathscr{F}^r_{M(m),N(nq)} \to \varphi_q^*(\mathscr{F}^r_{M(mq),N(n)}) \quad \text{and} \quad \lambda_q^* : \varphi_q^*(\mathscr{F}^r_{M(mq),N(n)}) \to \mathscr{F}^r_{M(m),N(nq)}.$$

These morphisms allow us to define

$$\begin{aligned} \Phi_{q*} &: H^i_{\mathrm{et}}(Y(M(m),N(nq))_S,\mathscr{F}^r_{M(m),N(nq)}) \to H^i_{\mathrm{et}}(Y(M(mq),N(n))_S,\mathscr{F}^r_{M(mq),N(n)}),\\ \Phi_q^* &: H^i_{\mathrm{et}}(Y(M(mq),N(n))_S,\mathscr{F}^r_{M(mq),N(n)}) \to H^i_{\mathrm{et}}(Y(M(m),N(nq))_S,\mathscr{F}^r_{M(m),N(nq)}), \end{aligned}$$

as the compositions

$$\Phi_{q*} = \varphi_{q*} \circ \lambda_{q*} \quad \text{and} \quad \Phi_q^* = \lambda_q^* \circ \varphi_q^*.$$

We define the Hecke operators T_q and the adjoint Hecke operators T_q' acting on the étale cohomology groups

$$H^i_{\mathrm{et}}(Y(M(m),N(nq))_S,\mathscr{F}^r_{M(m),N(nq)})$$

as the compositions

$$T_q = \check{\nu}_{q*} \circ \Phi_{q*} \circ \nu_q^* \quad \text{and} \quad T_q' = \nu_{q*} \circ \Phi_q^* \circ \check{\nu}_q^*.$$

If we define pullbacks

$$\begin{aligned} H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)}) &\xrightarrow{\pi_1^*} H^i_{\mathrm{et}}(Y(M(m),N(nq))_S,\mathscr{F}^r_{M(m),N(nq)}),\\ H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)}) &\xrightarrow{\pi_2^*} H^i_{\mathrm{et}}(Y(M(mq),N(n))_S,\mathscr{F}^r_{M(mq),N(n)}), \end{aligned}$$

and pushforward

$$\begin{aligned} H^i_{\mathrm{et}}(Y(M(m),N(nq))_S,\mathscr{F}^r_{M(m),N(nq)}) &\xrightarrow{\pi_{1*}} H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)}),\\ H^i_{\mathrm{et}}(Y(M(mq),N(n))_S,\mathscr{F}^r_{M(mq),N(n)}) &\xrightarrow{\pi_{2*}} H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)}), \end{aligned}$$

as

$$\pi_1^* = \nu_q^*, \quad \pi_2^* = \Phi_q^* \circ \check{\nu}_q^*, \quad \pi_{1*} = \nu_{q*} \quad \text{and} \quad \pi_{2*} = \check{\nu}_{q*} \circ \Phi_{q*},$$

then we can write

$$T_q = \pi_{2*} \circ \pi_1^* \quad \text{and} \quad T_q' = \pi_{1*} \circ \pi_2^*.$$

Now we introduce diamond operators. For $d \in (\mathbb{Z}/MN\mathbb{Z})^\times$, these are defined on the curves $Y(M(m),N(n))$ as the automorphisms $\langle d \rangle$ acting on the moduli space by

$$(E,P,Q,C,D) \mapsto (E,d^{-1} \cdot P,d \cdot Q,C,D).$$

We can also define the diamond operator $\langle d \rangle$ on the corresponding universal elliptic curve as the unique automorphism making the diagram

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$$\begin{array}{ccc} E(M(m),N(n))_S & \xrightarrow{\langle d\rangle} & E(M(m),N(n))_S \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} \\ Y(M(m),N(n))_S & \xrightarrow{\langle d\rangle} & Y(M(m),N(n))_S \end{array}$$

cartesian. This, in turn, induces automorphisms $\langle d\rangle = \langle d\rangle^*$ and $\langle d\rangle' = \langle d\rangle_*$ on the group $H^i_{\mathrm{et}}(Y(M(m),N(n))_S,\mathscr{F}^r_{M(m),N(n)})$ which are inverses of each other. In general, we will be interested in modular curves of the form $Y(1(m),N(n))$. In this case, the natural pairing $\mathscr{L}_r \otimes_{\mathbb{Z}_p} \mathscr{S}_r \to \mathbb{Z}_p$ together with cup-product yields a pairing

$$H^1_{\mathrm{et}}(Y(1(m),N(n))_S,\mathscr{L}_r(1)) \otimes_{\mathbb{Z}_p} H^1_{\mathrm{et},c}(Y(1(m),N(n))_S,\mathscr{S}_r) \to \mathbb{Z}_p$$

which becomes perfect after inverting p. The operators T_q, T'_q, $\langle d\rangle$, $\langle d\rangle'$ induce endomorphisms on compactly supported cohomology and

$$(T_q,T'_q), \quad (T'_q,T_q), \quad (\langle d\rangle,\langle d\rangle'), \quad \text{and} \quad (\langle d\rangle',\langle d\rangle)$$

are adjoint pairs under this pairing.

2.5. Galois representations

Let $f \in S_k(N_f,\chi_f)$ be a newform of weight $k = r+2 \geq 2$, level N_f, and character χ_f. Let p be a prime, and let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ containing the Fourier coefficients of f. By the work of Eichler–Shimura and Deligne, there is a two-dimensional representation

$$\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(E)$$

unramified outside pN_f and characterised by the property that

$$\operatorname{trace} \rho_f(\mathrm{Fr}_q) = a_q(f)$$

for all primes $q \nmid pN_f$, where Fr_q denotes an arithmetic Frobenius element at q (in fact, this is the dual of the p-adic representation constructed by Deligne).

It will be convenient for our purposes to work with the following geometric realisation of ρ_f. Let

$$H^1_{\mathrm{et}}(Y_1(N_f)_{\overline{\mathbb{Q}}},\mathscr{L}_r(1)) \otimes_{\mathbb{Z}_p} E \twoheadrightarrow V_f$$

be the maximal quotient on which T'_q and $\langle d\rangle'$ act as multiplication by $a_q(f)$ and $\chi_f(d)$ for all primes $q \nmid N_f$ and all $d \in (\mathbb{Z}/N_f\mathbb{Z})^\times$. Then V_f is a two-dimensional E-vector space affording the p-adic representation ρ_f, and we let $T_f \subset V_f$ be the lattice defined by the image of

$$H^1_{\mathrm{et}}(Y_1(N_f)_{\overline{\mathbb{Q}}},\mathscr{L}_r(1)) \otimes_{\mathbb{Z}_p} \mathcal{O}$$

under the above quotient map.

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3. Hecke algebras and ring class fields

In this section, we extend the results of [LLZ15, Section 5.2], including ring class field extensions of an imaginary quadratic field K. The resulting Corollary 3.6 will allow us to obtain classes over ring class field extensions of K from diagonal cycles over $\mathbb{Q}$ on triple products of modular curves of varying levels.

Thus, let K be an imaginary quadratic field of discriminant $-D<0$, and let ε_K be the corresponding quadratic character. Let ψ be a Grössencharacter of K of infinity type $(-1,0)$ and conductor $\mathfrak{f}$, taking values in a finite extension L/K, and let χ be the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$, such that $\psi((n))=n\chi(n)$ for integers n coprime to $N_{K/\mathbb{Q}}(\mathfrak{f})$. Put $N_\psi = N_{K/\mathbb{Q}}(\mathfrak{f})D$, and let $\theta_\psi \in S_2(N_\psi, \chi\varepsilon_K)$ be the newform attached to ψ, that is,

$$\theta_\psi = \sum_{(\mathfrak{a},\mathfrak{f})=1} \psi(\mathfrak{a}) q^{N_{K/\mathbb{Q}}(\mathfrak{a})}.$$

Fix a prime $p \geq 5$ unramified in K, a prime $\mathfrak{p}$ of K above p and a prime $\mathfrak{P}$ of L above $\mathfrak{p}$. Let $E = L_{\mathfrak{P}}$, and let $\mathcal{O} \subset E$ be the ring of integers. Let $\psi_{\mathfrak{P}}$ be the continuous E-valued character of $K^\times \backslash \mathbb{A}_{K,\mathrm{f}}^\times$ defined by

$$\psi_{\mathfrak{P}}(x) = x_{\mathfrak{p}}^{-1}\psi(x),$$

where $x_{\mathfrak{p}}$ is the projection of the idèle x to the component at $\mathfrak{p}$. We will also denote by $\psi_{\mathfrak{P}}$ the corresponding character of G_K obtained via the geometric Artin map. Then $\mathrm{Ind}_K^{\mathbb{Q}} E(\psi_{\mathfrak{P}}^{-1})$ is the p-adic representation attached to θ_ψ.

Definition 3.1. For an integral ideal $\mathfrak{n}$ of K, we denote by $H_{\mathfrak{n}}$ the maximal p-quotient of the corresponding ray class group, and by $K(\mathfrak{n})$ the maximal p-extension in the corresponding ray class field. We similarly define R_n and $K[n]$, for each integer $n>0$, as the maximal p-quotient in the corresponding ring class group and the maximal p-extension in the corresponding ring class field.

Let $\mathfrak{n}$ be an integral ideal of K divisible by $\mathfrak{f}$, and let $N = N_{K/\mathbb{Q}}(\mathfrak{n})D$, which is of course a multiple of N_ψ. Let $\mathbb{T}_1'(N)$ be the algebra generated by all the Hecke operators T_q', $\langle d\rangle'$ acting on $H^1(Y_1(N)(\mathbb{C}),\mathbb{Z})$.

Proposition 3.2. *With the previous definitions and notations, there exists a homomorphism* $\phi_{\mathfrak{n}} : \mathbb{T}_1'(N) \to \mathcal{O}[H_{\mathfrak{n}}]$ *defined on generators by*

$$\phi_{\mathfrak{n}}(T_q') = \sum_{\mathfrak{q}} \psi(\mathfrak{q})[\mathfrak{q}]$$

for every rational prime q*, where the sum runs over ideals coprime to* $\mathfrak{n}$ *of norm* q*; and*

$$\phi_{\mathfrak{n}}(\langle d\rangle') = \chi(d)\varepsilon_K(d)[(d)].$$

Proof. This follows immediately from [LLZ15, Proposition 3.2.1]. □

Now let $\mathfrak{n}' = \mathfrak{n}\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ above a rational prime q. Assume that $\mathfrak{n}'$ is coprime to p, and let $N' = N_{K/\mathbb{Q}}(\mathfrak{n}')D$. Following [LLZ15, Section 3.3], we define norm maps

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$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} : \mathcal{O}[H_{\mathfrak{n}'}] \otimes_{(\mathbb{T}'_1(N')\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}'})} H^1_{\text{et}}(Y_1(N')_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) \longrightarrow \mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}})} H^1_{\text{et}}(Y_1(N)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))$$

by the formulae:

- if $\mathfrak{q} \mid \mathfrak{n}$,
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \pi_{1*};$$
- if $\mathfrak{q} \nmid \mathfrak{n}$ and $\mathfrak{q}$ is ramified or split,
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \pi_{1*} - \frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q} \otimes \pi_{2*};$$
- if $\mathfrak{q} \nmid \mathfrak{n}$ and $\mathfrak{q}$ is inert,
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \pi_{1*} - \frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q^2} \otimes \pi_{2*}.$$

More generally, for $\mathfrak{n}' = \mathfrak{n}\mathfrak{r}$ with $\mathfrak{r}$ a product of (not necessarily distinct) prime ideals, we define the map $\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'}$ by composing in the natural way the previously defined norm maps. From now on, we assume that in the case where $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K, the following holds: If $\mathfrak{p} \mid \mathfrak{f}$, then $\overline{\mathfrak{p}} \nmid \mathfrak{f}$ and $\psi|_{\mathcal{O}_{K,\mathfrak{p}}^\times}$ is not congruent to the Teichmüller character modulo $\mathfrak{P}$.

Theorem 3.3. *Let A be the set of prime ideals of K coprime to $\overline{\mathfrak{p}}$ (respectively, p) if p splits (respectively, is inert) in K and divisible by $\mathfrak{f}$. Then there is a family of $G_{\mathbb{Q}}$-equivariant isomorphisms of $\mathcal{O}[H_{\mathfrak{n}}]$-modules*

$$\nu_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}})} H^1_{\text{et}}(Y_1(N)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) \xrightarrow{\simeq} \text{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}),$$

for all $\mathfrak{n} \in A$, such that for $\mathfrak{n} \mid \mathfrak{n}'$ the diagram

$$\begin{array}{ccc} \mathcal{O}[H_{\mathfrak{n}'}] \otimes_{(\mathbb{T}'_1(N')\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}'})} H^1_{\text{et}}(Y_1(N')_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_{\mathfrak{n}'}} & \text{Ind}_{K(\mathfrak{n}')}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\ \downarrow{\scriptstyle \mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'}} & & \downarrow \\ \mathcal{O}[H_{\mathfrak{n}}] \otimes_{(\mathbb{T}'_1(N)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}})} H^1_{\text{et}}(Y_1(N)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_{\mathfrak{n}}} & \text{Ind}_{K(\mathfrak{n})}^{\mathbb{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \end{array}$$

commutes, where the right vertical arrow is the natural norm map.

Proof. This is [LLZ15, Corollary 5.2.6]. □

Definition 3.4. For any positive integer n with $(n,p\mathfrak{f}) = 1$, we let $K(\mathfrak{f})[n]$ be the compositum of $K(\mathfrak{f})$ and $K[n]$, and put $R_{\mathfrak{f},n} = \text{Gal}(K(\mathfrak{f})[n]/K)$.

Let $\mathbb{T}'(1,N_\psi(n^2)) \subset \text{End}_{\mathbb{Z}}(H^1(Y(1,N_\psi(n^2))(\mathbb{C}),\mathbb{Z}))$ be the subalgebra generated by all Hecke operators T'_q and $\langle d\rangle'$.

Lemma 3.5. *There exists a homomorphism*

$$\phi_n : \mathbb{T}'(1,N_\psi(n^2)) \longrightarrow \mathcal{O}[R_{\mathfrak{f},n}]$$

defined on generators by the same formula as in Proposition 3.2.

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Proof. Take the modulus $\mathfrak{n} = \mathfrak{f}(n)$. By Proposition 5.1.2 and Remark 5.1.3 in [LLZ15], the kernel $\mathcal{I}$ of the composition

$$\mathbb{T}'_1(N_\psi n^2) \xrightarrow{\phi_{\mathfrak{n}}} \mathcal{O}[H_{\mathfrak{n}}] \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{P},$$

where $\phi_{\mathfrak{n}}$ is as in Proposition 3.2, is a non-Eisenstein maximal ideal of $\mathbb{T}'_1(N_\psi n^2)$ in the sense of [*op. cit.*, Definition 4.1.2]. Therefore, denoting $\mathcal{I}$-adic completions with the subscript $\mathcal{I}$, we have an isomorphism of $\mathbb{T}'_1(N_\psi n^2)_{\mathcal{I}}$-modules

$$H^1(Y_1(N_\psi n^2)(\mathbb{C}),\mathbb{Z})_{\mathcal{I}} \cong H^1_c(Y_1(N_\psi n^2)(\mathbb{C}),\mathbb{Z})_{\mathcal{I}}.$$

On the other hand, as in the proof of [LLZ15, Lemma 4.2.4], the natural pullback map yields an isomorphism

$$H^1_c(Y(1,N_\psi(n^2))(\mathbb{C}),\mathbb{Z}) \cong H^1_c(Y_1(N_\psi n^2)(\mathbb{C}),\mathbb{Z})^{\Delta},$$

where Δ is the set of diamond operators $\langle d\rangle'$ with $d \equiv 1 (\mathrm{mod}\ N_\psi)$. Since Δ maps to 1 under the composition

$$\mathbb{T}'_1(N_\psi n^2) \xrightarrow{\phi_{\mathfrak{n}}} \mathcal{O}[H_{\mathfrak{n}}] \longrightarrow \mathcal{O}[R_{\mathfrak{f},n}],$$

the result follows. □

Corollary 3.6. *Let B be the set of positive integers n coprime to $p\mathfrak{f}$. Then there is a family of $G_{\mathbb{Q}}$-equivariant isomorphisms of $\mathcal{O}[R_{\mathfrak{f},n}]$-modules*

$$\nu_n : \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1,N_\psi(n^2))\otimes\mathbb{Z}_p,\phi_n)} H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) \xrightarrow{\simeq} \mathrm{Ind}^{\mathbb{Q}}_{K(\mathfrak{f})[n]} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})$$

for all $n \in B$, such that for $n \mid n'$ the diagram

$$\begin{array}{ccc}
\mathcal{O}[R_{\mathfrak{f},n'}] \otimes_{(\mathbb{T}'(1,N_\psi(n'^2))\otimes\mathbb{Z}_p,\phi_{n'})} H^1_{\mathrm{et}}(Y(1,N_\psi(n'^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_{n'}} & \mathrm{Ind}^{\mathbb{Q}}_{K(\mathfrak{f})[n']} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \\
\downarrow{\scriptstyle \mathcal{N}^{\mathfrak{f},n'}_{\mathfrak{f},n}} & & \downarrow \\
\mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1,N_\psi(n^2))\otimes\mathbb{Z}_p,\phi_n)} H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1)) & \xrightarrow[\simeq]{\nu_n} & \mathrm{Ind}^{\mathbb{Q}}_{K(\mathfrak{f})[n]} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})
\end{array}$$

commutes, where $\mathcal{N}^{\mathfrak{f},n'}_{\mathfrak{f},n}$ is induced by $\mathcal{N}^{\mathfrak{f}(n)}_{\mathfrak{f}(n')}$ and the right vertical arrow is the natural norm map.

Proof. Let $\mathfrak{n} = \mathfrak{f}(n)$, $\mathcal{I}$ and Δ be as in the proof of Lemma 3.5. Since $\mathcal{I}$ is non-Eisenstein, the natural trace map

$$H^1_{\mathrm{et}}(Y_1(N_\psi n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))_{\Delta} \longrightarrow H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))$$

becomes an isomorphism after taking $\mathcal{I}$-adic completions. Since the map ϕ_n of Lemma 3.5 is induced by $\phi_{\mathfrak{n}}$ (as shown in the proof of that result), it follows that the $\mathcal{O}[R_{\mathfrak{f},n}]$-module

$$\mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1,N_\psi(n^2))\otimes\mathbb{Z}_p,\phi_n)} H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))$$

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is naturally isomorphic to

$$\mathcal{O}[R_{\mathfrak{f},n}]\otimes_{\mathcal{O}[H_{\mathfrak{n}}]}\left(\mathcal{O}[H_{\mathfrak{n}}]\otimes_{(\mathbb{T}'_1(N_\psi n^2)\otimes\mathbb{Z}_p,\phi_{\mathfrak{n}})}H^1_{\text{ét}}(Y_1(N_\psi n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))\right).$$

The result now follows from Theorem 3.3. □

4. Proof of the tame norm relations

We keep the notations introduced in Section 3. Fix two newforms (g,h) of weights (l,m) of the same parity, levels (N_g,N_h) and characters (χ_g,χ_h), such that $\chi\varepsilon_K\chi_g\chi_h=1$. Enlarging L if necessary, assume that it contains the Fourier coefficients of g and h.

Let $N=\mathrm{lcm}(N_\psi,N_g,N_h)$, and (since N will be fixed throughout) put $Y(m)=Y(1,N(m))$ for every positive integer m.

Definition 4.1. Let $\mathbf{r}=(r_1,r_2,r_3)$ be a triple of nonnegative integers, such that

$$r_1+r_2+r_3=2r$$

with $r\in\mathbb{Z}_{\geq 0}$, and $r_i+r_j\geq r_k$ for every permutation (i,j,k) of $(1,2,3)$. Put

$$\mathscr{L}_{[\mathbf{r}]}=\mathscr{L}_{1,N(m),r_1}(\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathscr{L}_{1,N(m),r_2}(\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathscr{L}_{1,N(m),r_3}(\mathbb{Z}_p),$$

and define

$$\kappa^{(1)}_{m,\mathbf{r}}\in H^1\left(\mathbb{Q},H^3_{\text{ét}}(Y(m)^3_{\overline{\mathbb{Q}}},\mathscr{L}_{[\mathbf{r}]})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(2-r)\right)$$

to be the class $\kappa_{N(m),\mathbf{r}}=\mathtt{s}_{\mathbf{r}*}\circ\mathtt{HS}\circ d_*(\mathtt{Det}^{\mathbf{r}}_{N(m)})$ constructed as in [BSV22, Section 3] for the modular curve $Y(m)$.

Lemma 4.2. *Let m be a positive integer, and let q be a prime number. Assume that both m and q are coprime to p and N. Then*

$$\begin{aligned}
&(\pi_2,\pi_1,\pi_1)_*\kappa^{(1)}_{mq,\mathbf{r}}=(T_q,1,1)\kappa^{(1)}_{m,\mathbf{r}};\quad (\pi_1,\pi_2,\pi_2)_*\kappa^{(1)}_{mq,\mathbf{r}}=q^{r-r_1}(T'_q,1,1)\kappa^{(1)}_{m,\mathbf{r}};\\
&(\pi_1,\pi_2,\pi_1)_*\kappa^{(1)}_{mq,\mathbf{r}}=(1,T_q,1)\kappa^{(1)}_{m,\mathbf{r}};\quad (\pi_2,\pi_1,\pi_2)_*\kappa^{(1)}_{mq,\mathbf{r}}=q^{r-r_2}(1,T'_q,1)\kappa^{(1)}_{m,\mathbf{r}};\\
&(\pi_1,\pi_1,\pi_2)_*\kappa^{(1)}_{mq,\mathbf{r}}=(1,1,T_q)\kappa^{(1)}_{m,\mathbf{r}};\quad (\pi_2,\pi_2,\pi_1)_*\kappa^{(1)}_{mq,\mathbf{r}}=q^{r-r_3}(1,1,T'_q)\kappa^{(1)}_{m,\mathbf{r}}.
\end{aligned}$$

If q is coprime to m, we also have

$$(\pi_1,\pi_1,\pi_1)_*\kappa^{(1)}_{mq,\mathbf{r}}=(q+1)\kappa^{(1)}_{m,\mathbf{r}};\quad (\pi_2,\pi_2,\pi_2)_*\kappa^{(1)}_{mq,\mathbf{r}}=(q+1)q^r\kappa^{(1)}_{m,\mathbf{r}}.$$

Proof. The same argument proving equations (174) and (176) in [BSV22] yields these identities, adding the prime q to the level rather than the prime p. □

We next consider the following 'asymmetric' diagonal classes.

Definition 4.3. For each squarefree positive integer n coprime to p and N, let

$$\kappa^{(2)}_{n,\mathbf{r}}=n^{r_2}(1,1,\langle n\rangle')(1,\pi_1,\pi_2)_*\kappa^{(1)}_{n^2,\mathbf{r}}\in H^1\left(\mathbb{Q},H^3_{\text{ét}}(Y(n^2)_{\overline{\mathbb{Q}}}\times Y(1)^2_{\overline{\mathbb{Q}}},\mathscr{L}_{[\mathbf{r}]})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(2-r)\right),$$

where $\pi_1,\pi_2:Y(n^2)\to Y(1)$ are the degeneracy maps in (2.2).

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Lemma 4.4. *Let n be as above, and let q be a rational prime coprime to p, N and n. Then*

$$\begin{aligned}
(\pi_{11},1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= \left\{q^{r_2}(1,1,T_qT_q') - (q+1)q^{r_2+r_3}(1,1,1)\right\}\kappa^{(2)}_{n,\mathbf{r}},\\
(\pi_{21},1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= \left\{q^{r}(1,T_q',T_q') - q^{r_2+r_3}(T_q',\langle q\rangle',\langle q\rangle')\right\}\kappa^{(2)}_{n,\mathbf{r}},\\
(\pi_{22},1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= \left\{q^{r_1+r_3}(1,T_q'^2,\langle q\rangle') - (q+1)q^{2r}(1,\langle q\rangle',\langle q\rangle')\right\}\kappa^{(2)}_{n,\mathbf{r}},
\end{aligned}$$

where $\pi_{ij} : Y(n^2q^2) \to Y(n^2)$ denotes the composite map

$$Y(n^2q^2) \xrightarrow{\pi_i} Y(n^2q) \xrightarrow{\pi_j} Y(n^2).$$

Proof. To better distinguish between the degeneracy maps π_i for different levels, in this proof, we use ϖ_i to denote the map π_i descending the level by q, so that $\varpi_j \circ \varpi_i$ is the degeneracy map π_{ij} in the statement of the lemma. Thus, we find

$$\begin{aligned}
(\varpi_1,1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= n^{r_2}q^{r_2}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_1,\varpi_1,\varpi_2)_*\kappa^{(1)}_{n^2q^2,\mathbf{r}}\\
&= n^{r_2}q^{r_2}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(1,1,T_q)\kappa^{(1)}_{n^2q,\mathbf{r}},
\end{aligned}$$

using Lemma 4.2 for the second equality; and similarly,

$$\begin{aligned}
(\varpi_2,1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= n^{r_2}q^{r_2}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_2,\varpi_1,\varpi_2)_*\kappa^{(1)}_{n^2q^2,\mathbf{r}}\\
&= n^{r_2}q^{r}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(1,T_q',1)\kappa^{(1)}_{n^2q,\mathbf{r}}.
\end{aligned}$$

Descending the level again by q, this gives

$$\begin{aligned}
(\pi_{11},1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= n^{r_2}q^{r_2}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_1,\varpi_1,\varpi_2)_*(1,1,T_q)\kappa^{(1)}_{n^2q,\mathbf{r}}\\
&= n^{r_2}q^{r_2}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_{1*},\varpi_{1*},T_q\varpi_{2*} - q^{r_3}\langle q\rangle\varpi_{1*})\kappa^{(1)}_{n^2q,\mathbf{r}}\\
&= n^{r_2}q^{r_2}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*\left\{(1,1,T_q^2) - (q+1)q^{r_3}(1,1,\langle q\rangle)\right\}\kappa^{(1)}_{n^2,\mathbf{r}}\\
&= q^{r_2}\left\{(1,1,T_qT_q') - (q+1)q^{r_3}(1,1,1)\right\}n^{r_2}(1,1,\langle n\rangle')(1,\pi_1,\pi_2)_*\kappa^{(1)}_{n^2,\mathbf{r}}\\
&= \left\{q^{r_2}(1,1,T_qT_q') - (q+1)q^{r_2+r_3}(1,1,1)\right\}\kappa^{(2)}_{n,\mathbf{r}},
\end{aligned}$$

and similarly

$$\begin{aligned}
(\pi_{21},1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= n^{r_2}q^{r}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_1,\varpi_1,\varpi_2)_*(1,T_q',1)\kappa^{(1)}_{n^2q,\mathbf{r}}\\
&= n^{r_2}q^{r}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_{1*},T_q'\varpi_{1*} - \langle q\rangle'\varpi_{2*},\varpi_{2*})\kappa^{(1)}_{n^2q,\mathbf{r}}\\
&= n^{r_2}q^{r}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*\left\{(1,T_q',T_q) - q^{r-r_1}(T_q',\langle q\rangle',1)\right\}\kappa^{(1)}_{n^2,\mathbf{r}}\\
&= q^{r}\left\{(1,T_q',T_q') - q^{r-r_1}(T_q',\langle q\rangle',\langle q\rangle')\right\}n^{r_2}(1,1,\langle n\rangle')(1,\pi_1,\pi_2)_*\kappa^{(1)}_{n^2,\mathbf{r}}\\
&= \left\{q^{r}(1,T_q',T_q') - q^{r_2+r_3}(T_q',\langle q\rangle',\langle q\rangle')\right\}\kappa^{(2)}_{n,\mathbf{r}},
\end{aligned}$$

and

$$\begin{aligned}
(\pi_{22},1,1)_*\kappa^{(2)}_{nq,\mathbf{r}} &= n^{r_2}q^{r}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_2,\varpi_1,\varpi_2)_*(1,T_q',1)\kappa^{(1)}_{n^2q,\mathbf{r}}\\
&= n^{r_2}q^{r}(1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_*(\varpi_{2*},T_q'\varpi_{1*} - \langle q\rangle'\varpi_{2*},\varpi_{2*})\kappa^{(1)}_{n^2q,\mathbf{r}}
\end{aligned}$$

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$$\begin{aligned}
&= n^{r_2} q^r (1,1,\langle nq\rangle')(1,\pi_1,\pi_2)_* \left\{ q^{r-r_2}(1,T_q'^2,1) - (q+1)q^r(1,\langle q\rangle',1) \right\} \kappa^{(1)}_{n^2,\mathbf{r}} \\
&= q^r \left\{ q^{r-r_2}(1,T_q'^2,\langle q\rangle') - (q+1)q^r(1,\langle q\rangle',\langle q\rangle') \right\} n^{r_2}(1,1,\langle n\rangle')(1,\pi_1,\pi_2)_* \kappa^{(1)}_{n^2,\mathbf{r}} \\
&= \left\{ q^{r_1+r_3}(1,T_q'^2,\langle q\rangle') - (q+1)q^{2r}(1,\langle q\rangle',\langle q\rangle') \right\} \kappa^{(2)}_{n,\mathbf{r}},
\end{aligned}$$

hence, the result. □

Projection of the classes $\kappa^{(2)}_{n,\mathbf{r}}$ to the $(1,1,1)$-component in the Künneth decomposition yields classes $\kappa^{(3)}_{n,\mathbf{r}}$ in

$$H^1\left(\mathbb{Q}, H^1_{\mathrm{et}}(Y(n^2)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_1}(1)) \otimes H^1_{\mathrm{et}}(Y(1)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_2}(1)) \otimes H^1_{\mathrm{et}}(Y(1)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_3}(1)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)\right).$$

Now set $(r_1,r_2,r_3) = (0,l-2,m-2)$. Fix test vectors

$$\breve{f} \in S_k(N,\chi\varepsilon_K)[\theta_\psi], \quad \breve{g} \in S_l(N,\chi_g)[g], \quad \breve{h} \in S_m(N,\chi_h)[h].$$

These test vectors determine maps

$$\begin{aligned}
&H^1_{\mathrm{et}}(Y(n^2)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \to H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \\
&H^1_{\mathrm{et}}(Y(1)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_2}(1)) \to H^1_{\mathrm{et}}(Y_1(N_g)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_2}(1)) \\
&H^1_{\mathrm{et}}(Y(1)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_3}(1)) \to H^1_{\mathrm{et}}(Y_1(N_h)_{\overline{\mathbb{Q}}}, \mathscr{L}_{r_3}(1))
\end{aligned}$$

which we use to project the classes $\kappa^{(3)}_{n,\mathbf{r}}$ to classes $\kappa^{(3)}_{n,\psi gh}$ in

$$H^1(\mathbb{Q}, \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1,N_\psi(n^2))\otimes\mathbb{Z}_p,\phi_n)} H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-r)).$$

Let

$$T^\psi_{g,h} = T_g \otimes_{\mathcal{O}} T_h(\psi^{-1}_{\mathfrak{P}})(-1-r), \quad V^\psi_{g,h} = T^\psi_{g,h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Using the isomorphisms

$$\nu_n : \mathcal{O}[R_{\mathfrak{f},n}] \otimes_{(\mathbb{T}'(1,N_\psi(n^2))\otimes\mathbb{Z}_p,\phi_n)} H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \mathrm{Ind}^{\mathbb{Q}}_{K(\mathfrak{f})[n]} \mathcal{O}(\psi^{-1}_{\mathfrak{P}})$$

of Corollary 3.6, and taking the projection of both sides via the quotient map $\mathcal{O}[R_{\mathfrak{f},n}] \to \mathcal{O}[R_n]$, we obtain new isomorphisms

$$\tilde{\nu}_n : \mathcal{O}[R_n] \otimes_{(\mathbb{T}'(1,N_\psi(n^2))\otimes\mathbb{Z}_p,\phi_n)} H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1)) \xrightarrow{\simeq} \mathrm{Ind}^{\mathbb{Q}}_{K[n]} \mathcal{O}(\psi^{-1}_{\mathfrak{P}}),$$

so that applying the corresponding projection map to the classes $\kappa^{(3)}_{n,\psi gh}$ and using Shapiro's lemma, we obtain classes

$$\tilde{\kappa}_{\psi,g,h,n} \in H^1(K[n], V^\psi_{g,h}).$$

Proposition 4.5. *Let n be as above, and let q be a rational prime coprime to p, N and n.*

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(i) *If q splits in K as $(q) = \mathfrak{q}\overline{\mathfrak{q}}$, then*

$$\begin{aligned}\mathrm{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq}) = q^{l+m-4}\Bigg\{&\chi_g(q)\chi_h(q)q\left(\frac{\psi(\mathfrak{q})}{q}\mathrm{Fr}_{\mathfrak{q}}^{-1}\right)^2 - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}}\left(\frac{\psi(\mathfrak{q})}{q}\mathrm{Fr}_{\mathfrak{q}}^{-1}\right)\\ &+ \frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-1}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-2}} - \frac{q^2+1}{q}\\ &- \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}}\left(\frac{\psi(\overline{\mathfrak{q}})}{q}\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1}\right) + \chi_g(q)\chi_h(q)q\left(\frac{\psi(\overline{\mathfrak{q}})}{q}\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1}\right)^2\Bigg\}\tilde{\kappa}_{\psi,g,h,n}.\end{aligned}$$

(ii) *If q is inert in K, then*

$$\mathrm{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq}) = q^{l+m-4}\left\{\frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-1}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-2}} - \frac{(q+1)^2}{q}\right\}\tilde{\kappa}_{\psi,g,h,n}.$$

Proof. We have the commutative diagram

$$\begin{array}{ccc} H^1(K[nq],V_{g,h}^{\psi}) & \xrightarrow{\cong} & H^1(\mathbb{Q},\mathrm{Ind}_{K[nq]}^{\mathbb{Q}}\mathcal{O}(\psi_{\mathfrak{P}}^{-1})\otimes_{\mathcal{O}}T_g\otimes_{\mathcal{O}}T_h\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-1-r)) \\ \downarrow{\scriptstyle \mathrm{cor}_{K[nq]/K[n]}} & & \downarrow \\ H^1(K[n],V_{g,h}^{\psi}) & \xrightarrow{\cong} & H^1(\mathbb{Q},\mathrm{Ind}_{K[n]}^{\mathbb{Q}}\mathcal{O}(\psi_{\mathfrak{P}}^{-1})\otimes_{\mathcal{O}}T_g\otimes_{\mathcal{O}}T_h\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-1-r)), \end{array}$$

where the horizontal isomorphisms are given by Shapiro's lemma and the right vertical arrow comes from the natural norm map between induced representations. Using the isomorphisms $\tilde{\nu}_n$ above, the vertical arrows in the previous diagram correspond to the map

$$\begin{array}{c} H^1(\mathbb{Q},\mathcal{O}[R_{nq}]\otimes_{\phi_{nq}}H^1_{\mathrm{et}}(Y(1,N_\psi(n^2q^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))\otimes_{\mathcal{O}}T_g\otimes_{\mathcal{O}}T_h\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-1-r)) \\ \downarrow{\scriptstyle \mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},nq}\otimes\mathrm{Id}\otimes\mathrm{Id}} \\ H^1(\mathbb{Q},\mathcal{O}[R_n]\otimes_{\phi_n}H^1_{\mathrm{et}}(Y(1,N_\psi(n^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))\otimes_{\mathcal{O}}T_g\otimes_{\mathcal{O}}T_h\otimes_{\mathbb{Z}_p}\mathbb{Q}_p(-1-r)). \end{array}$$

If q splits in K, the map $\mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},nq}$ is given by

$$\mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},nq} = \pi_{11*} - \left(\frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q} + \frac{\psi(\overline{\mathfrak{q}})[\overline{\mathfrak{q}}]}{q}\right)\pi_{21*} + \frac{\chi(q)}{q}\pi_{22*},$$

using the notations introduced in Lemma 4.4 for the degeneracy maps, and from the relations in that lemma, we find

$$\begin{aligned}\mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},nq}(\tilde{\kappa}_{\psi,g,h,nq}) = \Bigg[&1\otimes\left\{q^{r_2}(1,1,T_qT_q') - (q+1)q^{r_2+r_3}(1,1,1)\right\}\\ &- \left(\frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q} + \frac{\psi(\overline{\mathfrak{q}})[\overline{\mathfrak{q}}]}{q}\right)\otimes\left\{q^{r}(1,T_q',T_q') - q^{r_2+r_3}(T_q',\langle q\rangle',\langle g\rangle')\right\}\\ &+ \frac{\chi(q)}{q}\otimes\left\{q^{r_1+r_3}(1,T_q'^2,\langle q\rangle') - (q+1)q^{2r}(1,\langle q\rangle',\langle q\rangle')\right\}\Bigg]\tilde{\kappa}_{\psi,g,h,n}\end{aligned}$$

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$$= \Bigg[\chi_h(q)^{-1} a_q(h)^2 q^{r_2} + (q+1) q^{r_2+r_3} - \left(\frac{\psi(\mathfrak{q})[\mathfrak{q}]}{q} + \frac{\psi(\overline{\mathfrak{q}})[\overline{\mathfrak{q}}]}{q} \right) \left\{ a_q(g) a_q(h) q^r - \chi_g(q)\chi_h(q) q^{r_2+r_3} (\psi(\mathfrak{q})[\mathfrak{q}] + \psi(\overline{\mathfrak{q}})[\overline{\mathfrak{q}}]) \right\} + \frac{\chi(q)}{q} \left\{ \chi_h(q) a_q(g)^2 q^{r_1+r_3} - \chi_g(q)\chi_h(q)(q+1) q^{2r} \right\} \Bigg] \tilde{\kappa}_{\psi,g,h,n}.$$

This implies the result in this case. When q is inert in K, we have

$$\mathcal{N}_{\mathfrak{f},n}^{\mathfrak{f},nq} = \pi_{11*} - \frac{\chi(q)}{q} \pi_{22*},$$

and the result in this case follows by a very similar computation that we leave to the reader. □

In particular, restricting to positive integers n as above that are divisible only by primes q which split in K, Proposition 4.5 yields the following result (note that since in this section we assume ψ has infinity type $(-1,0)$, the balanced condition forces $l = m$).

Theorem 4.6. *Suppose the weights of g,h are $l = m$. Let $\mathcal{S}$ be the set of squarefree products of primes q which split in K and are coprime to p and N. Assume that $H^1(K[n], T_{g,h}^{\psi})$ is torsion-free for every $n \in \mathcal{S}$. There exists a collection of classes*

$$\left\{ \kappa_{\psi,g,h,n} \in H^1(K[n], T_{g,h}^{\psi}) : \ n \in \mathcal{S} \right\},$$

such that whenever $n, nq \in \mathcal{S}$ with q a prime, we have

$$\mathrm{cor}_{K[nq]/K[n]}(\kappa_{\psi,g,h,nq}) = P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\, \kappa_{\psi,g,h,n},$$

where $\mathfrak{q}$ is any of the primes of K above q, and $P_{\mathfrak{q}}(X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | (V_{g,h}^{\psi})^{\vee}(1))$.

Proof. We begin by noting that the only possible denominators of the classes $\tilde{\kappa}_{\psi,g,h,n}$ are divisors of $(l-2)!(m-2)!$ (as follows from [BSV22, Remark 3.3]), so after multiplying them by a suitable power of p, they all have coefficients in $T_{g,h}^{\psi}$.

Now given a prime $q \in \mathcal{S}$, we note that for any prime v of K above q, we have

$$P_v(X) = 1 - \frac{a_q(g)a_q(h)}{q^{(l+m-2)/2}} \frac{\psi(v)}{q} X + \left(\frac{\chi_g(q) a_q(h)^2}{q^{m-1}} + \frac{\chi_h(q) a_q(g)^2}{q^{l-1}} - 2\chi_g(q)\chi_h(q) \right) \frac{\psi(v)^2}{q^2} X^2 - \frac{\chi_g(q)\chi_h(q) a_q(g) a_q(h)}{q^{(l+m-2)/2}} \frac{\psi(v)^3}{q^3} X^3 + \chi_g(q)^2 \chi_h(q)^2 \frac{\psi(v)^4}{q^4} X^4.$$

Writing $(q) = \mathfrak{q}\overline{\mathfrak{q}}$ and using that $\psi(\mathfrak{q})\psi(\overline{\mathfrak{q}}) = \chi(q)q$ and $\chi_g(q)\chi_h(q)\chi(q) = 1$, we therefore find the congruences

$$\begin{aligned} P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \chi_g(q)\chi_h(q)\psi(\overline{\mathfrak{q}})^2 \mathrm{Fr}_{\overline{\mathfrak{q}}}^{-2} &\equiv P_{\overline{\mathfrak{q}}}(\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1}) \chi_g(q)\chi_h(q)\psi(\mathfrak{q})^2 \mathrm{Fr}_{\mathfrak{q}}^{-2} (\mathrm{mod}\ q-1) \\ &\equiv \chi_g(q)\chi_h(q)\psi(\mathfrak{q})^2 \mathrm{Fr}_{\mathfrak{q}}^{-2} - a_q(g)a_q(h)\psi(\mathfrak{q})\mathrm{Fr}_{\mathfrak{q}}^{-1} \\ &\quad + \chi_g(q)^{-1} a_q(g)^2 + \chi_h(q)^{-1} a_q(h)^2 - 2 \\ &\quad - a_q(g)a_q(h)\psi(\overline{\mathfrak{q}})\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1} + \chi_g(q)\chi_h(q)\psi(\overline{\mathfrak{q}})^2 \mathrm{Fr}_{\overline{\mathfrak{q}}}^{-2} (\mathrm{mod}\ q-1) \end{aligned}$$

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as endomorphisms of $H^1(K[n],T^{\psi}_{g,h})$. Since these expressions agree modulo $q-1$ with the factor appearing in the norm relation of Proposition 4.5(i), together with [Rub00, Lemmas 9.6.1 and 9.6.3], the result follows. □

Remark 4.7. The condition that $H^1(K[n],T^{\psi}_{g,h})$ is torsion-free for every $n \in \mathcal{S}$ holds, for example, under the assumptions in Lemma 8.9 below. Indeed, since $\mathrm{SL}_2(\mathbb{Z}_p)\times\mathrm{SL}_2(\mathbb{Z}_p)$ has no proper normal subgroups of finite p-power index, it follows from this lemma that the residual $G_{K[n]}$-representation attached to $T^{\psi}_{g,h}$ is absolutely irreducible for every $n \in \mathcal{S}$, so that $H^0(K[n],V^{\psi}_{g,h}/T^{\psi}_{g,h})$ is trivial for every $n \in \mathcal{S}$ and the condition follows.

Remark 4.8. In the inert case, writing $\mathfrak{q} = (q)$, we have

$$\begin{aligned}
&P_{\mathfrak{q}}(X) = \det(1-\mathrm{Fr}_{\mathfrak{q}}^{-1}X|(T^{\psi}_{g,h})^{\vee}(1))\\
&= 1-\left(\frac{a_q(g)^2}{q^{l-1}}-2\chi_g(q)\right)\left(\frac{a_h(q)^2}{q^{m-1}}-2\chi_h(q)\right)\frac{\psi(\mathfrak{q})}{q^2}X\\
&+\left(\chi_h(q)^2\left(\frac{a_q(g)^2}{q^{l-1}}-2\chi_g(q)\right)^2+\chi_g(q)^2\left(\frac{a_q(h)^2}{q^{m-1}}-2\chi_h(q)\right)^2-2\chi_g(q)^2\chi_h(q)^2\right)\frac{\psi(\mathfrak{q})^2}{q^4}X^2\\
&-\chi_g(q)^2\chi_h(q)^2\left(\frac{a_q(g)^2}{q^{l-1}}-2\chi_g(q)\right)\left(\frac{a_q(h)^2}{q^{m-1}}-2\chi_h(q)\right)\frac{\psi(\mathfrak{q})^3}{q^6}X^3+\chi_g(q)^4\chi_h(q)^4\frac{\psi(\mathfrak{q})^4}{q^8}X^4,
\end{aligned}$$

and similarly, as in the proof of Theorem 4.6, we find the congruence

$$\begin{aligned}
P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \equiv{}& \chi_g(q)^{-2}a_q(g)^4+\chi_h(q)^{-2}a_q(h)^4+2\chi_g(q)^{-1}\chi_h(q)^{-1}a_q(g)^2a_q(h)^2q\\
&-4\frac{\chi_g(q)^{-1}a_q(g)^2(q+1)}{q^{l-1}}-4\frac{\chi_h(q)^{-1}a_q(h)^2(q+1)}{q^{m-1}}+8(q+1)(\mathrm{mod}\ q^2-1)
\end{aligned}$$

as endomorphisms of $H^1(K[n],T^{\psi}_{g,h})$. Similarly, as above, this expression agrees modulo q^2-1 with the square of the Euler factor appearing in the norm relation of Proposition 4.5(ii).

Now assume that $(p)=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $f=\theta_{\psi}$ be the theta series associated to ψ. Assume also that both g and h are ordinary at p. Then, for $\phi \in \{f,g,h\}$, the $G_{\mathbb{Q}_p}$-representation V_{ϕ} admits a filtration

$$0 \longrightarrow V_{\phi}^{+} \longrightarrow V_{\phi} \longrightarrow V_{\phi}^{-} \longrightarrow 0,$$

where $V_{\phi}^{\pm}$ is one-dimensional and V_{ϕ}^{-} is unramified with Fr_p acting as multiplication by α_{ϕ}, the unit root of the Hecke polynomial of ϕ at p. Letting $V_{fgh}=V_f\otimes V_g\otimes V_h(-1-r)$, we can therefore consider the $G_{\mathbb{Q}_p}$-subrepresentation

$$\mathscr{F}^2V_{fgh}=(V_f\otimes V_g^{+}\otimes V_h^{+}+V_f^{+}\otimes V_g\otimes V_h^{+}+V_f^{+}\otimes V_g^{+}\otimes V_h)(-1-r)$$

and define the *balanced* local condition $H^1_{\mathrm{bal}}(\mathbb{Q}_p,V_{fgh})\subset H^1(\mathbb{Q}_p,V_{fgh})$ to be the image of the natural map $H^1(\mathbb{Q}_p,\mathscr{F}^2V_{fgh})\to H^1(\mathbb{Q}_p,V_{fgh})$.

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Setting

$$\mathcal{F}_{\mathfrak{p}}^{+}(V_{g,h}^{\psi}) = (V_g^{+} \otimes V_h + V_g \otimes V_h^{+})(\psi_{\mathfrak{P}}^{-1})(-1-r), \quad \mathcal{F}_{\overline{\mathfrak{p}}}^{+}(V_{g,h}^{\psi}) = (V_g^{+} \otimes V_h^{+})(\psi_{\mathfrak{P}}^{-1})(-1-r), \tag{4.1}$$

one readily checks that under the isomorphism $H^1(\mathbb{Q},V_{fgh}) \cong H^1(K,V_{g,h}^{\psi})$ of Shapiro's lemma, the balanced local condition $H^1_{\mathrm{bal}}(\mathbb{Q}_p,V_{fgh})$ corresponds to the natural image of

$$\bigoplus_{v|p} H^1(K_v,\mathcal{F}_v^{+}(V_{g,h}^{\psi})) \longrightarrow \bigoplus_{v|p} H^1(K_v,V_{g,h}^{\psi}).$$

This motivates the following definition. Let L/K be a finite extension, and for every finite prime v of L, put

$$H^1_{\mathrm{bal}}(L_v,V_{g,h}^{\psi}) = \begin{cases} \mathrm{im}\big(H^1(L_v,\mathcal{F}_v^{+}(V_{g,h}^{\psi})) \to H^1(L_v,V_{g,h}^{\psi})\big) & \text{if } v \mid p, \\ \ker\big(H^1(L_v,V_{g,h}^{\psi}) \to H^1(L_v^{\mathrm{nr}},V_{g,h}^{\psi})\big) & \text{if } v \nmid p, \end{cases}$$

where L_v^{nr} is the maximal unramified extension of L_v. We then let $H^1_{\mathrm{bal}}(L_v,T_{g,h}^{\psi})$ be the inverse image of $H^1_{\mathrm{bal}}(L_v,V_{g,h}^{\psi})$ under the natural map $H^1(L_v,T_{g,h}^{\psi}) \to H^1(L_v,V_{g,h}^{\psi})$, and let $\mathrm{Sel}_{\mathrm{bal}}(L,T_{g,h}^{\psi}) \subset H^1(L,T_{g,h}^{\psi})$ be the Greenberg Selmer group cut out by these local conditions (note that this is a special case of the more general construction discussed in Section 8.1).

Proposition 4.9. *For every $n \in \mathcal{S}$, the class $\kappa_{\psi,g,h,n}$ lies in the group $\mathrm{Sel}_{\mathrm{bal}}(K[n],T_{g,h}^{\psi})$.*

Proof. Fix $n \in \mathcal{S}$ and v a finite prime of $K[n]$. If $v \nmid p$, then it follows from the Weil conjectures that $V_{g,h}^{\psi}$ is pure of weight -1, and hence

$$H^1_{\mathrm{ur}}(K[n]_v,V_{g,h}^{\psi}) := \ker\big(H^1(K[n]_v,V_{g,h}^{\psi}) \to H^1(K[n]_v^{\mathrm{nr}},V_{g,h}^{\psi})\big) = 0. \tag{4.2}$$

By [Rub00, Corollary 1.3.3(i)] and local Tate duality (using the fact that the G_K-representation $V_{g,h}^{\psi}$ is conjugate self-dual), it follows that

$$H^0(K[n]_v,V_{g,h}^{\psi}) = H^2(K[n]_{\overline{v}},V_{g,h}^{\psi}) = 0.$$

Repeating the argument with the roles of v and $\overline{v}$ reversed, from (4.2) and [Rub00, Corollary 1.3.3(ii)], we conclude that

$$H^1(K[n]_v,V_{g,h}^{\psi}) = H^1_{\mathrm{ur}}(K[n]_v,V_{g,h}^{\psi}) = 0,$$

and so the inclusion $\mathrm{res}_v(\kappa_{\psi,g,h,n}) \in H^1_{\mathrm{bal}}(K[n]_v,T_{g,h}^{\psi})$ is automatic.

Now suppose $v \mid p$. As noted in [BSV22, Proposition 3.2], it follows from the results of [NN16] that the classes $\kappa_{m,\mathbf{r}}^{(1)}$ are geometric at p, and therefore the class $\mathrm{res}_v(\kappa_{\psi,g,h,n}) \in H^1(K[n]_v,T_{g,h}^{\psi})$ lands in the inverse image of

$$H^1_{\mathrm{geo}}(K[n]_v,V_{g,h}^{\psi}) = \ker\big(H^1(K[n]_v,V_{g,h}^{\psi}) \to H^1(K[n]_v,V_{g,h}^{\psi} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})\big)$$

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under the natural map $H^1(K[n]_v, T^{\psi}_{g,h}) \to H^1(K[n]_v, V^{\psi}_{g,h})$. Since $H^1_{\mathrm{geo}}(K[n]_v, V^{\psi}_{g,h})$ agrees with the Bloch–Kato finite subspace $H^1_{\mathrm{fin}}(K[n]_v, V^{\psi}_{g,h})$ (see [Nek93, Proposition 1.24(2)]), and the latter agrees with $H^1_{\mathrm{bal}}(K[n]_v, V^{\psi}_{g,h})$ (see Lemma 9.1 below), the result follows. □

5. Hida families and Galois representations

In the next section, we will prove that the classes $\kappa_{\psi,g,h,n}$ of Theorem 4.6 extend along the anticyclotomic $\mathbb{Z}_p$-extension of K, that is, they are anticyclotomic universal norms and explain the construction of $\kappa_{\psi,g,h,n}$ for more general weights. In this section, we collect the background results we shall need, closely following the treatment in [BSV22].

5.1. Hida families

Let $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$, and let

$$\mathcal{W} = \mathrm{Spf}(\Lambda)$$

be the weight space. Then, for any extension E of $\mathbb{Q}_p$, we have $\mathcal{W}(E) = \mathrm{Hom}_{\mathrm{cont}}(1+p\mathbb{Z}_p, E^{\times})$. Points of the form $\nu_{r,\epsilon}(n) = \epsilon(n)n^r$, where r is a nonnegative integer and ϵ is a finite order character, will be called *arithmetic*. We refer to $k = r+2$ as the *weight* of $\nu_{r,\epsilon}$. Arithmetic points of the form $\nu_r = \nu_{r,1}$ will be called *classical*.

More generally, let $\mathcal{R}$ be a normal domain finite flat over Λ, and let $\mathcal{W}_{\mathcal{R}} = \mathrm{Spf}(\mathcal{R})$. Then, a point $x \in \mathcal{W}_{\mathcal{R}}(\overline{\mathbb{Q}}_p)$ will be called *arithmetic* if it lies above an arithmetic point $\nu_{r,\epsilon}$ of $\mathcal{W}(\overline{\mathbb{Q}}_p)$, and *classical* if it lies above a classical point ν_r of $\mathcal{W}(\overline{\mathbb{Q}}_p)$. Again, we refer to $k = r+2$ as the *weight* of x.

Let M be a positive integer coprime to p. A *Hida family* of tame level M and character $\chi : (\mathbb{Z}/Mp\mathbb{Z})^{\times} \to \overline{\mathbb{Q}}_p^{\times}$ is a formal q-expansion

$$\mathbf{f} = \sum_{n \geq 1} a_n(\mathbf{f}) q^n \in \Lambda_{\mathbf{f}}[[q]],$$

where $\Lambda_{\mathbf{f}}$ is a normal domain finite flat over Λ, such that, for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ lying over some $\nu_{r,\epsilon}$, the corresponding specialisation is a p-ordinary eigenform $f_x \in S_k(Mp^s, \chi\epsilon\omega^{-r})$. As above, we have denoted by k the weight of x, and we can take $s = \max\{1, \mathrm{ord}_p(\mathrm{cond}(\epsilon))\}$. We say that a Hida family $\mathbf{f}$ is *primitive* if the specialisations f_x at arithmetic points x are p-stabilised newforms. We say that it is *normalised* if $a_1(\mathbf{f}) = 1$.

Let $\mathbf{f}$ be a normalised primitive Hida family of tame level M. For each arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$, let $\mathbf{f}_x$ denote the specialisation of $\mathbf{f}$ at x, and let f_x be the corresponding newform. There exists a locally free rank-two $\Lambda_{\mathbf{f}}$-module $\mathbb{V}_{\mathbf{f}}$ equipped with a continuous action of $G_{\mathbb{Q}}$, such that, for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$, the corresponding specialisation $\mathbb{V}_{\mathbf{f}} \otimes_{\Lambda_{\mathbf{f}},x} \overline{\mathbb{Q}}_p$ recovers the $G_{\mathbb{Q}}$-representation V_{f_x} attached to f_x. In particular, the representation $\mathbb{V}_{\mathbf{f}}$ is unramified at any prime $q \nmid Mp$ and $\mathrm{Tr}(\mathrm{Fr}_q) = a_q(\mathbf{f})$. We refer to $\mathbb{V}_{\mathbf{f}}$ as the *big Galois representation attached to* $\mathbf{f}$. If for some (equivalently all) arithmetic point $x_0 \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ the $G_{\mathbb{Q}}$-representation $T_{f_{x_0}}$ attached to f_{x_0} is residually irreducible, then $\mathbb{V}_{\mathbf{f}}$ is a free $\Lambda_{\mathbf{f}}$-module.

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5.2. Continuous functions and distributions

Define the semigroups

$$\Sigma_0(p) = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \quad \text{and} \quad \Sigma_0'(p) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

The sets $\mathsf{T} = \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $\mathsf{T}' = p\mathbb{Z}_p \times \mathbb{Z}_p^\times$ bear a right action of $\Sigma_0(p)$ and $\Sigma_0'(p)$, respectively.

Let ν be a character of $\mathbb{Z}_p^\times$ taking values in a finite extension E of $\mathbb{Q}_p$. Let $\mathcal{O}$ be the ring of integers of E and denote by $\mathfrak{m}$ its maximal ideal. Let $\mathrm{Cont}(\mathbb{Z}_p,\mathcal{O})$ denote the module of continuous functions on $\mathbb{Z}_p$ with values in $\mathcal{O}$. Define $\mathcal{O}$-modules

$$\mathcal{A}_\nu = \left\{ f : \mathsf{T} \to \mathcal{O} \mid f(1,z) \in \mathrm{Cont}(\mathbb{Z}_p,\mathcal{O}) \text{ and } f(a\cdot t) = \nu(a)\cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times, t \in \mathsf{T} \right\},$$

$$\mathcal{A}_\nu' = \left\{ f : \mathsf{T}' \to \mathcal{O} \mid f(pz,1) \in \mathrm{Cont}(\mathbb{Z}_p,\mathcal{O}) \text{ and } f(a\cdot t) = \nu(a)\cdot f(t) \text{ for all } a \in \mathbb{Z}_p^\times, t \in \mathsf{T}' \right\}$$

equipped with the $\mathfrak{m}$-adic topology, and $\mathcal{O}$-modules

$$\mathcal{D}_\nu = \mathrm{Hom}_{\mathrm{cont},\mathcal{O}}(\mathcal{A}_\nu,\mathcal{O}), \quad \mathcal{D}_\nu' = \mathrm{Hom}_{\mathrm{cont},\mathcal{O}}(\mathcal{A}_\nu',\mathcal{O})$$

equipped with the weak-$*$ topology. The right $\Sigma_0^{\cdot}(p)$-action on $\mathsf{T}^{\cdot}$ yields naturally a left $\Sigma_0^{\cdot}(p)$-action on $\mathcal{A}_\nu^{\cdot}$ and a right $\Sigma_0^{\cdot}(p)$-action on $\mathcal{D}_\nu^{\cdot}$.

5.3. Group cohomology and étale cohomology

Let N and m be coprime positive integers which are also coprime to p, let $Y = Y(1,N(pm))$, and let Γ be the corresponding modular group. Let $\mathcal{E} \to Y$ be the universal elliptic curve over Y, and denote by C_p the canonical cyclic p-subgroup. Let $\mathscr{T}$ be the relative p-adic Tate module of $\mathcal{E}$ over Y. Fix a geometric point $\eta : \mathrm{Spec}(\overline{\mathbb{Q}}) \to Y$, and choose an isomorphism $\mathscr{T}_\eta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, such that the Weil pairing on $\mathscr{T}_\eta$ corresponds to the natural determinant map on the right and the reduction modulo p of the element $(0,1)$ generates $C_{p,\eta}$.

Let $\mathcal{G} = \pi_1^{\mathrm{et}}(Y,\eta)$. The action of $\mathcal{G}$ on $\mathscr{T}$ yields an action of $\mathcal{G}$ on $\mathbb{Z}_p \oplus \mathbb{Z}_p$, and hence a continuous representation $\rho : \mathcal{G} \to \mathrm{GL}_2(\mathbb{Z}_p)$. More precisely, for any $g \in \mathcal{G}$,

$$g \cdot (a,b) = (a,b)\rho(g)^{-1}.$$

In fact, since the action of $\mathcal{G}$ preserves the canonical subgroup, we have a continuous representation $\rho : \mathcal{G} \to \Gamma_0(p\mathbb{Z}_p)$, where

$$\Gamma_0(p\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : p \mid c \right\}.$$

The anti-involution of $\mathrm{GL}_2(\mathbb{Z}_p)$ given by $\gamma \mapsto \gamma^\iota = \det(\gamma)\gamma^{-1}$ restricts to $\Gamma_0(p\mathbb{Z}_p)$ and allows us to think of this group as acting on the right or left as convenient.

Taking the stalk at η gives an equivalence of categories between the category $\mathbf{S}_f(Y_{\mathrm{et}})$ of locally constant constructible sheaves with finite stalk of p-power order at η and the category $\mathbf{M}_f(\mathcal{G})$ of finite $\mathcal{G}$-sets of p-power order. For any topological group G, define $\mathbf{M}_f(G)$ as we did for $\mathcal{G}$. Let $\mathbf{M}_{\mathrm{cont}}(G)$ be the category of G-modules which are filtred unions $\cup_{i\in I} M_i$ with $M_i \in \mathbf{M}_f(G)$, and let $\mathbf{M}(G) \subset \mathbf{M}_{\mathrm{cont}}(G)^{\mathbb{N}}$ be the category of inverse

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systems of objects in $\mathbf{M}_{\mathrm{cont}}(G)$. Define $\mathbf{S}(Y_{\mathrm{et}})$ similarly. Then, there is an equivalence of categories between $\mathbf{M}(\mathcal{G})$ and $\mathbf{S}(Y_{\mathrm{et}})$. Moreover, the representation ρ defined above yields a functor $\mathbf{M}(\Gamma_0(p\mathbb{Z}_p)) \to \mathbf{M}(\mathcal{G})$. Regarding this functor, we adopt the following criterion: if an object $\mathcal{F} \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$ is given as a left $\Gamma_0(p\mathbb{Z}_p)$-module, we define the left $\mathcal{G}$-action via the map $\rho : \mathcal{G} \to \Gamma_0(p\mathbb{Z}_p)$; if it is given as a right $\Gamma_0(p\mathbb{Z}_p)$-module, we define the left $\mathcal{G}$-action via the map $g \mapsto \rho(g)^{-1}$. Given an inverse system of sheaves $\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}_i)_{i\in\mathbb{N}} \in \mathbf{S}(Y_{\mathrm{et}})$, we use the notation $H^j_{\mathrm{et}}(Y,\boldsymbol{\mathcal{F}})$ for continuous étale cohomology as defined by Jannsen [Jan88], and write $\mathtt{H}^j_{\mathrm{et}}(Y,\boldsymbol{\mathcal{F}}) = \varprojlim_i H^j_{\mathrm{et}}(Y,\boldsymbol{\mathcal{F}}_i)$. There is a natural surjective morphism $H^j_{\mathrm{et}}(Y,\boldsymbol{\mathcal{F}}) \to \mathtt{H}^j_{\mathrm{et}}(Y,\boldsymbol{\mathcal{F}})$. The compactly supported cohomology groups $H^j_{\mathrm{et},c}(Y,\boldsymbol{\mathcal{F}})$ and $\mathtt{H}^j_{\mathrm{et},c}(Y,\boldsymbol{\mathcal{F}})$ are defined similarly.

There is an isomorphism $\pi_1^{\mathrm{et}}(Y_{\overline{\mathbb{Q}}},\eta) \cong \hat{\Gamma}$. Thus, if $\mathcal{F} \in \mathbf{M}_f(\mathcal{G})$ is a discrete $\mathcal{G}$-module and $\boldsymbol{\mathcal{F}}$ is the corresponding object in $\mathbf{S}_f(Y_{\mathrm{et}})$, there are natural isomorphisms

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) \cong H^1(\hat{\Gamma},\mathcal{F}) \cong H^1(\Gamma,\mathcal{F}). \tag{5.1}$$

Let $\mathcal{F} \in \mathbf{M}_f(\Gamma_0(p\mathbb{Z}_p))$ be a left $\Gamma_0(p\mathbb{Z}_p)$-module, and assume that the $\Gamma_0(p\mathbb{Z}_p)$-action on $\mathcal{F}$ extends to a left action of $\Sigma_0^{\cdot}(p)$. Let $S = \Sigma_0^{\cdot}(p) \cap \mathrm{GL}_2(\mathbb{Q})$. The pair (Γ,S) is then a Hecke pair in the sense of [AS86a, Section 1.1], and there is a covariant (left) action of the Hecke algebra $D(\Gamma,S)$ on $H^1(\Gamma,\mathcal{F})$. For each $g \in S$, let $T(g) = \Gamma g \Gamma$. Following [GS93, Section 1], we define, for each positive integer n, the Hecke operators

$$T_n = T\left(\begin{pmatrix} 1 & \\ & n \end{pmatrix}\right), \quad T'_n = T\left(\begin{pmatrix} n & \\ & 1 \end{pmatrix}\right).$$

Also, for each positive integer a coprime to p, let

$$[a]_p = T\left(\begin{pmatrix} a & \\ & a \end{pmatrix}\right), \quad [a]'_p = T\left(\begin{pmatrix} a & \\ & a \end{pmatrix}\right).$$

Finally, for each positive integer a coprime to N, choose β_a (respectively, β'_a) in $\Gamma_0(Npm)$ whose lower right entry is congruent to a (respectively, a^{-1}) modulo N, and let

$$[a]_N = T(\beta_a), \quad [a]'_N = T(\beta'_a).$$

The isomorphism (5.1) is compatible with Hecke actions in the following sense. To distinguish between different levels, we shall now write $\tilde{Y}(m)$ and $\tilde{\Gamma}(m)$ for the above Y and Γ, respectively. Let s be a positive integer. Choose, as above, a geometric point $\eta : \mathrm{Spec}(\overline{\mathbb{Q}}) \to \tilde{Y}(m)$, and let $\eta_s : \mathrm{Spec}(\overline{\mathbb{Q}}) \to \tilde{Y}(ms)$ be a geometric point lying above η. Let $r = 1 + \mathrm{ord}_p(s)$, and choose an isomorphism $\mathscr{T}_{\eta_s} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, such that the Weil pairing on $\mathscr{T}_{\eta_s}$ corresponds to the natural determinant map on the right, and the reduction modulo p^r of the element $(0,1)$ generates the canonical subgroup C_{p^r,η_s}. Using these choices to define the corresponding isomorphisms between group cohomology and étale cohomology, there are commutative diagrams

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$$\begin{array}{ccc} H^1_{\mathrm{et}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) & \xrightarrow{\pi_{1*}} & H^1_{\mathrm{et}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) \\ \downarrow\cong & & \downarrow\cong \\ H^1(\tilde{\Gamma}(ms),\mathcal{F}) & \xrightarrow{\mathrm{cor}} & H^1(\tilde{\Gamma}(m),\mathcal{F}) \end{array} \qquad \begin{array}{ccc} H^1_{\mathrm{et}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) & \xrightarrow{\pi_1^*} & H^1_{\mathrm{et}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) \\ \downarrow\cong & & \downarrow\cong \\ H^1(\tilde{\Gamma}(m),\mathcal{F}) & \xrightarrow{\mathrm{res}} & H^1(\tilde{\Gamma}(ms),\mathcal{F}). \end{array}$$

Also, if $\left(\begin{smallmatrix} s & \\ & 1 \end{smallmatrix}\right) \in \Sigma_0^{\cdot}(p)$, we have the commutative diagram

$$\begin{array}{ccccccc} H^1_{\mathrm{et}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) & \xrightarrow{\lambda_{s*}} & H^1_{\mathrm{et}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}},\varphi_s^*(\boldsymbol{\mathcal{F}})) & \xrightarrow{\varphi_{s*}} & H^1_{\mathrm{et}}(Y(1(s),N(pm)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) & \xrightarrow{\check{\nu}_{s*}} & H^1_{\mathrm{et}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H^1(\tilde{\Gamma}(ms),\mathcal{F}) & \xrightarrow{\lambda_{s*}} & H^1(\tilde{\Gamma}(ms),\varphi_s^*(\mathcal{F})) & \xrightarrow{\varphi_{s*}} & H^1(\Gamma(1(s),N(pm)),\mathcal{F}) & \xrightarrow{\mathrm{cor}} & H^1(\tilde{\Gamma}(m),\mathcal{F}), \end{array}$$

and, if $\left(\begin{smallmatrix} 1 & \\ & s \end{smallmatrix}\right) \in \Sigma_0^{\cdot}(p)$, the commutative diagram

$$\begin{array}{ccccccc} H^1_{\mathrm{et}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) & \xrightarrow{\check{\nu}_s^*} & H^1_{\mathrm{et}}(Y(1(s),N(pm))_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) & \xrightarrow{\varphi_s^*} & H^1_{\mathrm{et}}(\tilde{Y}(ms)_{\overline{\mathbb{Q}}},\varphi_s^*(\boldsymbol{\mathcal{F}})) & \xrightarrow{\lambda_s^*} & H^1_{\mathrm{et}}(\tilde{Y}(m)_{\overline{\mathbb{Q}}},\boldsymbol{\mathcal{F}}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H^1(\tilde{\Gamma}(m),\mathcal{F}) & \xrightarrow{\mathrm{res}} & H^1(\Gamma(1(s),N(pm)),\mathcal{F}) & \xrightarrow{\varphi_s^*} & H^1(\tilde{\Gamma}(ms),\varphi_s^*(\mathcal{F})) & \xrightarrow{\lambda_s^*} & H^1(\tilde{\Gamma}(m),\mathcal{F}). \end{array}$$

In the bottom lines of the previous two diagrams, $\varphi_s^*(\mathcal{F})$ is $\mathcal{F}$ with the action of $\Gamma_0(p^r\mathbb{Z}_p)$ conjugated by $\left(\begin{smallmatrix} s & \\ & 1 \end{smallmatrix}\right)$; the map λ_{s*} is induced by the map $\mathcal{F} \to \varphi_s^*(\mathcal{F})$ defined by $c \mapsto \left(\begin{smallmatrix} s & \\ & 1 \end{smallmatrix}\right) c$; φ_{s*} is induced by the pair of compatible maps $\Gamma(1(s),N(pm)) \to \tilde{\Gamma}(ms)$ and $\varphi_s^*(\mathcal{F}) \to \mathcal{F}$ defined by $\gamma \mapsto \left(\begin{smallmatrix} s^{-1} & \\ & 1 \end{smallmatrix}\right) \gamma \left(\begin{smallmatrix} s & \\ & 1 \end{smallmatrix}\right)$ and $c \mapsto c$, respectively; λ_s^* is induced by the map $\varphi_s(\mathcal{F}) \to \mathcal{F}$ defined by $c \mapsto \left(\begin{smallmatrix} 1 & \\ & s \end{smallmatrix}\right) c$ and φ_s^* is induced by the pair of compatible maps $\tilde{\Gamma}(m) \to \Gamma(1(s),N(pm))$ and $\mathcal{F} \to \varphi_s^*(\mathcal{F})$ defined by $\gamma \mapsto \left(\begin{smallmatrix} 1 & \\ & s^{-1} \end{smallmatrix}\right) \gamma \left(\begin{smallmatrix} 1 & \\ & s \end{smallmatrix}\right)$ and $c \mapsto c$, respectively.

We shall denote by π_{2*} and π_2^*, respectively, the composition of the maps in the rows of the previous two diagrams, both in étale cohomology and in group cohomology. Similarly, we shall also use π_{1*} and π_1^* to denote the corresponding corestriction and restriction maps.

For any rational prime q, a simple calculation shows that the following identities hold in group cohomology whenever the maps involved are defined:

$$T_q = \pi_{1*} \circ \pi_2^*, \quad T_q' = \pi_{2*} \circ \pi_1^*.$$

Therefore, under the isomorphism (5.1), the covariant action of the operators T_q, T_q' on étale cohomology corresponds to the covariant action of the operators T_q, T_q' on group cohomology, whenever defined. Similarly, the covariant action of the operators $\langle d \rangle$, $\langle d \rangle'$ on étale cohomology corresponds to the covariant action of the operators $[d]_N$, $[d]_N'$ on group cohomology.

The anti-involution ι extends to $\mathrm{Mat}_{2\times 2}(\mathbb{Z}_p)$ in the obvious way, and turns a left (respectively, right) action of $\Sigma_0(p)$ into a right (respectively, left) action of $\Sigma_0'(p)$. Thus,

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given an object $\mathcal{F} \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$ whose right $\Gamma_0(p\mathbb{Z}_p)$-action extends to a right $\Sigma_0^{\cdot}(p)$-action, there is an isomorphism $H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{F}}) \cong H^1(\Gamma, \mathcal{F})$ under which the contravariant action of the operators T_q, T_q', $\langle d \rangle$, $\langle d \rangle'$ on étale cohomology corresponds to the contravariant action of the operators T_q, T_q', $[d]_N$, $[d]_N'$ on group cohomology, whenever defined.

Consider the modules $\mathcal{A}_\nu^{\cdot}$ and $\mathcal{D}_\nu^{\cdot}$ defined earlier in this section. The action of $\Gamma_0(p\mathbb{Z}_p)$ on T' is transitive and the stabiliser of the element $(0,1) \in \mathsf{T}'$ is the subgroup

$$P(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \right\},$$

so we can identify T' with $P(\mathbb{Z}_p)\backslash\Gamma_0(p\mathbb{Z}_p)$. Similarly, the action of $\Gamma_0(p\mathbb{Z}_p)$ on T is transitive and the stabiliser of the element $(1,0) \in \mathsf{T}$ is the subgroup

$$P(\mathbb{Z}_p)^w = \left\{ \begin{pmatrix} 1 & 0 \\ pc & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \right\},$$

so we can identify T with $P(\mathbb{Z}_p)^w\backslash\Gamma_0(p\mathbb{Z}_p)$. For any positive integer j, let

$$\Gamma_1(p^j\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p^j}, d \equiv 1 \pmod{p^j} \right\},$$

$$\Gamma_1(p^j\mathbb{Z}_p)^w = \left\{ \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : a \equiv 1 \pmod{p^j}, b \equiv 0 \pmod{p^{j-1}} \right\}.$$

Then, for any positive integers i,j, we can define

$$\mathcal{A}'_{\nu,i,j} = \big\{ f : \Gamma_1(p^j\mathbb{Z}_p)\backslash\Gamma_0(p\mathbb{Z}_p) \to \mathcal{O}/\mathfrak{m}^i \mid f(a\cdot\gamma) = \nu(a)\cdot f(\gamma) \\ \text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^j\mathbb{Z}_p)\backslash\Gamma_0(p\mathbb{Z}_p) \big\},$$

$$\mathcal{A}_{\nu,i,j} = \big\{ f : \Gamma_1(p^j\mathbb{Z}_p)^w\backslash\Gamma_0(p\mathbb{Z}_p) \to \mathcal{O}/\mathfrak{m}^i \mid f(a\cdot\gamma) = \nu(a)\cdot f(\gamma) \\ \text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^j\mathbb{Z}_p)^w\backslash\Gamma_0(p\mathbb{Z}_p) \big\}.$$

The objects $\mathcal{A}_{\nu,i,j}^{\cdot}$ can be regarded as left $\mathcal{O}[\Sigma_0^{\cdot}(p)]$-modules. Let $\mathcal{A}_{\nu,i}^{\cdot} = \varinjlim_j \mathcal{A}_{\nu,i,j}^{\cdot}$. Then $\mathcal{A}_\nu^{\cdot} \cong \varprojlim_i \mathcal{A}_{\nu,i}^{\cdot}$. We denote by $\boldsymbol{\mathcal{A}}_\nu^{\cdot}$ the object in $\mathbf{S}(Y_{\mathrm{et}})$ corresponding to $\{\mathcal{A}_{\nu,i}^{\cdot}\}_i \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$. We also define $\mathcal{D}_{\nu,i}^{\cdot} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}_{\nu,i,i}^{\cdot}, \mathcal{O}/\mathfrak{m}^i)$. These objects can be regarded as right $\mathcal{O}[\Sigma_0^{\cdot}(p)]$-modules, and we have $\mathcal{D}_\nu^{\cdot} \cong \varprojlim_i \mathcal{D}_{\nu,i}^{\cdot}$. We denote by $\boldsymbol{\mathcal{D}}_\nu^{\cdot}$ the object in $\mathbf{S}(Y_{\mathrm{et}})$ corresponding to $\{\mathcal{D}_{\nu,i}^{\cdot}\}_i \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$. There are natural morphisms of $\mathcal{O}$-modules

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}_\nu^{\cdot}) \to \mathtt{H}^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}_\nu^{\cdot}) \cong H^1(\Gamma, \mathcal{A}_\nu^{\cdot})$$

and

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}_\nu^{\cdot}) \cong \mathtt{H}^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}_\nu^{\cdot}) \cong H^1(\Gamma, \mathcal{D}_\nu^{\cdot})$$

compatible with the action of Hecke operators. We also have Hecke-equivariant isomorphisms

$$H^1_{\mathrm{et},c}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}_\nu^{\cdot}) \cong \mathtt{H}^1_{\mathrm{et},c}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}_\nu^{\cdot}) \cong H^1_c(\Gamma, \mathcal{D}_\nu^{\cdot}),$$

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where $H^j_c(\Gamma,-) = H^{j-1}(\Gamma, \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})),-))$. These isomorphisms allow us to define continuous $G_{\mathbb{Q}}$-actions on the groups $H^1(\Gamma,\mathcal{A}^{\cdot}_{\nu})$, $H^1(\Gamma,\mathcal{D}^{\cdot}_{\nu})$ and $H^1_c(\Gamma,\mathcal{D}^{\cdot}_{\nu})$.

Given a character $\chi : \mathbb{Z}_p^{\times} \to \mathcal{O}^{\times}$, let $\mathcal{O}(\chi)$ be the module $\mathcal{O}$ with $\Gamma_0(p\mathbb{Z}_p)$ acting via $\chi \circ \det$, where $\det : \Gamma_0(p\mathbb{Z}_p) \to \mathbb{Z}_p^{\times}$ is the determinant map.

The natural $\mathcal{G}$-equivariant evaluation map $\mathcal{A}^{\cdot}_{\nu} \otimes_{\mathcal{O}} \mathcal{D}^{\cdot}_{\nu} \to \mathcal{O}$ yields a $G_{\mathbb{Q}}$-equivariant cup-product pairing

$$H^1(\Gamma,\mathcal{A}^{\cdot}_{\nu}) \otimes_{\mathcal{O}} H^1_c(\Gamma,\mathcal{D}^{\cdot}_{\nu}) \longrightarrow \mathcal{O}(-1), \tag{5.2}$$

under which the Hecke operators T_q, T'_q, $[d]_N$, $[d]'_N$ acting covariantly on the left, whenever defined, are adjoint to these same operators acting contravariantly on the right.

Let $\det : \mathsf{T}' \times \mathsf{T} \to \mathbb{Z}_p^{\times}$ be the function defined by $\det((x_1,x_2),(y_1,y_2)) = x_1y_2 - x_2y_1$, and let $\det_{\nu}$ be the composition of this function with $\nu : \mathbb{Z}_p^{\times} \to \mathcal{O}$. Evaluation at this function defines a $\mathcal{G}$-equivariant map $\mathcal{D}'_{\nu} \otimes_{\mathcal{O}} \mathcal{D}_{\nu} \to \mathcal{O}(-\nu)$ which yields a $G_{\mathbb{Q}}$-equivariant cup-product pairing

$$H^1(\Gamma,\mathcal{D}'_{\nu}) \otimes_{\mathcal{O}} H^1_c(\Gamma,\mathcal{D}_{\nu}) \longrightarrow \mathcal{O}(\boldsymbol{\nu})(-1), \tag{5.3}$$

where $\boldsymbol{\nu} = \nu \circ \epsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathcal{O}^{\times}$. Under this pairing, the Hecke operators T_q, T'_q, $[d]_N$, $[d]'_N$ acting contravariantly on the left, whenever defined, are adjoint to the Hecke operators T'_q, T_q, $[d]'_N$, $[d]_N$ acting contravariantly on the right. We obtain a similar pairing interchanging the roles of $\mathcal{D}_{\nu}$ and $\mathcal{D}'_{\nu}$.

5.4. Ordinary cohomology

For any $\mathbb{Z}_p$-algebra B, let $S_r(B)$ be the set of two-variable homogeneous polynomials of degree r in $B[x_1,x_2]$. It is a left $B[\Sigma^{\cdot}_0(p)]$-module with the action of $\Sigma^{\cdot}_0(p)$ defined by

$$gP(x_1,x_2) = P((x_1,x_2)\cdot g)$$

for all $g \in \Sigma^{\cdot}_0(p)$ and $P(x_1,x_2) \in S_r(B)$. To the p-adic $\Gamma_0(p\mathbb{Z}_p)$-representation $S_r = S_r(\mathbb{Z}_p)$, there corresponds the locally constant p-adic sheaf $\mathscr{S}_r$ on Y_{et} defined in Section 2.3. Therefore, we have an isomorphism

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}},\mathscr{S}_r) \cong H^1(\Gamma,S_r)$$

which is Hecke-equivariant when we consider the covariant action of Hecke operators on both sides, and we use this isomorphism to define an action of $G_{\mathbb{Q}}$ on $H^1(\Gamma,S_r)$.

We also define $L_r(B) = \mathrm{Hom}_B(S_r(B),B)$, which we regard as a right $B[\Sigma^{\cdot}_0(p)]$-module defining the $\Sigma^{\cdot}_0(p)$-action by

$$(\mu\cdot g)(P(x_1,x_2)) = \mu(gP(x_1,x_2))$$

for all $g \in \Sigma^{\cdot}_0(p)$, $\mu \in L_r(B)$ and $P(x_1,x_2) \in S_r(B)$. To the p-adic $\Gamma_0(p\mathbb{Z}_p)$-representation $L_r = L_r(\mathbb{Z}_p)$, there corresponds the locally constant p-adic sheaf $\mathscr{L}_r$ on Y_{et}. Therefore, we have an isomorphism

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}},\mathscr{L}_r) \cong H^1(\Gamma,L_r)$$

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which is Hecke-equivariant when we consider the contravariant action of Hecke operators on both sides, and we use this isomorphism to define an action of $G_{\mathbb{Q}}$ on $H^1(\Gamma, L_r)$.

The natural $\Gamma_0(p\mathbb{Z}_p)$-equivariant evaluation map $S_r \otimes_{\mathbb{Z}_p} L_r \to \mathbb{Z}_p$ yields a $G_{\mathbb{Q}}$-equivariant cup-product pairing

$$H^1(\Gamma, S_r) \otimes_{\mathbb{Z}_p} H^1_c(\Gamma, L_r) \longrightarrow \mathbb{Z}_p(-1), \tag{5.4}$$

under which the Hecke operators T_q, T'_q, $[d]_N$, $[d]'_N$ acting covariantly on the left, whenever defined, are adjoint to these same operators acting contravariantly on the right. This pairing becomes perfect after inverting p.

Let $\nu_r : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ be the character defined by $\nu_r(z) = z^r$. Evaluation at the polynomial $(x_1y_2 - x_2y_1)^r \in S_r \otimes_{\mathbb{Z}_p} S_r$ defines a $\Gamma_0(p\mathbb{Z}_p)$-equivariant map $L_r \otimes_{\mathbb{Z}_p} L_r \to \mathbb{Z}_p(-\nu_r)$ and thus yields a $G_{\mathbb{Q}}$-equivariant cup-product pairing

$$H^1(\Gamma, L_r) \otimes_{\mathbb{Z}_p} H^1_c(\Gamma, L_r) \longrightarrow \mathbb{Z}_p(r-1), \tag{5.5}$$

under which the Hecke operators T_q, T'_q, $[d]_N$, $[d]'_N$ acting contravariantly on the left, whenever defined, are adjoint to the Hecke operators T'_q, T_q, $[d]'_N$, $[d]_N$ acting contravariantly on the right. This pairing becomes perfect after inverting p.

Combining these two pairings, we can define a morphism

$$\mathtt{s}_{r*} : H^1(\Gamma, S_r(\mathbb{Q}_p)) \longrightarrow H^1(\Gamma, L_r(\mathbb{Q}_p))(-r).$$

This map is $G_{\mathbb{Q}}$-equivariant and intertwines the covariant action of the operators T_q, $[d]_N$, $[a]_p$ on the source with the contravariant action of the operators T'_q, $[d]'_N$, $[a]'_p$ on the target. We can also define $\mathtt{s}_{r*}$ directly via the isomorphism $S_r(\mathbb{Q}_p) \cong L_r(\mathbb{Q}_p)(\nu_r)$ arising from the perfect pairing $L_r(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} L_r(\mathbb{Q}_p) \to \mathbb{Q}_p(-\nu_r)$ defined by evaluation at $(x_1y_2 - x_2y_1)^r$. Therefore, the denominators introduced by this map are bounded by $r!$, that is, an element in

$$\mathrm{im}\left(H^1(\Gamma, S_r) \to H^1(\Gamma, S_r(\mathbb{Q}_p))\right)$$

is mapped to an element in

$$\frac{1}{r!}\,\mathrm{im}\left(H^1(\Gamma, L_r) \to H^1(\Gamma, L_r(\mathbb{Q}_p))\right),$$

as follows from [BSV22, Remark 3.3].

To slightly simplify the notation, we will write $\mathcal{A}_r^\cdot$ and $\mathcal{D}_r^\cdot$ for $\mathcal{A}_{\nu_r}^\cdot$ and $\mathcal{D}_{\nu_r}^\cdot$, respectively. Regarding two-variable polynomials as functions on $\mathsf{T}^\cdot$, we obtain a natural morphism of left $\mathbb{Z}_p[\Sigma_0^\cdot(p)]$-modules $S_r \to \mathcal{A}_r^\cdot$. Also, dualizing this map, we obtain a morphism of right $\mathbb{Z}_p[\Sigma_0^\cdot(p)]$-modules $\mathcal{D}_r^\cdot \to L_r$. Thus, we have $G_{\mathbb{Q}}$-equivariant and Hecke-equivariant morphisms

$$H^1(\Gamma, S_r) \to H^1(\Gamma, \mathcal{A}_r^\cdot) \quad \text{and} \quad H^1(\Gamma, \mathcal{D}_r^\cdot) \to H^1(\Gamma, L_r).$$

Applying Hida's (anti-)ordinary projector $e_{\mathrm{ord}}^\cdot := \lim_{n\to\infty}(T_p^\cdot)^{n!}$, the previous morphisms become isomorphisms

$$e_{\mathrm{ord}}^\cdot H^1(\Gamma, S_r) \cong e_{\mathrm{ord}}^\cdot H^1(\Gamma, \mathcal{A}_r^\cdot), \quad e_{\mathrm{ord}}^\cdot H^1(\Gamma, \mathcal{D}_r^\cdot) \cong e_{\mathrm{ord}}^\cdot H^1(\Gamma, L_r).$$

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Under these isomorphisms, the pairings (5.4) and (5.5) correspond to the pairings (5.2) and (5.3), respectively, after applying the corresponding (anti-)ordinary projector to every term involved.

5.5. Λ-adic Poincaré pairing

It will be convenient to write $\langle a;b\rangle$, with $a\in(\mathbb{Z}/N\mathbb{Z})^\times$ and $b\in(\mathbb{Z}/p^r\mathbb{Z})^\times$, for the diamond operator $\langle d\rangle$, where $d\in(\mathbb{Z}/Np^r)^\times$ is congruent to a modulo N and to b modulo p^r. We also write $\epsilon_N : G_{\mathbb{Q}}\to(\mathbb{Z}/N\mathbb{Z})^\times$ for the mod N cyclotomic character.

For any positive integer r, let

$$G_r = 1+p(\mathbb{Z}/p^r\mathbb{Z}),\quad \tilde{G}_r=(\mathbb{Z}/p^r\mathbb{Z})^\times,$$

and define

$$\Lambda_r=\mathbb{Z}_p[G_r],\quad \tilde{\Lambda}_r=\mathbb{Z}_p[\tilde{G}_r],\quad \Lambda=\varprojlim_r\Lambda_r=\mathbb{Z}_p[[1+p\mathbb{Z}_p]],\quad \tilde{\Lambda}=\varprojlim_r\tilde{\Lambda}_r=\mathbb{Z}_p[[\mathbb{Z}_p^\times]].$$

We have natural factorisations $(\mathbb{Z}/p^r\mathbb{Z})^\times=\mu_{p-1}\times(1+p\mathbb{Z}/p^r\mathbb{Z})$ and $\mathbb{Z}_p^\times=\mu_{p-1}\times(1+p\mathbb{Z}_p)$ which give natural embeddings $\Lambda_r\hookrightarrow\tilde{\Lambda}_r$ and $\Lambda\hookrightarrow\tilde{\Lambda}$. We define idempotents

$$e_i=\frac{1}{p-1}\sum_{\zeta\in\mu_{p-1}}\zeta^{-i}[\zeta]$$

for any integer i modulo $p-1$. Let $\kappa_i:\mathbb{Z}_p^\times\to\Lambda^\times$ be the character defined by $z\mapsto\omega^i(z)[\langle z\rangle]$, and let $\boldsymbol{\kappa}_i=\kappa_i\circ\epsilon_{\mathrm{cyc}}:G_{\mathbb{Q}}\to\Lambda^\times$.

We will shorten notation by writing

$$X_r(m)=X(1,Np^r(m)),\quad H^1_{\mathrm{et}}(X_\infty(m)_{\overline{\mathbb{Q}}},\mathbb{Z}_p)=\varprojlim_r H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}},\mathbb{Z}_p).\tag{5.6}$$

We have a natural action of $\tilde{\Lambda}_r$ and $\tilde{\Lambda}$ on the previous groups defined by letting group-like elements $[u]$ act like the diamond operators $\langle 1;u\rangle'$.

Fix compatible primitive p-power roots of unity ζ_{p^r} and a primitive N-th root of unity ζ_N. Then one can define Atkin–Lehner automorphisms w_r and w for the curve $X_r(m)$ similarly as in [DR17, Section 1.2]. More precisely, $X_r(m)$ parameterises quadruples (E,P,Q,C), where E is an elliptic curve, P is a point of order N, Q is a point of order p^r and C is a cyclic subgroup of E of order Nm containing P. Then, we define

$$w_r(E,P,Q,C)=(E/C_Q,P+C_Q,Q'+C_Q,C+C_Q/C_Q),$$

where $C_Q\subseteq E$ is the subgroup generated by Q, and $Q'\in E[p^r]$ satisfies $\langle Q,Q'\rangle=\zeta_{p^r}$. Similarly, we define

$$w(E,P,Q,C)=(E/C,P'+C,Q+C,E[Nm]/C),$$

where $P'\in E[N]$ satisfies $\langle P,P'\rangle=\zeta_N$. These Atkin–Lehner automorphisms satisfy, for any $\sigma\in G_{\mathbb{Q}}$,

$$w_r^\sigma=\langle 1;\epsilon_{\mathrm{cyc}}(\sigma)\rangle w_r,\quad w^\sigma=\langle\epsilon_N(\sigma);1\rangle w.$$

We let w and w_r act on cohomology via pullback.

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Define $G_{\mathbb{Q}}$-equivariant pairings

$$\langle , \rangle_{G_r} : e_i H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_{-i} H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \longrightarrow \Lambda_r(-1)$$

by the formula

$$\langle a, b \rangle_{G_r} = \sum_{\sigma \in G_r} \langle a^\sigma, b \rangle_r \cdot \sigma^{-1},$$

where $\langle , \rangle_r$ stands for the natural Poincaré pairing. These pairings are Λ_r-linear and antilinear in the first and second argument, respectively. Then we get $G_{\mathbb{Q}}$-equivariant Λ_r-pairings

$$[,]_{G_r} : e_i H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(\langle \epsilon_N^{-1}; 1 \rangle') \longrightarrow \Lambda_r(\boldsymbol{\kappa}_i)(-1)$$

via the following modification of the previous pairing:

$$[a, b]_{G_r} = \langle a, w w_r \cdot (T_p')^r \cdot b \rangle_{G_r}.$$

These pairings are compatible in the sense that the diagram

$$\begin{array}{ccc} e_i H^1_{\mathrm{et}}(X_{r+1}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H^1_{\mathrm{et}}(X_{r+1}(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(\langle \epsilon_N^{-1}; 1 \rangle') & \xrightarrow{[,]_{G_{r+1}}} & \Lambda_{r+1}(\boldsymbol{\kappa}_i)(-1) \\ \downarrow{\scriptstyle \pi_{1*} \times \pi_{1*}} & & \downarrow \\ e_i H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \times e_i H^1_{\mathrm{et}}(X_r(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(\langle \epsilon_N^{-1}; 1 \rangle') & \xrightarrow{[,]_{G_r}} & \Lambda_r(\boldsymbol{\kappa}_i)(-1) \end{array}$$

commutes, which can be proved as in [DR17, Lemma 1.1]. This yields a Λ-adic perfect $G_{\mathbb{Q}}$-equivariant pairing

$$e_i H^1_{\mathrm{et}}(X_\infty(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\mathrm{ord}} \times e_i H^1_{\mathrm{et}}(X_\infty(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\mathrm{ord}}(\langle \epsilon_N^{-1}; 1 \rangle') \longrightarrow \Lambda(\boldsymbol{\kappa}_i)(-1), \tag{5.7}$$

where $H^1_{\mathrm{et}}(X_\infty(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\mathrm{ord}} = e'_{\mathrm{ord}} H^1_{\mathrm{et}}(X_\infty(m)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)$. All Hecke operators are self-adjoint for this pairing.

5.6. Big Galois representations

Let $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ, let $\mathrm{Cont}(\mathbb{Z}_p, \Lambda)$ be the Λ-module of continuous functions on $\mathbb{Z}_p$ with values in Λ and let κ be any of the κ_i above. Define the Λ-module

$$\mathcal{A}'_\kappa = \big\{ f : \mathsf{T}' \to \Lambda \mid f(pz, 1) \in \mathrm{Cont}(\mathbb{Z}_p, \Lambda) \text{ and } f(a \cdot \gamma) = \kappa(a) \cdot f(\gamma) \text{ for all } a \in \mathbb{Z}_p^\times, \gamma \in \mathsf{T}' \big\},$$

equipped with the $\mathfrak{m}_\Lambda$-adic topology, and the Λ-module

$$\mathcal{D}'_\kappa = \mathrm{Hom}_{\mathrm{cont}, \Lambda}(\mathcal{A}'_\kappa, \Lambda),$$

equipped with the weak-$*$ topology. As in Section 5.2, we can regard $\mathcal{A}'_\kappa$ (respectively, $\mathcal{D}'_\kappa$) as a left (respectively, right) $\Lambda[\Sigma'_0(p)]$-module.

Similarly to what we did in Section 5.3, define, for any positive integers j, r,

$$\mathcal{A}'_{\kappa, j, r} = \big\{ f : \Gamma_1(p^r \mathbb{Z}_p) \backslash \Gamma_0(p\mathbb{Z}_p) \to \Lambda / \mathfrak{m}_\Lambda^j \mid f(a \cdot \gamma) = \kappa(a) \cdot f(\gamma) \\ \text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^r \mathbb{Z}_p) \backslash \Gamma_0(p\mathbb{Z}_p) \big\}$$

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and $\mathcal{A}'_{\kappa,j} = \varinjlim_r \mathcal{A}'_{\kappa,j,r}$. Then $\mathcal{A}'_\kappa = \varprojlim_j \mathcal{A}'_{\kappa,j}$. We denote by $\boldsymbol{\mathcal{A}}'_\kappa$ the object in $\mathbf{S}(Y_{\mathrm{et}})$ corresponding to $\{\mathcal{A}'_{\kappa,j}\}_j \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$. We also define $\mathcal{D}'_{\kappa,j} = \mathrm{Hom}_\Lambda(\mathcal{A}'_{\kappa,j,j}, \Lambda/\mathfrak{m}_\Lambda^j)$, so that $\mathcal{D}'_\kappa = \varprojlim_j \mathcal{D}'_{\kappa,j}$, and denote by $\boldsymbol{\mathcal{D}}'_\kappa$ the object in $\mathbf{S}(Y_{\mathrm{et}})$ corresponding to $\{\mathcal{D}'_{\kappa,j}\}_j \in \mathbf{M}(\Gamma_0(p\mathbb{Z}_p))$. There are natural Hecke-equivariant morphisms of Λ-modules

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}'_\kappa) \to \mathtt{H}^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{A}}'_\kappa) \cong H^1(\Gamma, \mathcal{A}'_\kappa),$$

$$H^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong \mathtt{H}^1_{\mathrm{et}}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong H^1(\Gamma, \mathcal{D}'_\kappa),$$

$$H^1_{\mathrm{et},c}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong \mathtt{H}^1_{\mathrm{et},c}(Y_{\overline{\mathbb{Q}}}, \boldsymbol{\mathcal{D}}'_\kappa) \cong H^1_c(\Gamma, \mathcal{D}'_\kappa),$$

which allow us to define continuous $G_{\mathbb{Q}}$-actions on the groups $H^1(\Gamma, \mathcal{A}'_\kappa)$, $H^1(\Gamma, \mathcal{D}'_\kappa)$ and $H^1_c(\Gamma, \mathcal{D}'_\kappa)$.

The evaluation map $\mathcal{A}'_\kappa \otimes_\Lambda \mathcal{D}'_\kappa \to \Lambda$ yields a $G_{\mathbb{Q}}$-equivariant cup-product pairing

$$H^1(\Gamma, \mathcal{A}'_\kappa) \otimes_\Lambda H^1_c(\Gamma, \mathcal{D}'_\kappa) \longrightarrow \Lambda(-1), \tag{5.8}$$

under which the Hecke operators T_q, T'_q, $[d]_N$, $[d]'_N$ acting covariantly on the left, whenever defined, are adjoint to these same operators acting contravariantly on the right.

Recall that in this section, we have set $\Gamma = \Gamma(1, N(pm))$, and let $S = \Sigma'_0(p) \cap \mathrm{GL}_2(\mathbb{Q})$. For any positive integer r, define

$$\Sigma'_1(p^r) = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p^r\mathbb{Z}_p & 1 + p^r\mathbb{Z}_p \end{pmatrix}, \quad S_r = \Sigma'_1(p^r) \cap \mathrm{GL}_2(\mathbb{Q}), \quad \Gamma_r = \Gamma(1, Np^r(m)).$$

We define compatibility of Hecke pairs as in [AS86a, Definition 1.1.2] but changing left-right conventions. More precisely, we say that the Hecke pair $(\Gamma_\alpha, S_\alpha)$ is compatible to the Hecke pair (Γ_β, S_β) if $(\Gamma_\alpha, S_\alpha) \subseteq (\Gamma_\beta, S_\beta)$, $S_\alpha \Gamma_\beta = S_\beta$ and $\Gamma_\beta \cap S_\alpha^{-1} S_\alpha = \Gamma_\alpha$. With this definition, the Hecke pair (Γ_r, S_r) is compatible to the Hecke pair (Γ_t, S_t), if $r \geq t$, and to the Hecke pair (Γ, S).

Suppose that the Hecke pair $(\Gamma_\alpha, S_\alpha)$ is compatible to (Γ_β, S_β), and that Γ_α has finite index in Γ_β. For any right S_α-module E, we define

$$\mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_\beta} E = \big\{ \varphi : \Gamma_\beta \to E \mid \varphi(xy) = \varphi(y)x^{-1} \text{ for all } x \in \Gamma_\alpha, y \in \Gamma_\beta \big\}.$$

This module is equipped with a right action of S_β: given $\varphi \in \mathrm{Ind}_{\Gamma_\alpha}^{\Gamma_\beta} E$ and $g \in S_\beta$

$$(\varphi g)(x) = \sum \varphi(\gamma) \gamma g x^{-1},$$

where the sum is over representatives γ for the cosets in $\Gamma_\alpha \backslash \Gamma \cap S_\alpha x g^{-1}$.

Now define

$$A'_{\kappa,r} = \big\{ f : \Gamma_1(p^r\mathbb{Z}_p) \backslash \Gamma_0(p\mathbb{Z}_p) \to \Lambda_r \mid f(a \cdot \gamma) = \kappa(a) \cdot f(\gamma) \\ \text{for all } a \in \mathbb{Z}_p^\times, \gamma \in \Gamma_1(p^r\mathbb{Z}_p) \backslash \Gamma_0(p\mathbb{Z}_p) \big\},$$

and let $D'_{\kappa,r} = \mathrm{Hom}_{\Lambda_r}(A'_{\kappa,r}, \Lambda_r)$. With these definitions, $\mathcal{D}'_\kappa = \varprojlim_r D'_{\kappa,r}$. Let S_r act trivially on $\mathbb{Z}_p$ and consider the right $\mathbb{Z}_p[S_1]$-module $\mathrm{Ind}_{\Gamma_r}^{\Gamma_1} \mathbb{Z}_p$. Let R be a set of representatives for the cosets in $\Gamma_r \backslash \Gamma_1$. The map $\mathrm{Ind}_{\Gamma_r}^{\Gamma_1} \mathbb{Z}_p \to D'_{\kappa,r}$ defined by

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$$\varphi \mapsto \big[f \mapsto \sum_{r\in R} \varphi(r) f(r)\big]$$

is an isomorphism of right $\mathbb{Z}_p[S_1]$-modules. Therefore, there are natural isomorphisms

$$H^1(\Gamma_1, \mathcal{D}'_\kappa) \cong \varprojlim_r H^1(\Gamma_1, D'_{\kappa,r}) \cong \varprojlim_r H^1(\Gamma_r, \mathbb{Z}_p).$$

According to [AS86a, Lemma 1.1.3] and [AS86a, Lemma 1.1.4], both corestriction and the Shapiro isomorphism commute with the action of $D(\Gamma, S)$ via restriction of Hecke algebras, so the previous isomorphisms are Hecke-equivariant.

Similarly to (5.6), but omitting m from the notation, we let $Y_r = Y(1, Np^r(m))$ and put

$$H^1_{\mathrm{et}}(Y_{\infty,\overline{\mathbb{Q}}}, \mathbb{Z}_p) := \varprojlim_r H^1_{\mathrm{et}}(Y_{r,\overline{\mathbb{Q}}}, \mathbb{Z}_p),$$

where the inverse limit is with respect to the maps π_{1*}. Then

$$H^1(\Gamma_1, \mathcal{D}'_\kappa) \cong \varprojlim_r H^1(\Gamma_r, \mathbb{Z}_p) \cong H^1_{\mathrm{et}}(Y_{\infty,\overline{\mathbb{Q}}}, \mathbb{Z}_p),$$

where the last isomorphism is defined by choosing a compatible system of geometric points for the curves Y_r and suitable compatible bases for the corresponding Tate modules. Under the isomorphisms above, the contravariant operators T'_q, $[d]'_N$, $[a]'_p$ on the first term correspond to the contravariant operators T'_q, $\langle d;1\rangle'$, $\langle 1;a\rangle'$ defined on the last term via the compatibility of these operators with the pushforward maps π_{1*}.

Also, according to [AS86a, Lemma 1.1.5], the restriction map yields a Hecke-equivariant isomorphism

$$H^1(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1(\Gamma_1, \mathcal{D}'_\kappa)$$

(recall that we have set $\kappa = \kappa_i$). Combining this isomorphism with the previous ones, we obtain a Hecke-equivariant isomorphism

$$H^1(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1_{\mathrm{et}}(Y_{\infty,\overline{\mathbb{Q}}}, \mathbb{Z}_p).$$

Similarly, using [AS86b, Proposition 4.2], one proves that there is a Hecke-equivariant isomorphism

$$H^1_c(\Gamma, \mathcal{D}'_\kappa) \cong e_i H^1_{\mathrm{et},c}(Y_{\infty,\overline{\mathbb{Q}}}, \mathbb{Z}_p). \tag{5.9}$$

6. Proof of the wild norm relations

Assume that p splits in K as $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, and that it does not divide the class number h_K.

We keep most of the notations from Section 4. In particular, (g,h) is a pair of newforms of weights (l,m) of the same parity, levels (N_g, N_h) and characters (χ_g, χ_h), and we assume that the ring of integers $\mathcal{O} \subset E = L_{\mathfrak{P}}$ contains the Fourier coefficients of g and h. In addition, we assume that p does not divide N_g nor N_h, and that both g and h are ordinary at p.

We now allow the Grössencharacter ψ to have infinity type $(1-k,0)$ for any even integer $k \geq 2$, and let $\mathfrak{f}$ be the conductor of ψ, which we assume to be coprime to p. Let χ be the

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unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$, such that $\psi((n)) = n^{k-1}\chi(n)$ for integers n coprime to $N_{K/\mathbb{Q}}(\mathfrak{f})$.

As in [BL18, Section 3.2.1], we denote by ψ_0 the unique Grössencharacter of infinity type $(-1,0)$, conductor $\mathfrak{p}$ and whose associated p-adic Galois character factors through $\Gamma_{\mathfrak{p}}$, the Galois group of the unique $\mathbb{Z}_p$-extension of K unramified outside $\mathfrak{p}$. Then we can uniquely write $\psi = \alpha\psi_0^{k-1}$, where α is a ray class character of conductor dividing $\mathfrak{fp}$. Since $(\mathfrak{f},p) = 1$ and k is even, it easily follows that ψ is *non-Eisenstein* and *p-distinguished*, meaning that

$$\alpha\psi_0|_{\mathcal{O}_{K,\mathfrak{p}}^\times} \not\equiv \omega \,(\mathrm{mod}\,\mathfrak{P}), \tag{6.1}$$

where ω is the Teichmüller character.

Let $\psi_{\mathfrak{P}}$ be the continuous E-valued character of $K^\times\backslash\mathbb{A}_{K,\mathrm{f}}^\times$ defined by

$$\psi_{\mathfrak{P}}(x) = x_{\mathfrak{p}}^{1-k}\psi(x),$$

where $x_{\mathfrak{p}}$ is the projection of the idèle x to the component at $\mathfrak{p}$. We will also denote by $\psi_{\mathfrak{P}}$ the corresponding character of G_K obtained via the geometric Artin map. Then $\mathrm{Ind}_K^{\mathbb{Q}} E(\psi_{\mathfrak{P}}^{-1})$ is the p-adic representation attached to θ_ψ, and we note that by (6.1), the associated residual representation is absolutely irreducible and p-distinguished (see [LLZ15, Remark 5.1.4]).

Consider the q-expansion

$$\Theta = \sum_{(\mathfrak{a},\mathfrak{fp})=1} [\mathfrak{a}]q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \mathcal{O}[[H_{\mathfrak{fp}^\infty}]][[q]],$$

where $H_{\mathfrak{fp}^\infty}$ denotes the maximal pro-p quotient of the ray class group of K of conductor $\mathfrak{fp}^\infty$, and $[\mathfrak{a}]$ is the image of $\mathfrak{a}$ in $H_{\mathfrak{fp}^\infty}$ under the geometric Artin map. Since we assume that p does not divide h_K, we can factor $H_{\mathfrak{fp}^\infty} \cong H_{\mathfrak{f}} \times \Gamma_{\mathfrak{p}}$. Hence, we have $\Theta \in \mathcal{O}[H_{\mathfrak{f}}] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]][[q]]$, and we can specialise this to

$$\mathbf{f} = \sum_{(\mathfrak{a},\mathfrak{fp})=1} \alpha([\mathfrak{a}])\psi_0([\mathfrak{a}])[\mathfrak{a}]q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in \Lambda_{\mathbf{f}}[[q]], \tag{6.2}$$

where $\Lambda_{\mathbf{f}} = \mathcal{O}[[\Gamma_{\mathfrak{p}}]]$. We identify $\Gamma_{\mathfrak{p}}$ with $\Gamma = 1 + p\mathbb{Z}_p$ via the isomorphism $\Gamma \cong \mathcal{O}_{K,\mathfrak{p}}^{(1)} \to \Gamma_{\mathfrak{p}}$ defined by $u \mapsto \mathrm{art}_{\mathfrak{p}}(u)^{-1}$, where $\mathrm{art}_{\mathfrak{p}}$ stands for the geometric local Artin map, and in this way, we identify $\Lambda_{\mathbf{f}}$ with $\Lambda_{\mathcal{O}} = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}$. We can therefore regard $\mathbf{f}$ as a primitive Hida family specialising to

$$\mathbf{f}_{k'} = \sum_{(\mathfrak{a},\mathfrak{fp})=1} \alpha([\mathfrak{a}])\psi_0([\mathfrak{a}])^{k'-1}q^{N_{K/\mathbb{Q}}(\mathfrak{a})} \in S_{k'}^{\mathrm{ord}}(N_\psi p, \chi_\alpha\varepsilon_K\omega^{1-k'})$$

at the arithmetic point $\nu_{k'-2}$, where $N_\psi = DN_{K/\mathbb{Q}}(\mathfrak{f})$ and $\chi_\alpha(n) = \alpha((n))$. Note that $\mathbf{f}$ has character $\chi = \chi_\alpha\omega^{1-k}$, and $\mathbf{f}_k = \theta_\psi^{(p)}$ is the ordinary p-stabilisation of θ_ψ.

Let $\chi_{\mathbb{Q}}$ be the adelic character attached to χ, let $\chi_K = \chi_{\mathbb{Q}} \circ N_{K/\mathbb{Q}}$ and let $\psi^* = \chi_K^{-1}\psi$. We can define a primitive Hida family $\mathbf{f}^*$ attached to the Grössencharacter ψ^* in the same way that we defined the Hida family $\mathbf{f}$ attached to ψ. This is just the Hida family $\mathbf{f} \otimes \chi^{-1}$.

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We assume that $\chi\varepsilon_K\chi_g\chi_h = 1$, that is, the product of the characters of θ_ψ, g and h is trivial. Similarly to what we did in Section 4, set $(r_1,r_2,r_3) = (k-2,l-2,m-2)$. For every positive integer m, let

$$\tilde{Y}(m) = Y(1,N(pm)), \quad \text{where } N = \mathrm{lcm}(N_\psi,N_g,N_h),$$

and denote by $\tilde{\Gamma}(m)$ the corresponding modular group. Let $\kappa = \kappa_{r_1} : \mathbb{Z}_p^\times \to \Lambda^\times$, and choose a square root of this character defined by $\kappa^{1/2}(u) = \omega(u)^{(k-2)/2}[\langle u\rangle^{1/2}]$.

We can define classes

$$\mathbf{Det}_m^{\mathbf{f}gh} \in H^0_{\mathrm{et}}(\tilde{Y}(m),\boldsymbol{\mathcal{A}}'_\kappa \otimes \boldsymbol{\mathcal{A}}_{r_2} \otimes \boldsymbol{\mathcal{A}}_{r_3}(-\kappa^{1/2} - \nu_{(r_2+r_3)/2})),$$

as in [BSV22, Section 8.1], but replacing the Hida families $\mathbf{g},\mathbf{h}$ in their construction by our g,h and working with modules of continuous functions instead of modules of locally analytic functions. Similarly to what is done in *loc. cit.*, and adopting some of the notations there, we define the cohomology classes

$$\kappa^{(1)}_{m,\mathbf{f}gh} = (e'_{\mathrm{ord}} \otimes e_{\mathrm{ord}} \otimes e_{\mathrm{ord}}) \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathbf{Det}_m^{\mathbf{f}gh}),$$

inside the group

$$H^1\left(\mathbb{Q}, H^1(\tilde{\Gamma}(m),\mathcal{A}'_\kappa)^{\mathrm{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(m),\mathcal{A}_{r_2})^{\mathrm{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(m),\mathcal{A}_{r_3})^{\mathrm{ord}}(\boldsymbol{\kappa}^{1/2} + 2 + (r_2+r_3)/2)\right),$$

where $\boldsymbol{\kappa}^{1/2} = \kappa^{1/2} \circ \epsilon_{\mathrm{cyc}}$; and, for each squarefree positive integer n coprime to p and N, we define

$$\kappa^{(2)}_{n,\mathbf{f}gh} = \chi\varepsilon_K(n)\kappa(n)^{-1}n^{r_2}(\mathrm{Id}\otimes\mathrm{Id}\otimes[n]_N)(\mathrm{Id}\otimes\pi_{1*}\otimes\pi_{2*})\kappa^{(1)}_{n^2,\mathbf{f}gh}$$

lying in the group

$$H^1\left(\mathbb{Q}, H^1(\tilde{\Gamma}(n^2),\mathcal{A}'_\kappa)^{\mathrm{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(1),\mathcal{A}_{r_2})^{\mathrm{ord}} \hat{\otimes} H^1(\tilde{\Gamma}(1),\mathcal{A}_{r_3})^{\mathrm{ord}}(\boldsymbol{\kappa}^{1/2} + 2 + (r_2+r_3)/2)\right).$$

Now we can prove norm relations for Λ-adic classes, as we did for the classes in the previous section.

Lemma 6.1. *Let m be a positive integer, and let q be a prime number. Assume that both m and q are coprime to p and N. Then*

$$\begin{aligned}
(\pi_{2*}\otimes\pi_{1*}\otimes\pi_{1*})\kappa^{(1)}_{mq,\mathbf{f}gh} &= (T'_q\otimes\mathrm{Id}\otimes\mathrm{Id})\kappa^{(1)}_{m,\mathbf{f}gh};\\
(\pi_{1*}\otimes\pi_{2*}\otimes\pi_{2*})\kappa^{(1)}_{mq,\mathbf{f}gh} &= \kappa^{-1/2}(q)q^{(r_2+r_3)/2}(T_q\otimes\mathrm{Id}\otimes\mathrm{Id})\kappa^{(1)}_{m,\mathbf{f}gh};\\
(\pi_{1*}\otimes\pi_{2*}\otimes\pi_{1*})\kappa^{(1)}_{mq,\mathbf{f}gh} &= (\mathrm{Id}\otimes T'_q\otimes\mathrm{Id})\kappa^{(1)}_{m,\mathbf{f}gh};\\
(\pi_{2*}\otimes\pi_{1*}\otimes\pi_{2*})\kappa^{(1)}_{mq,\mathbf{f}gh} &= \kappa^{1/2}(q)q^{(r_3-r_2)/2}(\mathrm{Id}\otimes T_q\otimes\mathrm{Id})\kappa^{(1)}_{m,\mathbf{f}gh};\\
(\pi_{1*}\otimes\pi_{1*}\otimes\pi_{2*})\kappa^{(1)}_{mq,\mathbf{f}gh} &= (\mathrm{Id}\otimes\mathrm{Id}\otimes T'_q)\kappa^{(1)}_{m,\mathbf{f}gh};\\
(\pi_{2*}\otimes\pi_{2*}\otimes\pi_{1*})\kappa^{(1)}_{mq,\mathbf{f}gh} &= \kappa^{1/2}(q)q^{(r_2-r_3)/2}(\mathrm{Id}\otimes\mathrm{Id}\otimes T_q)\kappa^{(1)}_{m,\mathbf{f}gh}.
\end{aligned}$$

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If q is coprime to m, we also have

$$(\pi_{1*}\otimes\pi_{1*}\otimes\pi_{1*})\kappa^{(1)}_{mq,\mathbf{f}gh} = (q+1)\kappa^{(1)}_{m,\mathbf{f}gh};$$
$$(\pi_{2*}\otimes\pi_{2*}\otimes\pi_{2*})\kappa^{(1)}_{mq,\mathbf{f}gh} = (q+1)\kappa^{1/2}(q)q^{(r_2+r_3)/2}\kappa^{(1)}_{m,\mathbf{f}gh}.$$

Proof. As in Lemma 4.2, the same arguments proving equations (174) and (176) in [BSV22] apply *mutatis mutandis* to yield the proof of these identities. □

Lemma 6.2. *Let n be a squarefree positive integer coprime to p and N, and let q be a rational prime coprime to p, N and n. Then*

$$\begin{aligned}(\pi_{11*}\otimes\mathrm{Id}\otimes\mathrm{Id})\kappa^{(2)}_{nq,\mathbf{f}gh} &= \big\{\chi(q)\kappa(q)^{-1}q^{r_2}(\mathrm{Id}\otimes\mathrm{Id}\otimes[q]_N^{-1}T_q^2)\\ &\quad-\chi(q)\kappa(q)^{-1}(q+1)q^{r_2+r_3}(\mathrm{Id}\otimes\mathrm{Id}\otimes\mathrm{Id})\big\}\kappa^{(2)}_{n,\mathbf{f}gh},\\ (\pi_{21*}\otimes\mathrm{Id}\otimes\mathrm{Id})\kappa^{(2)}_{nq,\mathbf{f}gh} &= \big\{\chi(q)\kappa^{-1/2}(q)q^{(r_2+r_3)/2}(\mathrm{Id}\otimes T_q\otimes T_q)\\ &\quad-\chi(q)\kappa(q)^{-1}q^{r_2+r_3}(([q]'_N)^{-1}T'_q\otimes[q]_N\otimes[q]_N)\big\}\kappa^{(2)}_{n,\mathbf{f}gh},\\ (\pi_{22*}\otimes\mathrm{Id}\otimes\mathrm{Id})\kappa^{(2)}_{nq,\mathbf{f}gh} &= \big\{\chi(q)q^{r_3}(\mathrm{Id}\otimes T_q^2\otimes[q]_N)\\ &\quad-\chi(q)(q+1)q^{r_2+r_3}(\mathrm{Id}\otimes[q]_N\otimes[q]_N)\big\}\kappa^{(2)}_{n,\mathbf{f}gh},\end{aligned}$$

where π_{ij*} denotes the composition

$$H^1(\tilde\Gamma(n^2q^2),\mathcal{F})\xrightarrow{\pi_{i*}}H^1(\tilde\Gamma(n^2q),\mathcal{F})\xrightarrow{\pi_{j*}}H^1(\tilde\Gamma(n^2),\mathcal{F}).$$

Proof. This can be deduced from Lemma 6.1 by the same calculation as in Lemma 4.4. □

Let $\Gamma(m)=\Gamma(1,Np(m))$, and write $Y(m)$ and $X(m)$ for the corresponding affine and projective modular curves. The pairing in equation (5.8) yields a map

$$H^1(\tilde\Gamma(m),\mathcal{A}'_\kappa)\to\mathrm{Hom}_\Lambda(H^1_c(\Gamma(m),\mathcal{D}'_\kappa),\Lambda)(-1)\cong\mathrm{Hom}_\Lambda(e_{r_1}H^1_{\mathrm{et},c}(Y_\infty(m)_{\overline{\mathbb{Q}}},\mathbb{Z}_p),\Lambda)(-1),$$

where the isomorphism comes from equation (5.9). Let $\mathcal{I}_n$ be the maximal ideal in Hida's big ordinary Hecke algebra $\mathbb{T}(1,Np^\infty(n^2))'_{\mathrm{ord}}$ corresponding to the Hida family $\mathbf{f}^*$; by (6.1) this ideal corresponds to a non-Eisenstein maximal ideal in $\mathbb{T}(1,Np(n^2))'$, so there are isomorphisms

$$H^1_{\mathrm{et},c}(Y_\infty(n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p)^{\mathrm{ord}}_{\mathcal{I}_n}\cong H^1_{\mathrm{et}}(X_\infty(n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p)^{\mathrm{ord}}_{\mathcal{I}_n}\cong H^1_{\mathrm{et}}(Y_\infty(n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p)^{\mathrm{ord}}_{\mathcal{I}_n}.$$

Hence, the pairings (5.7) and (5.8) together with the isomorphism (5.9) yield a morphism

$$\mathbf{s}_{\mathbf{f},n*}:H^1(\tilde\Gamma(n^2),\mathcal{A}'_\kappa)^{\mathrm{ord}}\longrightarrow e_{r_1}H^1_{\mathrm{et}}(Y_\infty(n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p)^{\mathrm{ord}}_{\mathcal{I}_n}(\langle\epsilon_N^{-1};1\rangle')(-\boldsymbol{\kappa}).$$

This map is $G_{\mathbb{Q}}$-equivariant and intertwines the covariant action of the operators T'_q, $[d]'_N$, $[a]'_p$ on the source with the contravariant action of the operators T'_q, $\langle d;1\rangle'$, $\langle 1;a\rangle'$ on the target.

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Fix a level-N test vector $\breve{\mathbf{f}}$ for $\mathbf{f}$, and let $\breve{\mathbf{f}}^* = \breve{\mathbf{f}} \otimes \chi^{-1}\varepsilon_K$. Fix also test vectors

$$\breve{g} \in S_l(N,\chi_g)[g], \quad \breve{h} \in S_m(N,\chi_h)[h],$$

and write $\breve{g}_\alpha$ and $\breve{h}_\alpha$ for the corresponding ordinary p-stabilisations.

Define maps

$$\phi_{n,r} : \mathbb{T}(1,N_\psi p^r(n^2))'_{\mathrm{ord}} \longrightarrow \mathcal{O}[R_{\bar{\mathfrak{f}}\mathfrak{p}^r,n}]$$

attached to the Grössencharacter $\alpha\chi_K^{-1}\psi_0$ as in Lemma 3.5, and let

$$\phi_{n,\infty} : \mathbb{T}(1,N_\psi p^\infty(n^2))'_{\mathrm{ord}} \longrightarrow \mathcal{O}[[R_{\bar{\mathfrak{f}}\mathfrak{p}^\infty,n}]] = \mathcal{O}[R_{\bar{\mathfrak{f}},n}] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]]$$

be the inverse limit $\varprojlim_r \phi_{n,r}$. The test vector $\breve{\mathbf{f}}^*$ determines a degeneracy map

$$H^1_{\mathrm{et}}(Y_\infty(n^2)_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))^{\mathrm{ord}} \to H^1_{\mathrm{et}}(Y(1,N_\psi p^\infty(n^2))_{\overline{\mathbb{Q}}},\mathbb{Z}_p(1))^{\mathrm{ord}}.$$

Composing this degeneracy map with the natural quotient map, we get a morphism

$$\pi_{\mathbf{f}^*} : e_{r_1} H^1_{\mathrm{et}}(Y_\infty(n^2)_{\overline{\mathbb{Q}}},\mathcal{O}(1))^{\mathrm{ord}}_{\mathcal{I}_n} \to (\mathcal{O}[R_n] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]]) \otimes_{\phi_{n,\infty}} H^1_{\mathrm{et}}(Y(1,N_\psi p^\infty(n^2))_{\overline{\mathbb{Q}}},\mathcal{O}(1))^{\mathrm{ord}}.$$

The test vectors $\breve{g}_\alpha$ and $\breve{h}_\alpha$ determine degeneracy maps

$$H^1_{\mathrm{et}}(\tilde{Y}(1)_{\overline{\mathbb{Q}}},\mathscr{L}_{r_2}(1)) \xrightarrow{\mu_p^*} H^1_{\mathrm{et}}(Y_1(Np)_{\overline{\mathbb{Q}}},\mathscr{L}_{r_2}(1)) \to H^1_{\mathrm{et}}(Y_1(N_g)_{\overline{\mathbb{Q}}},\mathscr{L}_{r_2}(1))$$
$$H^1_{\mathrm{et}}(\tilde{Y}(1)_{\overline{\mathbb{Q}}},\mathscr{L}_{r_3}(1)) \xrightarrow{\mu_p^*} H^1_{\mathrm{et}}(Y_1(Np)_{\overline{\mathbb{Q}}},\mathscr{L}_{r_3}(1)) \to H^1_{\mathrm{et}}(Y_1(N_h)_{\overline{\mathbb{Q}}},\mathscr{L}_{r_3}(1)).$$

Composing these maps with projection to the g-isotypic and h-isotypic quotient, respectively, we obtain

$$\pi_g : e'_{\mathrm{ord}} H^1(\tilde{\Gamma}(1),L_{r_2}(1)) \otimes_{\mathbb{Z}_p} \mathcal{O} \longrightarrow T_g$$
$$\pi_h : e'_{\mathrm{ord}} H^1(\tilde{\Gamma}(1),L_{r_3}(1)) \otimes_{\mathbb{Z}_p} \mathcal{O} \longrightarrow T_h.$$

For the ease of notation, we write

$$H^1(\psi,\bar{\mathfrak{f}},n) = (\mathcal{O}[R_{\bar{\mathfrak{f}},n}] \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma_{\mathfrak{p}}]]) \otimes_{\phi_{n,\infty}} H^1_{\mathrm{et}}(Y(1,N_\psi p^\infty(n^2))_{\overline{\mathbb{Q}}},\mathcal{O})^{\mathrm{ord}}(\langle\epsilon_N^{-1};1\rangle')(\boldsymbol{\kappa}^{-1/2})$$

and put $H^1(\psi,n) = \mathcal{O}[R_n] \otimes_{\mathcal{O}[R_{\bar{\mathfrak{f}},n}]} H^1(\psi,\bar{\mathfrak{f}},n)$. Then we define the class

$$\kappa^{(3)}_{n,\mathbf{f}gh} = (\pi_{\mathbf{f}^*} \otimes \pi_g \otimes \pi_h) \circ (\mathfrak{s}_{\mathbf{f}*} \otimes \mathfrak{s}_{r_2*} \otimes \mathfrak{s}_{r_3*}) \kappa^{(2)}_{n,\mathbf{f}gh} \tag{6.3}$$

lying in the group

$$H^1\left(\mathbb{Q}, H^1(\psi,n) \hat{\otimes}_{\mathcal{O}} (T_g \otimes_{\mathcal{O}} T_h) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-(r_2+r_3)/2)\right).$$

Let Γ_{ac} be the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K. We can identify this group with the antidiagonal in $(1+p\mathbb{Z}_p) \times (1+p\mathbb{Z}_p) \cong \mathcal{O}^{(1)}_{K,\mathfrak{p}} \times \mathcal{O}^{(1)}_{K,\bar{\mathfrak{p}}}$ via the geometrically normalised Artin map. Let $\kappa_{\mathrm{ac}} : \Gamma_{\mathrm{ac}} \to \mathbb{Z}_p^\times$ be the character defined by mapping $((1+p)^{-1/2},(1+p)^{1/2})$ to $1+p$, and let $\boldsymbol{\kappa}_{\mathrm{ac}} : \Gamma_{\mathrm{ac}} \to \Lambda^\times$ be the character defined by mapping $((1+p)^{-1/2},(1+p)^{1/2})$ to the group-like element $[1+p]$. We use the same

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notation for the corresponding characters of $G_{\mathbb{Q}}$. There is a $G_{\mathbb{Q}}$-equivariant isomorphism of $\Lambda_{\mathcal{O}}[R_n]$-modules

$$H^1(\psi,n) \cong \mathrm{Ind}_{K[n]}^{\mathbb{Q}} \Lambda_{\mathcal{O}}(\psi_{\mathfrak{P}}^{-1}\kappa_{\mathrm{ac}}^{r_1/2}\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})(-r_1/2). \tag{6.4}$$

Let

$$T_{g,h}^{\psi} = T_g \otimes_{\mathcal{O}} T_h(\psi_{\mathfrak{P}}^{-1})(-1-r), \quad V_{g,h}^{\psi} = T_{g,h}^{\psi} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In light of the isomorphism (6.4), using Shapiro's lemma, the classes $\kappa_{n,\mathbf{f}gh}^{(3)}$ yield classes

$$\tilde{\kappa}_{\psi,g,h,n,\infty} \in H^1(K[n],\Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\hat{\otimes}_{\mathcal{O}} T_{g,h}^{\psi}(\kappa_{\mathrm{ac}}^{r_1/2})) \otimes_{\mathcal{O}} E \tag{6.5}$$

for every squarefree integer n coprime to p and N.

Proposition 6.3. *Let n be as above, and let q be a rational prime coprime to p, N and n. Then:*

(i) *If q splits in K as $(q) = \mathfrak{q}\overline{\mathfrak{q}}$,*

$$\begin{aligned}\mathrm{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq,\infty}) = q^{l+m-4}\Bigg\{ & \chi_g(q)\chi_h(q)q\left(\frac{\kappa_{\mathrm{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})}{q^{k-1}}\mathrm{Fr}_{\mathfrak{q}}^{-1}\right)^2 \\ & - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}}\left(\frac{\kappa_{\mathrm{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})}{q^{k-1}}\mathrm{Fr}_{\mathfrak{q}}^{-1}\right) \\ & + \frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-2}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-1}} - \frac{q^2+1}{q} \\ & - \frac{a_q(g)a_q(h)}{q^{(l+m-4)/2}}\left(\frac{\kappa_{\mathrm{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1})}{q^{k-1}}\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1}\right) \\ & + \chi_g(q)\chi_h(q)q\left(\frac{\kappa_{\mathrm{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1})}{q^{k-1}}\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1}\right)^2\Bigg\}\tilde{\kappa}_{\psi,g,h,n,\infty}.\end{aligned}$$

(ii) *If q is inert in K,*

$$\begin{aligned}&\mathrm{cor}_{K[nq]/K[n]}(\tilde{\kappa}_{\psi,g,h,nq,\infty}) \\ &\quad = q^{l+m-4}\left\{\frac{\chi_g(q)^{-1}a_q(g)^2}{q^{l-2}} + \frac{\chi_h(q)^{-1}a_q(h)^2}{q^{m-1}} - \frac{(q+1)^2}{q}\right\}\tilde{\kappa}_{\psi,g,h,n,\infty}.\end{aligned}$$

Proof. The proof of this proposition is similar to the proof of Proposition 4.5. We just remark that the maps $\mathbf{s}_{\mathbf{f},n*}$ interchange the degeneracy maps π_1 and π_2, and under the isomorphism

$$\begin{aligned}&H^1(K[n],\Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\hat{\otimes}_{\mathcal{O}} T_{g,h}^{\psi}(\kappa_{\mathrm{ac}}^{(k-2)/2})) \otimes_{\mathcal{O}} E \\ &\qquad \cong H^1(\mathbb{Q}, H^1(\psi,n)\hat{\otimes}_{\mathcal{O}}(T_g \otimes_{\mathcal{O}} T_h) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(-1-(r_2+r_3)/2))\end{aligned}$$

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arising from (6.4), the corestriction $\mathrm{cor}_{K[nq]/K[n]}$ corresponds, in the case where $(q) = \mathfrak{q}\overline{\mathfrak{q}}$ splits in K, to the map

$$\mathcal{N}_n^{nq} = \pi_{11*} - \chi^{-1}(q)\omega^{(k-2)/2}(q)\left(\frac{\kappa_{\mathrm{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})[\mathfrak{q}]}{q^{k/2}} + \frac{\kappa_{\mathrm{ac}}^{-(k-2)/2}\psi_{\mathfrak{P}}(\mathrm{Fr}_{\overline{\mathfrak{q}}}^{-1})[\overline{\mathfrak{q}}]}{q^{k/2}}\right)\pi_{21*} + \frac{\chi^{-1}(q)\omega^{k-2}(q)}{q}\pi_{22*},$$

and similarly in the case where q is inert in K. Since the result can be deduced from Lemma 6.1 by virtually the same calculation as in the proof of Lemma 4.4, we omit the details. □

Definition 6.4. For any E-valued G_K-representation V, put

$$H^1_{\mathrm{Iw}}(K[np^\infty],T) := \varprojlim_r H^1(K[np^r],T), \quad H^1_{\mathrm{Iw}}(K[np^\infty],V) := H^1_{\mathrm{Iw}}(K[np^\infty],T)\otimes_{\mathcal{O}} E,$$

where $T \subset V$ is a Galois stable $\mathcal{O}$-lattice.

By another application of Shapiro's lemma, the classes $\kappa_{\psi,g,h,n,\infty}$ in (6.5) naturally live in $H^1_{\mathrm{Iw}}(K[np^\infty],V_{g,h}^{\psi}(\kappa_{\mathrm{ac}}^{(k-2)/2}))$. We thus arrive at the following theorem, which is the main result of this section.

Theorem 6.5. *Suppose that:*

- $l \geq m \geq 2$ *have the same parity and* $k \geq 2$ *is even,*
- p *splits in* K*,*
- p *does not divide the class number of* K*.*

Let $\mathcal{S}$ *be the set of squarefree products of primes* q *which split in* K *and are coprime to* p *and* N*. Assume that* $H^1(K[np^s],T_{g,h}^{\psi})$ *is torsion-free for every* $n \in \mathcal{S}$ *and for every* $s \geq 0$*. There exists a collection of classes*

$$\left\{\kappa_{\psi,g,h,n,\infty} \in H^1_{\mathrm{Iw}}(K[np^\infty],T_{g,h}^{\psi}) : n \in \mathcal{S}\right\},$$

such that whenever $n,nq \in \mathcal{S}$ *with* q *a prime, we have*

$$\mathrm{cor}_{K[nq]/K[n]}(\kappa_{\psi,g,h,nq,\infty}) = P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\,\kappa_{\psi,g,h,n,\infty},$$

where $\mathfrak{q}$ *is any of the primes of* K *above* q*, and* $P_{\mathfrak{q}}(X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1}X|(V_{g,h}^{\psi})^\vee(1))$*.*

Proof. The same argument as in the proof of Theorem 4.6 (but using Proposition 6.3) yields a system of Iwasawa cohomology classes with the stated norm-compatibilities for the representation $V_{g,h}^{\psi}(\kappa_{\mathrm{ac}}^{(k-2)/2})$. By the twisting result of [Rub00, Theorem 6.3.5], the theorem follows. □

We conclude this section by proving that the classes $\kappa_{\psi,g,h,n,\infty}$ land in the *balanced Selmer group*

$$\mathrm{Sel}_{\mathrm{bal}}(K[np^\infty],T_{g,h}^{\psi}) := \varprojlim_r \mathrm{Sel}_{\mathrm{bal}}(K[np^r],T_{g,h}^{\psi});$$

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in the terminology introduced in Section 8.1 below, this is the same as the Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K[np^\infty],T^\psi_{g,h})$ associated to the G_{K_v}-invariant subspaces $\mathcal{F}^+_v(V^\psi_{g,h}) \subset V^\psi_{g,h}$ in (4.1) at the primes $v \mid p$.

Proposition 6.6. *For all $n \in \mathcal{S}$, we have $\kappa_{\psi,g,h,n,\infty} \in \mathrm{Sel}_{\mathrm{bal}}(K[np^\infty],T^\psi_{g,h})$.*

Proof. Let $v \nmid p$ be a finite prime of $K[np^\infty]$, and for every $r \geq 0$, denote also by v the prime of $K[np^r]$ below v. As in the proof of Proposition 4.9, we have

$$H^1(K[np^r]_v,V^\psi_{g,h}) = H^1_{\mathrm{Gr}}(K[np^r]_v,V^\psi_{g,h}) = 0,$$

and hence

$$H^1(K[np^r]_v,T^\psi_{g,h}) = H^1(K[np^r]_v,T^\psi_{g,h})_{\mathrm{tors}} = H^1_{\mathrm{Gr}}(K[np^r]_v,T^\psi_{g,h}),$$

where the first equality follows from the local Euler characteristic formula. Hence, the inclusion $\mathrm{res}_v(\kappa_{\psi,g,h,n,\infty}) \in \varprojlim_r H^1_{\mathrm{Gr}}(K[np^r]_v,T^\psi_{g,h})$ follows. Since, by [BSV22, Corollary 8.2], it follows that the classes $\kappa_{\psi,g,h,n,\infty}$ satisfy the balanced local condition at the primes above p, this concludes the proof. □

Part 2. Arithmetic applications

7. Iwasawa main conjectures

In this section, we formulate Iwasawa main conjectures (IMCs) for triple products of modular forms. We give two formulations: one in terms of the triple product p-adic L-function (Conjecture 7.7) and another in terms of diagonal cycle classes (Conjecture 7.9). In Theorem 7.15, we establish the equivalence of the two formulations.

7.1. Triple product p-adic L-function

Fix a triple $(\mathbf{f},g,h)$ consisting of a primitive Hida family $\mathbf{f}$ of tame level $N_{\mathbf{f}}$ and character $\chi_{\mathbf{f}}$ and two p-ordinary newforms g,h of weights $l,m \geq 2$, levels N_g,N_h prime-to-p and nebentypus χ_g,χ_h. Assume that $\mathbf{f}$ has coefficients in a ring $\Lambda_{\mathbf{f}}$ as in Section 5.1. Assume that $\chi_{\mathbf{f}}\chi_g\chi_h = \omega^{r_1}$ for some even integer r_1, and put

$$N = \mathrm{lcm}(N_{\mathbf{f}},N_g,N_h).$$

Let $\mathbf{g}$ and $\mathbf{h}$ be primitive Hida families with coefficients in $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ passing through g and h, respectively. More precisely, there exist arithmetic points $y_0 \in \mathcal{W}_{\Lambda_{\mathbf{g}}}(\overline{\mathbb{Q}}_p)$ and $z_0 \in \mathcal{W}_{\Lambda_{\mathbf{h}}}(\overline{\mathbb{Q}}_p)$, such that g_{y_0} and h_{z_0} are the ordinary p-stabilisations of g and h, respectively. The rings $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ need not be regular. However, for our purposes, we can consider the Λ-adic families, denoted again $\mathbf{g}$ and $\mathbf{h}$, that result from embedding $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ in the rings of functions of suitable wide open connected subsets $U_{\mathbf{g}}$ and $U_{\mathbf{h}}$ of $\mathcal{W}(\overline{\mathbb{Q}}_p) = \mathrm{Spf}(\Lambda)(\overline{\mathbb{Q}}_p)$ defined over some finite extension E of $\mathbb{Q}_p$ and containing the points y_0 and z_0, respectively. From now on, it is these rings of functions that we will denote by $\Lambda_{\mathbf{g}}$

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and $\Lambda_{\mathbf{h}}$. These rings are now noncanonically isomorphic to $\mathcal{O}[[T]]$, where $\mathcal{O}$ is the ring of integers of E; in particular, they are regular. Let $\boldsymbol{l}-l$ and $\boldsymbol{m}-m$ be generators in $\Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ of the prime ideals corresponding to the points y_0 and z_0, respectively.

We can and will assume that $\Lambda_{\mathbf{f}}$ is a finite flat extension of $\Lambda_{\mathcal{O}}$, and we will only consider arithmetic points in $\mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ lying in $\mathrm{Hom}_{\mathrm{cont},\mathcal{O}}(\Lambda_{\mathbf{f}},\overline{\mathbb{Q}}_p)$

Recall that in Section 5.5, we defined a character $\kappa_{r_1}:\mathbb{Z}_p^\times \to \Lambda^\times$ given by $u \mapsto \omega^{r_1}(u)[\langle u\rangle]$, and in Section 6, we fixed a square root $\kappa_{r_1}^{1/2}$ of this character given by $u \mapsto \omega^{r_1/2}(u)[\langle u\rangle^{1/2}]$. We let $\kappa_{\mathbf{f}}$ and $\kappa_{\mathbf{f}}^{1/2}$ be the composition of κ_{r_1} and $\kappa_{r_1}^{1/2}$, respectively, with the embedding $\Lambda^\times \hookrightarrow \Lambda_{\mathbf{f}}^\times$. We also define a character $\kappa_{\mathbf{gh}}:\mathbb{Z}_p^\times \to (\Lambda_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\Lambda_{\mathbf{h}})^\times$ by

$$\kappa_{\mathbf{gh}}(u) = \omega(u)^{l+m-4}\langle u\rangle^{\boldsymbol{l}+\mathbf{m}-4}$$

and choose a square root of this character defined by $\kappa_{\mathbf{gh}}^{1/2}(u) = \omega(u)^{(l+m-4)/2}\langle u\rangle^{(\boldsymbol{l}+\mathbf{m}-4)/2}$. Let $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\Lambda_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\Lambda_{\mathbf{h}}$, and consider the $\Lambda_{\mathbf{fgh}}[G_{\mathbb{Q}}]$-module

$$\mathbb{V}_{\mathbf{fgh}}^\dagger := \mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{h}}(\Xi_{\mathbf{fgh}}), \quad \text{where } \Xi_{\mathbf{fgh}} = \epsilon_{\mathrm{cyc}}^{-1}\kappa_{\mathbf{f}}^{-1/2}\kappa_{\mathbf{gh}}^{-1/2} \tag{7.1}$$

and $\mathbb{V}_{\mathbf{f}}$, $\mathbb{V}_{\mathbf{g}}$ and $\mathbb{V}_{\mathbf{h}}$ are the big Galois representations attached to $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, respectively. Then $\mathbb{V}_{\mathbf{fgh}}^\dagger$ is a self-dual twist of the tensor product of these representations. Consider also the $\Lambda_{\mathbf{f}}[G_{\mathbb{Q}}]$-module

$$\mathbb{V}_{\mathbf{f}gh}^\dagger := \mathbb{V}_{\mathbf{f}}\otimes_{\mathcal{O}} T_g \otimes_{\mathcal{O}} T_h(\Xi_{\mathbf{f}gh}), \quad \text{where } \Xi_{\mathbf{f}gh} = \epsilon_{\mathrm{cyc}}^{(2-l-m)/2}\kappa_{\mathbf{f}}^{-1/2}.$$

Given test vectors $(\breve{\mathbf{f}},\breve{g},\breve{h})$ for $(\mathbf{f},g,h)$ of level N, as explained in [HT01] and [DR14, Section 4.2], a generalisation of Hida's p-adic Rankin–Selberg convolution produces an element $\mathscr{L}_p(\breve{\mathbf{f}},\breve{g},\breve{h})$ in the fraction field of $\Lambda_{\mathbf{f}}$ whose specialisations to arithmetic points $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of even weight $k \geq l+m$ recover (a square root of) the central critical values of the triple product L-function $L(\mathbb{V}_{\mathbf{f}_x gh}^\dagger, s)$ for the specialisation of $\mathbb{V}_{\mathbf{f}gh}^\dagger$ at x by virtue of Harris–Kudla's proof of Jacquet's conjecture, [HK91]. A recent result by Hsieh [Hsi21] constructs test vectors $(\underline{\breve{\mathbf{f}}},\underline{\breve{g}},\underline{\breve{h}})$ for which a precise interpolation property for the resulting $\mathscr{L}_p(\underline{\breve{\mathbf{f}}},\underline{\breve{g}},\underline{\breve{h}})$ is proved. To recall the result in the form that will be used here, for any arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ as above, we set

$$\mathbf{f}_k := \mathbf{f}_x, \quad \alpha_k := a_p(\mathbf{f}_k), \quad \beta_k := \chi_{\mathbf{f}}(p)p^{k-1}\alpha_k^{-1},$$

let α_g,β_g be the roots of the Hecke polynomial of g at p with $\mathrm{ord}_p(\alpha_g) = 0$, and define α_h,β_h similarly. As recalled in [*op. cit.*, Section 1.4], when the residual Galois representation $\bar{\rho}_{\mathbf{f}}$ associated to $\mathbf{f}$ is absolutely irreducible and p-distinguished, the local ring $\Lambda_{\mathbf{f}}$ is known to be Gorenstein and, by a result of Hida's the congruence module of $\mathbf{f}$, is isomorphic to $\Lambda_{\mathbf{f}}/(\xi)$ for some nonzero $\xi \in \Lambda_{\mathbf{f}}$. We call (ξ) the *congruence ideal* of $\mathbf{f}$. Finally, denote by $\varepsilon_\ell(\mathbb{V}_{\mathbf{f}_k gh}^\dagger) \in \{\pm 1\}$ the epsilon factor of the Weil–Deligne representation attached to the restriction of $\mathbb{V}_{\mathbf{f}_k gh}^\dagger$ to $G_{\mathbb{Q}_\ell}$.

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Theorem 7.1. *In addition to $\chi_{\mathbf{f}}\chi_g\chi_h = \omega^{r_1}$, assume that:*

(a) *$\bar{\rho}_{\mathbf{f}}$ is absolutely irreducible and p-distinguished,*

(b) *for some arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$, we have $\varepsilon_\ell(\mathbb{V}^\dagger_{\mathbf{f}_k gh}) = +1$ for all primes $\ell \mid N$,*

(c) *$\gcd(N_{\mathbf{f}}, N_g, N_h)$ is squarefree.*

Let ξ be a generator of the congruence ideal of $\mathbf{f}$. There exist test vectors $(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})$ for $(\mathbf{f}, g, h)$ of level N, and an element

$$\mathscr{L}_p^\xi(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}}) \in \Lambda_{\mathbf{f}},$$

such that for all arithmetic points $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of even weight $k \geq l+m$ with $k \equiv r_1 + 2 \pmod{2(p-1)}$ we have

$$\left(\frac{\mathscr{L}_p^\xi(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})(x)}{\xi_x}\right)^2 = \frac{\Gamma(k,l,m)}{2^{\alpha(k,l,m)}} \cdot \frac{\mathcal{E}(\mathbf{f}_k, g, h)^2}{\mathcal{E}_0(\mathbf{f}_k)^2 \cdot \mathcal{E}_1(\mathbf{f}_k)^2} \cdot \prod_{\ell \mid N} \tau_\ell \cdot \frac{L(\mathbb{V}^\dagger_{\mathbf{f}_k gh}, 0)}{\pi^{2k} \cdot \langle \mathbf{f}_k^\sharp, \mathbf{f}_k^\sharp \rangle^2},$$

where:

- $\Gamma(k,l,m) = (c-1)! \cdot (c-m)! \cdot (c-l)! \cdot (c+1-l-m)!$, *with* $c = (k+l+m-2)/2$,
- $\alpha(k,l,m) \in \Lambda_{\mathbf{f}}$ *is a linear form in the variables k, l, m,*
- $\mathcal{E}(\mathbf{f}_k, g, h) = (1 - \frac{\beta_k \alpha_g \alpha_h}{p^c})(1 - \frac{\beta_k \beta_g \alpha_h}{p^c})(1 - \frac{\beta_k \alpha_g \beta_h}{p^c})(1 - \frac{\beta_k \beta_g \beta_h}{p^c})$,
- $\mathcal{E}_0(\mathbf{f}_k) = (1 - \frac{\beta_k}{\alpha_k})$, $\mathcal{E}_1(\mathbf{f}_k) = (1 - \frac{\beta_k}{p\alpha_k})$,
- τ_ℓ *is an explicit nonzero rational number independent of k,*
- $\mathbf{f}_k^\sharp$ *is the newform associated to the p-stabilised newform $\mathbf{f}_k$,*

and $\|\mathbf{f}_k^\sharp\|^2$ is the Petersson norm of $\mathbf{f}_k^\sharp$.

Proof. Letting $\mathbf{g}, \mathbf{h}$ be the primitive Hida families of tame level N_g, N_h passing through the ordinary p-stabilisations of g, h, this follows by specialising the three-variable p-adic L-function in [Hsi21, Theorem A] attached to $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ and the congruence ideal generator ξ. □

Definition 7.2. For the test vectors $(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})$ of level N provided by Theorem 7.1, we set

$$L_p(\mathbf{f}, g, h) := \mathscr{L}_p^\xi(\check{\underline{\mathbf{f}}}, \check{\underline{g}}, \check{\underline{h}})^2,$$

where ξ is any fixed generator of the congruence ideal of $\mathbf{f}$.

Note that $L_p(\mathbf{f}, g, h)$ depends on the choice of ξ, but the principal ideal in $\Lambda_{\mathbf{f}}$ it generates is of course independent of that choice.

7.2. Reciprocity law for diagonal cycles

Keep the notations in the previous subsection and without loss of generality assume that $l \geq m$ (reordering g and h if necessary).

Assume that the Galois representations attached to $\mathbf{f}$, g and h are all residually irreducible and p-distinguished. Let $\boldsymbol{\phi} \in \{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$. As a $G_{\mathbb{Q}_p}$-representation, $\mathbb{V}_{\boldsymbol{\phi}}$ admits a filtration

$$0 \to \mathbb{V}^+_{\boldsymbol{\phi}} \to \mathbb{V}_{\boldsymbol{\phi}} \to \mathbb{V}^-_{\boldsymbol{\phi}} \to 0 \tag{7.2}$$

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with each $\mathbb{V}_{\boldsymbol{\phi}}^{\pm}$ free of rank one over $\Lambda_{\boldsymbol{\phi}}$, and with the $G_{\mathbb{Q}_p}$-action on $\mathbb{V}_{\boldsymbol{\phi}}^{-}$ given by the unramified character sending $\mathrm{Fr}_p \mapsto a_p(\boldsymbol{\phi})$. This induces an obvious three-step filtration

$$0 \subset \mathscr{F}^3\mathbb{V}^{\dagger}_{\mathbf{fgh}} \subset \mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{fgh}} \subset \mathscr{F}^1\mathbb{V}^{\dagger}_{\mathbf{fgh}} \subset \mathbb{V}^{\dagger}_{\mathbf{fgh}}$$

by $G_{\mathbb{Q}_p}$-stable $\Lambda_{\mathbf{fgh}}$-submodules of ranks 1, 4 and 7, respectively, given by

$$\begin{aligned}
\mathscr{F}^1\mathbb{V}^{\dagger}_{\mathbf{fgh}} &= (\mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{h}})(\Xi_{\mathbf{fgh}}),\\
\mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{fgh}} &= (\mathbb{V}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{h}} + \mathbb{V}^{+}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}_{\mathbf{h}})(\Xi_{\mathbf{fgh}}),\\
\mathscr{F}^3\mathbb{V}^{\dagger}_{\mathbf{fgh}} &= \mathbb{V}^{+}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{h}}(\Xi_{\mathbf{fgh}}).
\end{aligned} \tag{7.3}$$

The middle term $\mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ will play a special role in the following, and we note that

$$\mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{fgh}}/\mathscr{F}^3\mathbb{V}^{\dagger}_{\mathbf{fgh}} \cong \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}} \oplus \mathbb{V}^{\mathbf{fh}}_{\mathbf{g}} \oplus \mathbb{V}^{\mathbf{fg}}_{\mathbf{h}}, \tag{7.4}$$

where $\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}} := (\mathbb{V}^{-}_{\mathbf{f}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{g}}\hat{\otimes}_{\mathcal{O}}\mathbb{V}^{+}_{\mathbf{h}})(\Xi_{\mathbf{fgh}})$ and similarly for the other two direct summands. We similarly denote the induced subquotients on the specialisations of $\mathbb{V}^{\dagger}_{\mathbf{fgh}}$ (that is, $\mathscr{F}^i\mathbb{V}^{\dagger}_{\mathbf{f}gh}, \mathbb{V}^{gh}_{\mathbf{f}}$, etc.).

Consider the class $\kappa^{(3)}_{1,\mathbf{f}gh}$ defined in (6.3) for the choice of level-N test vectors $(\breve{\underline{\mathbf{f}}},\breve{\underline{g}},\breve{\underline{h}})$ given by Theorem 7.1, and let $\kappa(\mathbf{f},g,h) \in H^1(\mathbb{Q},\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ be the image of this class via the morphism obtained from the augmentation map $\mathcal{O}[R_1] \to \mathcal{O}$. By [BSV22, Corollary 8.2], the image of $\kappa(\mathbf{f},g,h)$ under the restriction map at p is contained in

$$H^1_{\mathrm{bal}}(\mathbb{Q}_p,\mathbb{V}^{\dagger}_{\mathbf{f}gh}) := \mathrm{im}\big(H^1(\mathbb{Q}_p,\mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{f}gh}) \to H^1(\mathbb{Q}_p,\mathbb{V}^{\dagger}_{\mathbf{f}gh})\big).$$

It is easily seen that this map is an injection, so we may and will view $\mathrm{res}_p(\kappa(\mathbf{f},g,h))$ as a class in $H^1(\mathbb{Q}_p,\mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{f}gh})$. Let

$$\mathrm{pr}_{(\mathbf{f},g,h)} : \mathscr{F}^2\mathbb{V}^{\dagger}_{\mathbf{f}gh} \longrightarrow \mathbb{V}^{gh}_{\mathbf{f}}$$

be the map induced by the projection onto the first direct summand in (7.4). The 'reciprocity law' from [BSV22, DR22] recalled in Theorem 7.4 below relates the image of $\mathrm{res}_p(\kappa(\mathbf{f},g,h))$ under the natural projection

$$\mathrm{pr}_{(\mathbf{f},g,h)*} : H^1_{\mathrm{bal}}(\mathbb{Q}_p,\mathbb{V}^{\dagger}_{\mathbf{f}gh}) \longrightarrow H^1(\mathbb{Q}_p,\mathbb{V}^{gh}_{\mathbf{f}})$$

to the triple product p-adic L-function of Section 7.1. Recall that $\xi \in \Lambda_{\mathbf{f}}$ denotes a generator of the congruence ideal of $\mathbf{f}$.

Proposition 7.3. *There is an injective* $\Lambda_{\mathbf{f}}$*-module homomorphism with pseudo-null cokernel*

$$\mathfrak{Log}^{\xi} : H^1(\mathbb{Q}_p,\mathbb{V}^{gh}_{\mathbf{f}}) \longrightarrow \Lambda_{\mathbf{f}}$$

characterised by the following interpolation property: for all $\mathfrak{Z} \in H^1(\mathbb{Q}_p,\mathbb{V}^{gh}_{\mathbf{f}})$ *and all classical points* $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ *of weight* $k \geq l+m$ *with* $k \equiv r_1+2 (\mathrm{mod}\ 2(p-1))$*, we have*

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$$\frac{\mathfrak{Log}^{\xi}(\mathfrak{Z})(x)}{\xi_x} = (p-1)\alpha_k\left(1-\frac{\beta_k\alpha_g\alpha_h}{p^c}\right)\left(1-\frac{\alpha_k\beta_g\beta_h}{p^c}\right)^{-1} \times \begin{cases} \frac{(-1)^{c-k}}{(c-k)!}\cdot\left\langle \log_p(\mathfrak{Z}_k), \eta_{\mathbf{f}_k}\otimes\omega_{\mathbf{g}_l}\otimes\omega_{\mathbf{h}_m}\right\rangle_{\mathrm{dR}}, & \text{if } l-m<k<l+m, \\ (k-c-1)!\cdot\left\langle \exp_p^*(\mathfrak{Z}_k), \eta_{\mathbf{f}_k}\otimes\omega_{\mathbf{g}_l}\otimes\omega_{\mathbf{h}_m}\right\rangle_{\mathrm{dR}}, & \text{if } k\geq l+m, \end{cases}$$

where $c=(k+l+m-2)/2$.

Proof. The construction of $\mathfrak{Log}^{\xi}$ will follow by specialising the three-variable p-adic regulator constructed in [BSV22, Section 7.1] (building on a generalisation of the construction in [LZ14] given by Kings–Loeffler–Zerbes [KLZ17]).

Let $\vartheta_{gh}:\Lambda_{\mathbf{fgh}}\to\Lambda_{\mathbf{f}}$ be the map given by reduction modulo $(\boldsymbol{l}-l,\boldsymbol{m}-m)$. This induces isomorphisms

$$\mathbb{V}^{\dagger}_{\mathbf{fgh}}\otimes_{\Lambda_{\mathbf{fgh}}}\Lambda_{\mathbf{f}}\cong\mathbb{V}^{\dagger}_{\mathbf{f}gh},\quad \mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}\otimes_{\Lambda_{\mathbf{fgh}}}\Lambda_{\mathbf{f}}\cong\mathbb{V}^{gh}_{\mathbf{f}},$$

and a natural map

$$\vartheta_{gh*}: H^1(\mathbb{Q}_p,\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}})\otimes_{\Lambda_{\mathbf{fgh}}}\Lambda_{\mathbf{f}}\longrightarrow H^1(\mathbb{Q}_p,\mathbb{V}^{gh}_{\mathbf{f}}).$$

This map is clearly injective, and its surjectivity can be shown easily by an application of local Tate duality and the Ramanujan–Petersson conjecture (*cf.* proof of [BSV22, (154)]). Letting

$$\mathscr{L}^{\xi}_{\mathbf{f}}: H^1(\mathbb{Q}_p,\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}})\longrightarrow\Lambda_{\mathbf{fgh}}$$

be the p-adic regulator $\mathscr{L}_{\mathbf{f}}$ defined as in [BSV22, Proposition 7.3] and multiplied by ξ, the map defined by the composition

$$\mathfrak{Log}^{\xi}: H^1(\mathbb{Q}_p,\mathbb{V}^{gh}_{\mathbf{f}})\xrightarrow{\vartheta^{-1}_{gh*}} H^1(\mathbb{Q}_p,\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}})\otimes_{\Lambda_{\mathbf{fgh}}}\Lambda_{\mathbf{f}}\xrightarrow{\mathscr{L}^{\xi}_{\mathbf{f}}\otimes\mathrm{id}}\Lambda_{\mathbf{f}}$$

satisfies the interpolation properties in the statement of the proposition.

It remains to see that $\mathfrak{Log}^{\xi}$ is injective with pseudo-null cokernel. By definition, we have

$$\mathbb{V}^{\mathbf{gh}}_{\mathbf{f}}=\mathbb{U}^{\mathbf{gh}}_{\mathbf{f}}(\epsilon_{\mathrm{cyc}}\kappa_{\mathbf{f}}^{-1/2}\kappa_{\mathbf{gh}}^{1/2}),$$

where $\mathbb{U}^{\mathbf{gh}}_{\mathbf{f}}$ is an unramified $G_{\mathbb{Q}_p}$-module on which an arithmetic Frobenius Fr_p acts as multiplication by $\chi_f^{-1}(p)a_p(\mathbf{f})a_p(\mathbf{g})^{-1}a_p(\mathbf{h})^{-1}$, and $\mathscr{L}_{\mathbf{f}}$ is obtained by specialising the four-variable p-adic regulator map in [KLZ17, Theorem 8.2.3] for the module $\mathbb{U}^{\mathbf{gh}}_{\mathbf{f}}$, paired against the differential $\eta_{\mathbf{f}}\otimes\omega_{\mathbf{g}}\otimes\omega_{\mathbf{h}}$. In light of [KLZ17, Remark 8.2.4], the fact that $\mathfrak{Log}^{\xi}$ has the above properties can therefore be deduced from the vanishing of $H^0(\mathbb{Q}_p,\mathbb{U}^{gh}_{\mathbf{f}})$, where $\mathbb{U}^{gh}_{\mathbf{f}}$ is the image of $\mathbb{U}^{\mathbf{gh}}_{\mathbf{f}}$ under ϑ_{gh}. □

Theorem 7.4 (Reciprocity law). *We have the following equality*

$$\mathfrak{Log}^{\xi}(\mathrm{res}_p(\kappa(\mathbf{f},g,h)))=\mathscr{L}^{\xi}_p(\breve{\underline{\mathbf{f}}},\breve{\underline{g}},\breve{\underline{h}}).$$

Proof. This is the specialisation of the three-variable reciprocity law of Theorem A in [BSV22] to $(\mathbf{f},g,h)$ (see also [DR22, Theorem 10]). □

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7.3. Selmer groups and formulation of the main conjectures

Let $(\mathbf{f},g,h)$ be as in the preceding subsection. Throughout the rest of this section, we assume that hypotheses (a)–(c) in Theorem 7.1 hold, so the p-adic L-function $L_p(\mathbf{f},g,h)$ in Definition 7.2 is available.

Recall the $G_{\mathbb{Q}_p}$-stable rank-four $\Lambda_{\mathbf{fgh}}$-submodule $\mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{fgh}} \subset \mathbb{V}^\dagger_{\mathbf{fgh}}$ in (7.3), and set

$$\mathbb{V}^f_{\mathbf{fgh}} := \mathbb{V}^+_{\mathbf{f}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathcal{O}} \mathbb{V}_{\mathbf{h}}(\Xi_{\mathbf{fgh}}).$$

As before, we let $\mathscr{F}^2\mathbb{V}_{\mathbf{f}gh}$ and $\mathbb{V}^f_{\mathbf{f}gh}$ denote the corresponding specialisations.

Fix a finite set Σ of places of $\mathbb{Q}$ containing ∞ and the primes dividing Np, and let $\mathbb{Q}^\Sigma$ be the maximal extension of $\mathbb{Q}$ unramified outside Σ.

Definition 7.5. For $\mathcal{L} \in \{\mathrm{bal}, \mathcal{F}\}$ define the Selmer group $\mathrm{Sel}_{\mathcal{L}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ by

$$\mathrm{Sel}_{\mathcal{L}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) = \ker\left(H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh}) \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh})}{H^1_{\mathcal{L}}(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh})} \right),$$

where

$$H^1_{\mathcal{L}}(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) = \begin{cases} \ker\big(H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}/\mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{f}gh})\big) & \text{if } \mathcal{L} = \mathrm{bal}, \\ \ker\big(H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) \longrightarrow H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}/\mathbb{V}^f_{\mathbf{f}gh})\big) & \text{if } \mathcal{L} = \mathcal{F}. \end{cases}$$

We call $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ (respectively, $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$) the *balanced* (respectively, *f-unbalanced*) Selmer group.

Remark 7.6. The pairs

$$\big(\mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{f}gh}, \{k \in \mathbb{Z}_{\geq 2} : \ l-m < k < l+m\}\big), \quad \big(\mathbb{V}^f_{\mathbf{f}gh}, \{k \in \mathbb{Z} : \ k \geq l+m\}\big)$$

satisfy the *Panchishkin condition* in [Gre94]. Thus, $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ and $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ may be viewed as instances of Greenberg's Selmer groups attached to different ranges of critical specialisations of $\mathbb{V}^\dagger_{\mathbf{f}gh}$.

Let

$$\mathbb{A}^\dagger_{\mathbf{f}gh} = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{V}^\dagger_{\mathbf{f}gh}, \mu_{p^\infty}).$$

Then for $\mathcal{L} \in \{\mathrm{bal}, \mathcal{F}\}$, we define the Selmer groups $\mathrm{Sel}_{\mathcal{L}}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ as above, taking $H^1_{\mathcal{L}}(\mathbb{Q}_p, \mathbb{A}^\dagger_{\mathbf{f}gh})$ to be the orthogonal complement of $H^1_{\mathcal{L}}(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh})$ under the local Tate duality

$$H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) \times H^1(\mathbb{Q}_p, \mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

and set

$$X_{\mathcal{L}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) := \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Sel}_{\mathcal{L}}(\mathbb{A}^\dagger_{\mathbf{f}gh}), \mathbb{Q}_p/\mathbb{Z}_p).$$

In light of Remark 7.6, the next conjecture may be viewed as an instance of the Iwasawa–Greenberg main conjectures [Gre94]. In the two formulations below, we also

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assume conditions (b) and (c) from Theorem 7.1, so that the p-adic L-function $L_p(\mathbf{f},g,h)$ in (7.2) is defined.

Conjecture 7.7 (IMC 'with p-adic L-functions'). *The modules* $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ *and* $X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$ *are both* $\Lambda_{\mathbf{f}}$*-torsion, and*

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})) = (L_p(\mathbf{f},g,h))$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Remark 7.8. An integral formulation of the equality of ideals in Conjecture 7.7 would, in general, involve certain Tamagawa factors, accounting for the fact that the construction of $L_p(\mathbf{f},g,h)$ uses Hida's congruence number, while, by definition, the classes in the Selmer group $X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$ are trivial at the places $v \in \Sigma \setminus \{p,\infty\}$, rather than just unramified (*cf.* [PW11]).

Under the local root number hypothesis (b) in Theorem 7.1, for all arithmetic point $x \in \mathcal{W}_{\Lambda_{\mathbf{f}}}(\overline{\mathbb{Q}}_p)$ of even weight $k \geq 2$ with $l-m < k < l-m$, the sign in the functional equation for $L(\mathbb{V}^{\dagger}_{\mathbf{f}_x gh},s)$ is -1, so that the central value $L(\mathbb{V}^{\dagger}_{\mathbf{f}_x gh},0)$ vanishes. Therefore, in the spirit of Perrin-Riou's main conjecture [PR87, Conjecture B] in the setting of Heegner points, a natural formulation of the Iwasawa main conjecture for $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ takes the following form.

Note that it follows from [BSV22, Corollary 8.2] that $\kappa(\mathbf{f},g,h)$ lands in $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})$.

Conjecture 7.9 (IMC 'without p-adic L-functions'). *Suppose* $\kappa(\mathbf{f},g,h) \in \mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q},\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ *is not* $\Lambda_{\mathbf{f}}$*-torsion. Then the modules* $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ *and* $X_{\mathrm{bal}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$ *have both rank one, and*

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathrm{bal}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})_{\mathrm{tors}}) = \mathrm{Char}_{\Lambda_{\mathbf{f}}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f},g,h)}\right)^2$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, *where the subscript* tors *denotes the* $\Lambda_{\mathbf{f}}$*-torsion submodule.*

Remark 7.10. Working under different hypotheses on the local signs ensuring that $L(\mathbb{V}^{\dagger}_{\mathbf{f}_k gh},s)$ has sign $+1$ (rather than -1) for weights $k \geq 2$ with $l-m < k < l-m$, the Iwasawa main conjecture would relate the characteristic ideal of $X_{\mathrm{bal}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$ to the balanced triple product p-adic L-function constructed in [Hsi21, Theorem B] (see also [GS20]), rather than diagonal classes. In this setting, the f-unbalanced Selmer group $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ should have $\Lambda_{\mathbf{f}}$-rank one, but the expected nontorsion Selmer class seems to not have been constructed yet.

7.4. Equivalence of the formulations

In this subsection, we show that the two formulations of the Iwasawa main conjecture in the previous subsection are essentially equivalent, focusing on the case where $\mathbf{f}$ is a CM Hida family[1] as in Section 6. Similar equivalences between IMC 'with' and 'without'

[1]This case will suffice for our applications in this paper, and makes some of the arguments simpler, but we expect the equivalence to hold in general.

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p-adic L-functions appear in [Kat04, Section 17], [KLZ17, Section 11] and, in a setting more germane to ours, [Wan20] and [Cas17, Appendix].

The following intermediate Selmer groups will allow us to bridge between $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ and $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ in the comparison. Set

$$\mathbb{V}^{f\cap+}_{\mathbf{f}gh} = \mathbb{V}^f_{\mathbf{f}gh} \cap \mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{f}gh}, \quad \mathbb{V}^{f\cup+}_{\mathbf{f}gh} = \mathbb{V}^f_{\mathbf{f}gh} + \mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{f}gh},$$

which are $G_{\mathbb{Q}_p}$-stable $\Lambda_{\mathbf{f}}$-submodules of $\mathbb{V}^\dagger_{\mathbf{f}gh}$ of ranks 3 and 5, respectively. Define $\mathrm{Sel}_{\mathcal{L}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ for $\mathcal{L} \in \{\mathcal{F}\cap+, \mathcal{F}\cup+\}$ by the same recipe as in Definition 7.5, with

$$H^1_{\mathcal{L}}(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) = \begin{cases} \ker\bigl(H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) \to H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}/\mathbb{V}^{f\cap+}_{\mathbf{f}gh})\bigr) & \text{if } \mathcal{L} = \mathcal{F}\cap+, \\ \ker\bigl(H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}) \to H^1(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh}/\mathbb{V}^{f\cup+}_{\mathbf{f}gh})\bigr) & \text{if } \mathcal{L} = \mathcal{F}\cup+. \end{cases}$$

We define the Selmer groups $\mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ and $\mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ taking $H^1_{\mathcal{F}\cap+}(\mathbb{Q}_p, \mathbb{A}^\dagger_{\mathbf{f}gh})$ and $H^1_{\mathcal{F}\cup+}(\mathbb{Q}_p, \mathbb{A}^\dagger_{\mathbf{f}gh})$ to be the orthogonal complements of $H^1_{\mathcal{F}\cup+}(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh})$ and $H^1_{\mathcal{F}\cap+}(\mathbb{Q}_p, \mathbb{V}^\dagger_{\mathbf{f}gh})$, respectively. As in the preceding section, we also define the corresponding $X_{\mathcal{F}\cap+}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ and $X_{\mathcal{F}\cup+}(\mathbb{A}^\dagger_{\mathbf{f}gh})$.

Throughout this subsection, we keep the setting from Section 6. In particular, $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ is the CM Hida family in (6.2) associated with the Hecke character ψ of conductor $\mathfrak{f}$. In addition, we assume conditions (b) and (c) from Theorem 7.1, so the p-adic L-function $L_p(\mathbf{f}, g, h) \in \Lambda_{\mathbf{f}}$ is defined, and let $\kappa(\mathbf{f}, g, h) \in H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh})$ be as above.

For every height one prime $\mathfrak{Q}$ of $\Lambda_{\mathbf{f}}$ away from p, let $S_{\mathfrak{Q}}$ be the integral closure of $\Lambda_{\mathbf{f}}/\mathfrak{Q}$, and let $\Phi_{\mathfrak{Q}}$ be the fraction field of $S_{\mathfrak{Q}}$. Let $\mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}$ be the extension of scalars of $\mathbb{V}^\dagger_{\mathbf{f}gh}/\mathfrak{Q}\mathbb{V}^\dagger_{\mathbf{f}gh}$ to $S_{\mathfrak{Q}}$, and let $\mathbb{A}^\dagger_{\mathbf{f}gh,\mathfrak{Q}} = \mathrm{Hom}(\mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}, \mu_{p^\infty})$. Following [MR04], define

$$H^1_{\underline{\mathrm{bal}}}(\mathbb{Q}_v, \mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}) := \begin{cases} \ker\bigl(H^1(\mathbb{Q}_v, \mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}) \to H^1(\mathbb{Q}_v^{\mathrm{nr}}, \mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}} \otimes \Phi_{\mathfrak{Q}})\bigr), & \text{if } v \nmid p, \\ \ker\bigl(H^1(\mathbb{Q}_v, \mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}) \to H^1(\mathbb{Q}_v, (\mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}/\mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}}) \otimes \Phi_{\mathfrak{Q}})\bigr), & \text{if } v \mid p, \end{cases} \tag{7.5}$$

and let $H^1_{\underline{\mathrm{bal}}}(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}})$ be the associated Selmer group. Taking $H^1_{\underline{\mathrm{bal}}}(\mathbb{Q}_v, \mathbb{A}^\dagger_{\mathbf{f}gh,\mathfrak{Q}})$ to be the orthogonal complement of $H^1_{\underline{\mathrm{bal}}}(\mathbb{Q}_v, \mathbb{V}^\dagger_{\mathbf{f}gh,\mathfrak{Q}})$ under local Tate duality, we define the Selmer group $H^1_{\underline{\mathrm{bal}}}(\mathbb{Q}, \mathbb{A}^\dagger_{\mathbf{f}gh,\mathfrak{Q}})$ similarly.

Define $\mathbb{V}^\psi_{g,h} = \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T^\psi_{g,h}(\kappa_{\mathrm{ac}}^{r_1/2})$, and let $\mathbb{A}^\psi_{g,h} = \mathrm{Hom}((\mathbb{V}^\psi_{g,h})^c, \mu_{p^\infty})$, where $(\mathbb{V}^\psi_{g,h})^c$ denotes $\mathbb{V}^\psi_{g,h}$ with the G_K-action twisted by complex conjugation. Note that $\mathbb{V}^\dagger_{\mathbf{f}gh} = \mathrm{Ind}_K^{\mathbb{Q}} \mathbb{V}^\psi_{g,h}$, so we can define Selmer conditions for $\mathbb{V}^\psi_{g,h}$ using Shapiro's lemma and for $\mathbb{A}^\psi_{g,h}$ by duality. Define $\mathbb{A}^\psi_{g,h,\mathfrak{Q}} = \mathrm{Hom}((\mathbb{V}^\psi_{g,h,\mathfrak{Q}})^c, \mu_{p^\infty})$. We have natural maps

$$\mathbb{V}^\psi_{g,h}/\mathfrak{Q}\mathbb{V}^\psi_{g,h} \to \mathbb{V}^\psi_{g,h,\mathfrak{Q}}, \quad \mathbb{A}^\psi_{g,h,\mathfrak{Q}} \to \mathbb{A}^\psi_{g,h}[\mathfrak{Q}] \tag{7.6}$$

preserving both the G_K and the Λ-modules structure in the same way as in [How04, p. 1461]. Note that in the quotient $\mathbb{V}^\psi_{g,h}/\mathfrak{Q}\mathbb{V}^\psi_{g,h}$ and in the submodule $\mathbb{A}^\psi_{g,h}[\mathfrak{Q}]$, we can

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define Selmer conditions by propagating the balanced conditions for $\mathbb{V}^{\psi}_{g,h}$ and $\mathbb{A}^{\psi}_{g,h}$, respectively, and we denote these conditions in the same way.

Lemma 7.11. *For every height one prime $\mathfrak{Q} \subset \Lambda_{\mathbf{f}}$ as above and every place v of K, the maps (7.6) induce natural maps*

$$H^1_{\mathrm{bal}}(K_v, \mathbb{V}^{\psi}_{gh}/\mathfrak{Q}\mathbb{V}^{\psi}_{gh}) \longrightarrow H^1_{\underline{\mathrm{bal}}}(K_v, \mathbb{V}^{\psi}_{gh,\mathfrak{Q}}),$$
$$H^1_{\underline{\mathrm{bal}}}(K_v, \mathbb{A}^{\psi}_{gh,\mathfrak{Q}}) \longrightarrow H^1_{\mathrm{bal}}(K_v, \mathbb{A}^{\psi}_{gh}[\mathfrak{Q}])$$

with finite kernel and cokernel, of order bounded by constants depending only on $[S_{\mathfrak{Q}} : \Lambda_{\mathbf{f}}/\mathfrak{Q}]$.

Proof. For the primes $v \nmid p$, the same argument as in the proof of [MR04, Lemma 5.3.13] applies, so it remains to consider the case $v \mid p$. Put

$$\mathcal{F}^+_{\mathfrak{p}}(T^{\psi}_{g,h}) = (T^+_g \otimes T_h + T_g \otimes T^+_h)(\psi^{-1}_{\mathfrak{P}})(-1-r),$$
$$\mathcal{F}^+_{\overline{\mathfrak{p}}}(T^{\psi}_{g,h}) = (T^+_g \otimes T^+_h)(\psi^{-1}_{\mathfrak{P}})(-1-r).$$

Under the isomorphism $H^1(\mathbb{Q}, \mathbb{V}^{\dagger}_{\mathbf{f}gh}) \cong H^1(K, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\hat{\otimes}_{\mathcal{O}} T^{\psi}_{g,h}(\kappa_{\mathrm{ac}}^{r_1/2}))$ coming from Shapiro's lemma, the balanced local condition $H^1_{\mathrm{bal}}(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh})$ corresponds to

$$H^1(K_{\mathfrak{p}}, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\hat{\otimes}_{\mathcal{O}} \mathcal{F}^+_{\mathfrak{p}}(T^{\psi}_{g,h})(\kappa_{\mathrm{ac}}^{r_1/2})) \oplus H^1(K_{\overline{\mathfrak{p}}}, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\hat{\otimes}_{\mathcal{O}} \mathcal{F}^+_{\overline{\mathfrak{p}}}(T^{\psi}_{g,h})(\kappa_{\mathrm{ac}}^{r_1/2})).$$

Let $A^-_g = T^-_g \otimes \mathbb{Q}_p/\mathbb{Z}_p$, and define A^+_g, A^-_h and A^+_h similarly. Arguing as in the proof of [How04, Lemma 2.2.7], we reduce to showing that the groups

$$H^0(K_{\infty,\mathfrak{p}}, (A^-_g \otimes A^-_h)(\psi^{-1}_{\mathfrak{P}}\kappa_{\mathrm{ac}}^{r_1/2})(-1-r)), \quad H^0(K_{\infty,\overline{\mathfrak{p}}}, (A^+_g \otimes A^-_h)(\psi^{-1}_{\mathfrak{P}}\kappa_{\mathrm{ac}}^{r_1/2})(-1-r))$$

are both finite, which follows from the fact that $\alpha_g\alpha_h\psi(\mathfrak{p})/p^{k-1} \neq 1$ and $\beta_g\alpha_h\psi(\overline{\mathfrak{p}}) \neq 1$, and this is a consequence of the Ramanujan–Petersson conjecture since we are assuming that $p \nmid N$. Note that the other pieces in the quotient decomposition can be treated similarly. This yields the required bounds on the kernel and cokernel of the first map in the statement of the lemma, and the result for the second map follows as well by local duality. □

Let Σ_Λ be the set of height one primes of $\Lambda_{\mathbf{f}}$ consisting of p and those for which either $H^2(\mathbb{Q}^{\Sigma}/\mathbb{Q}, \mathbb{V}^{\dagger}_{\mathbf{f}gh})[\mathfrak{Q}]$ is infinite or $H^2(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh})[\mathfrak{Q}]$ is infinite. Since $H^2(\mathbb{Q}^{\Sigma}/\mathbb{Q}, \mathbb{V}^{\dagger}_{\mathbf{f}gh})$ and $H^2(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh})$ are both finitely generated Λ-modules, the set Σ_Λ is finite.

Proposition 7.12. *For every height one prime $\mathfrak{Q} \notin \Sigma_\Lambda$, the maps (7.6) induce natural maps*

$$\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})/\mathfrak{Q}\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh}) \longrightarrow \mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}}),$$
$$\mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}}) \longrightarrow \mathrm{Sel}_{\mathrm{bal}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})[\mathfrak{Q}]$$

with finite kernel and cokernel bounded by a constant depending only on $[S_{\mathfrak{Q}} : \Lambda_{\mathbf{f}}/\mathfrak{Q}]$.

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Proof. This follows from Lemma 7.11 as in the proof of [MR04, Proposition 5.3.14] (see also [How04, Lemma 2.2.8] and [How07, Lemma 3.2.10]). □

For every height one prime $\mathfrak{Q} \subset \Lambda_{\mathbf{f}}$ as above, let $\mathfrak{m}_{\mathfrak{Q}} = (\pi_{\mathfrak{Q}})$ be the maximal ideal of $S_{\mathfrak{Q}}$.

Lemma 7.13. *Assume that there is a perfect pairing* $T^{\psi}_{g,h} \times T^{\psi}_{g,h} \to \mathcal{O}(1)$, *such that* $\langle x^{\sigma}, y^{c\sigma c}\rangle = \langle x,y\rangle^{\sigma}$ *for all* $x,y \in T^{\psi}_{g,h}$ *and for all* $\sigma \in G_K$, *where* c *stands for complex conjugation. The following hold:*

(1) $\mathrm{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh}) = \mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathrm{bal}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$,

(2) $\mathrm{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh}) = \mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$,

(3) $\mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F}\cup+}(\mathbb{A}^{\dagger}_{\mathbf{f}gh}) = 1 + \mathrm{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F}\cap+}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})$ *and*

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}\cup+}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})_{\mathrm{tors}}) = \mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}\cap+}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})_{\mathrm{tors}}),$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. For part (1), it suffices to show that for all height one primes $\mathfrak{Q} \subset \Lambda_{\mathbf{f}}$ with $\mathfrak{Q} \notin \Sigma_{\Lambda}$, the modules $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})/\mathfrak{Q}\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh})$ and $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})[\mathfrak{Q}]$ have the same rank over $\Lambda_{\mathbf{f}}/\mathfrak{Q}$. Since $\mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{V}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}})$ is the $\pi_{\mathfrak{Q}}$-adic Tate module of $\mathrm{Sel}_{\underline{\mathrm{bal}}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}})$ (indeed, this is a consequence of [How04, Lemma 1.3.3] since $\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}} \cong \mathbb{V}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}} \otimes \mathbb{Q}_p/\mathbb{Z}_p$), the result thus follows from Proposition 7.12.

For part (2), under the isomorphism $H^1(\mathbb{Q}, \mathbb{V}^{\dagger}_{\mathbf{f}gh}) \cong H^1(K, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T^{\psi}_{g,h}(\kappa_{\mathrm{ac}}^{r_1/2}))$, the f-unbalanced local condition $H^1_{\mathcal{F}}(\mathbb{Q}_p, \mathbb{V}^{\dagger}_{\mathbf{f}gh})$ corresponds to

$$H^1(K_{\mathfrak{p}}, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T^{\psi}_{g,h}) \oplus \{0\},$$

and hence an analogue of Lemma 7.11 for the f-unbalanced Selmer groups follows from the finiteness of $H^0(K_{\infty,\overline{\mathfrak{p}}}, A_g \otimes A_h(\psi^{-1}_{\mathfrak{P}} \kappa_{\mathrm{ac}}^{r_1/2})(-1-r))$. By the same reason as above, this yields the equality of ranks in part (2).

Finally, for the proof of part (3), we can argue similarly as in [AH06, Theorem 1.2.2]. Keeping with the above notations, let $\mathrm{Sel}_{\underline{\mathcal{F}\cup+}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}})$ and $\mathrm{Sel}_{\underline{\mathcal{F}\cap+}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}})$ be the Selmer groups defined by the obvious analogues of (7.5), so from another application of the argument in Lemma 7.11, we obtain natural maps

$$\mathrm{Sel}_{\underline{\mathcal{F}\cup+}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}}) \longrightarrow \mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})[\mathfrak{Q}],$$
$$\mathrm{Sel}_{\underline{\mathcal{F}\cap+}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}}) \longrightarrow \mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{A}^{\dagger}_{\mathbf{f}gh})[\mathfrak{Q}]$$

with finite kernel and cokernel bounded by a constant depending only on $[S_{\mathfrak{Q}} : \Lambda_{\mathbf{f}}/\mathfrak{Q}]$. Since the local condition $\mathcal{F}\cap+$ is the orthogonal complement of $\mathcal{F}\cup+$ under the local Tate pairing at p induced by the self-duality of $\mathbb{V}^{\dagger}_{\mathbf{f}gh}$, from [MR04, Theorem 4.1.13], we obtain

$$\mathrm{Sel}_{\underline{\mathcal{F}\cup+}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}})[\pi^i_{\mathfrak{Q}}] \cong (\Phi_{\mathfrak{Q}}/S_{\mathfrak{Q}})^r[\pi^i_{\mathfrak{Q}}] \oplus \mathrm{Sel}_{\underline{\mathcal{F}\cap+}}(\mathbb{A}^{\dagger}_{\mathbf{f}gh,\mathfrak{Q}})[\pi^i_{\mathfrak{Q}}] \tag{7.7}$$

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for all i, where r is given (by the Greenberg–Wiles formula in [MR04, Proposition 2.3.5]) by

$$\operatorname{corank}_{S_\Omega} H^1(\mathbb{Q}_p, \mathbb{A}^{f\cup+}_{\mathbf{f}gh,\Omega}) - \operatorname{corank}_{S_\Omega} H^0(\mathbb{R}, \mathbb{A}^\dagger_{\mathbf{f}gh,\Omega}),$$

so $r = 5-4 = 1$. The proof of part (3) now follows from (7.7) as in [AH06, Lemma 1.2.6]. □

Remark 7.14. The existence of the pairing in the previous lemma is not too restrictive. In particular, this holds automatically if g and h are non-Eisenstein.

We are now ready to establish that both formulations of the Iwasawa main conjecture are equivalent.

Theorem 7.15. *Keep the assumptions of the previous lemma and suppose $\kappa(\mathbf{f},g,h)$ is not $\Lambda_{\mathbf{f}}$-torsion. Then the following are equivalent:*

(1) $\operatorname{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) = \operatorname{rank}_{\Lambda_{\mathbf{f}}} X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) = 1$;

(2) $\operatorname{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) = \operatorname{rank}_{\Lambda_{\mathbf{f}}} X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) = 0$;

and, in that case, we have $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) = \mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ and

$$\operatorname{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh})) \cdot \operatorname{Char}_{\Lambda_{\mathbf{f}}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})}{\Lambda_{\mathbf{f}}\cdot\kappa(\mathbf{f},g,h)}\right)^2 = \operatorname{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh})_{\mathrm{tors}}) \cdot (L_p(\mathbf{f},g,h))$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In particular, Conjectures 7.7 and 7.9 are equivalent.

Proof. The Poitou–Tate global duality gives rise to the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{V}^\dagger_{\mathbf{f}gh}) \longrightarrow \mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) &\xrightarrow{\mathrm{res}_p} H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_{\mathbf{f}}) \\ &\longrightarrow X_{\mathcal{F}\cup+}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow 0. \end{aligned} \tag{7.8}$$

Assume that $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ and $X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ have both $\Lambda_{\mathbf{f}}$-rank one. Since $H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_{\mathbf{f}})$ has $\Lambda_{\mathbf{f}}$-rank one, from (7.8) and Theorem 7.4, we see that $\mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ is $\Lambda_{\mathbf{f}}$-torsion and $X_{\mathcal{F}\cup+}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ has $\Lambda_{\mathbf{f}}$-rank one. By Lemma 7.13(3), it follows that $X_{\mathcal{F}\cap+}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ is $\Lambda_{\mathbf{f}}$-torsion, and from the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) \longrightarrow \mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{V}^\dagger_{\mathbf{f}gh}) &\xrightarrow{\mathrm{res}_p} H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_{\mathbf{f}}) \\ &\longrightarrow X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow X_{\mathcal{F}\cap+}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow 0 \end{aligned}, \tag{7.9}$$

we get that $X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ and $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ are both $\Lambda_{\mathbf{f}}$-torsion by Lemma 7.13(2). This proves the implication (1) ⇒ (2) in the statement of the theorem, and the converse is shown similarly. Moreover, from (7.9), we see that $\operatorname{rank}_{\Lambda_{\mathbf{f}}} \mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{V}^\dagger_{\mathbf{f}gh}) = 1$, and hence the quotient $\mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{V}^\dagger_{\mathbf{f}gh})/\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ is a torsion $\Lambda_{\mathbf{f}}$-module injecting into $H^1(\mathbb{Q}_p, \mathbb{V}^{f\cup+}_{\mathbf{f}gh}/\mathscr{F}^2\mathbb{V}^\dagger_{\mathbf{f}gh})$; since this is $\Lambda_{\mathbf{f}}$-torsion free by Proposition 7.3, it follows that

$$\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh}) = \mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{V}^\dagger_{\mathbf{f}gh}). \tag{7.10}$$

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Now suppose that either (1) or (2) in the statement of theorem holds. Since $\bar{\rho}_{\mathbf{f}}$ is absolutely irreducible by our hypotheses, the module $H^1(\mathbb{Q}^\Sigma/\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh})$ is $\Lambda_{\mathbf{f}}$-torsion free by [PR00, Section 1.3.3]. Being $\Lambda_{\mathbf{f}}$-torsion, it follows that the module $\mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ vanishes, and therefore from (7.8), we deduce the exact sequence

$$0 \longrightarrow \frac{\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f},g,h)} \longrightarrow \frac{H^1(\mathbb{Q}_p, \mathbb{V}^{gh}_{\mathbf{f}})}{\Lambda_{\mathbf{f}} \cdot p_{\mathbf{f}*}(\mathrm{res}_p(\kappa(\mathbf{f},g,h)))} \longrightarrow \mathrm{coker}(\mathrm{res}_p) \longrightarrow 0. \tag{7.11}$$

Together with Theorem 7.4, it follows that

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f},g,h)}\right) \cdot \mathrm{Char}_{\Lambda_{\mathbf{f}}}(\mathrm{coker}(\mathrm{res}_p)) = (\mathscr{L}_p^{\xi}(\underline{\breve{\mathbf{f}}}, \underline{\breve{g}}, \underline{\breve{h}})). \tag{7.12}$$

On the other hand, in light of (7.10), from (7.8) and (7.9), we deduce exact sequences

$$0 \longrightarrow \mathrm{coker}(\mathrm{res}_p) \longrightarrow X_{\mathcal{F}\cup+}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{coker}(\mathrm{res}_p) \longrightarrow X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow X_{\mathcal{F}\cap+}(\mathbb{A}^\dagger_{\mathbf{f}gh}) \longrightarrow 0.$$

Taking characteristic ideals, these imply

$$\begin{aligned} \mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh}) &= \mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}\cap+}(\mathbb{A}^\dagger_{\mathbf{f}gh})) \cdot \mathrm{Char}_{\Lambda_{\mathbf{f}}}(\mathrm{coker}(\mathrm{res}_p)) \\ &= \mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}\cup+}(\mathbb{A}^\dagger_{\mathbf{f}gh})_{\mathrm{tors}}) \cdot \mathrm{Char}_{\Lambda_{\mathbf{f}}}(\mathrm{coker}(\mathrm{res}_p)) \\ &= \mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh})_{\mathrm{tors}}) \cdot \mathrm{Char}_{\Lambda_{\mathbf{f}}}(\mathrm{coker}(\mathrm{res}_p))^2, \end{aligned} \tag{7.13}$$

using Lemma 7.13(3) for the second equality. Multiplying (7.13) by the square of a generator of the characteristic ideal of $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})/\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f},g,h)$ and using (7.12), the result follows. □

8. Anticyclotomic Euler systems

In this section, we highlight results from the recent work of Jetchev–Nekovář–Skinner [JNS], where a general theory of Euler systems germane to [Rub00] is developed in the anticyclotomic setting.

8.1. The general theory

Let K be an imaginary quadratic field, and let p be an odd prime. If $\mathfrak{n}$ is an integral prime ideal of K, we denote by $K(\mathfrak{n})^\circ$ the ray class field of conductor $\mathfrak{n}$; as in the previous sections, we write $K(\mathfrak{n})$ for the maximal p-subextension in $K(\mathfrak{n})^\circ$. For any positive integer n, we denote by $K[n]$ the maximal p-subextension in the ring class field of K of conductor n. We denote by K_∞ the anticyclotomic $\mathbb{Z}_p$-extension of K.

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Let T be a free $\mathcal{O}$-module of finite rank endowed with a continuous G_K-action unramified outside a finite set of primes, and let $V = T \otimes_{\mathcal{O}} E$. Assume that there exists a nondegenerate symmetric $\mathcal{O}$-bilinear pairing

$$\langle , \rangle : T \times T \longrightarrow \mathcal{O}(1),$$

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such that $\langle x^\sigma, y^{c\sigma c}\rangle = \langle x,y\rangle^\sigma$ for all $x,y \in T$ and $\sigma \in G_K$, where c is complex conjugation. Thus, $V^c \simeq V^\vee(1)$, where V^c denotes the representation V with the G_K-action twisted by c, and, if the above pairing is perfect, we also have $T^c \simeq T^\vee(1)$. We also define the G_K-module $A = V/T$.

If L is a finite extension of K and v is a finite place of L, we write $\overline{v} = v^c$. Then, the pairing above induces a local pairing

$$H^1(L_v,V) \times H^1(L_{\overline{v}},V) \longrightarrow E,$$

and similarly replacing V by T and E by $\mathcal{O}$. The pair of compatible maps $G_{L_v} \to G_{L_{\overline{v}}}$ and $V \to V^c$ defined by $\sigma \mapsto c\sigma c$ and $w \mapsto w$, respectively, induces an isomorphism $H^1(L_{\overline{v}},V) \cong H^1(L_v,V^c) \cong H^1(L_v,V^\vee(1))$, whereby the above local pairing is just the natural cup-product pairing.

For the results we shall discuss, we consider two different types of 'big image' hypotheses, (HW) for the weaker ones, and (HS) for the stronger ones.

Hypothesis (HW).

(1) V is absolutely irreducible as a G_K-representation.

(2) There exists an element $\sigma_0 \in \mathrm{Gal}(\bar{K}/K(1)^\circ K(\mu_{p^\infty}))$, such that the E-dimension of $V/(\sigma_0 - 1)V$ is one.

Hypothesis (HS).

(1') The residual representation $\bar{T} = T/\mathfrak{m}T$ is absolutely irreducible.

(2') There exists an element $\sigma_0 \in \mathrm{Gal}(\bar{K}/K(p^\infty)^\circ)$, such that $T/(\sigma_0 - 1)T \simeq \mathcal{O}$ is a free $\mathcal{O}$-module of rank one.

(3') There exists an element $\tau_0 \in G_K$, such that $\tau_0 - 1$ acts on T as multiplication by a unit $a_{\tau_0} \in \mathcal{O}^\times$ with $a_{\tau_0} - 1 \in \mathcal{O}^\times$.

(4') The above pairing $T \times T \longrightarrow \mathcal{O}(1)$ is perfect.

For each prime $\mathfrak{p}$ of K above p, choose a $G_{K_\mathfrak{p}}$-stable $\mathcal{O}$-submodule $\mathcal{F}^+_\mathfrak{p}(T)$ of T, and let $\mathcal{F}^-_\mathfrak{p}(T) = T/\mathcal{F}^+_\mathfrak{p}(T)$. We also define $\mathcal{F}^+_\mathfrak{p}(V) = \mathcal{F}^+_\mathfrak{p}(T) \otimes_\mathcal{O} E \subseteq V$ and $\mathcal{F}^-_\mathfrak{p}(V) = V/\mathcal{F}^+_\mathfrak{p}(V)$. Let L be a finite extension of K. For each place v of L, we define a local condition

$$H^1_{\mathrm{Gr}}(L_v,V) = \begin{cases} \ker\left(H^1(L_v,V) \to H^1(L_v^{\mathrm{nr}},V)\right) & \text{if } v \nmid p, \\ \ker\left(H^1(L_v,V) \to H^1(L_v,\mathcal{F}^-_\mathfrak{p}(V))\right) & \text{if } v \mid \mathfrak{p} \text{ for some } \mathfrak{p} \mid p. \end{cases}$$

We define the *Greenberg Selmer group*

$$\mathrm{Sel}_{\mathrm{Gr}}(L,V) = \ker\Big(H^1(L,V) \to \prod_v H^1(L_v,V)/H^1_{\mathrm{Gr}}(L_v,V)\Big),$$

where the product is over all finite places of L.

We also define local conditions for T and A by propagation of the local conditions for V, that is, for each place v of L, we define

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- $H^1_{\mathrm{Gr}}(L_v,T)$ as the preimage of $H^1_{\mathrm{Gr}}(L_v,V)$ by the map $H^1(L_v,T) \to H^1(L_v,V)$, and
- $H^1_{\mathrm{Gr}}(L_v,A)$ as the image of $H^1_{\mathrm{Gr}}(L_v,V)$ by the map $H^1(L_v,V) \to H^1(L_v,A)$,

and use these to define the Selmer groups $\mathrm{Sel}_{\mathrm{Gr}}(L,T)$ and $\mathrm{Sel}_{\mathrm{Gr}}(L,A)$ as above. Finally, for each positive integer n, we also put

$$\mathrm{Sel}_{\mathrm{Gr}}(K[np^\infty],T) = \varprojlim_r \mathrm{Sel}_{\mathrm{Gr}}(K[np^r],T) \quad \text{and} \quad \mathrm{Sel}_{\mathrm{Gr}}(K[np^\infty],A) = \varinjlim_r \mathrm{Sel}_{\mathrm{Gr}}(K[np^r],A),$$

where the limits are with respect to the corestriction and restriction maps, respectively, and we define

$$X_{\mathrm{Gr}}(K[np^\infty],A) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Sel}_{\mathrm{Gr}}(K[np^\infty],A),\mathbb{Q}_p/\mathbb{Z}_p).$$

Let $\mathcal{N}$ be an ideal of K divisible by p and all the primes at which T is ramified, and let $\mathcal{S}$ be the set of all squarefree products of primes of $\mathbb{Q}$ which *split* in K and are coprime to $\mathcal{N}$.

Definition 8.1. A *'split' anticyclotomic Euler system* for $(T,\{\mathcal{F}^+_{\mathfrak{p}}(T)\}_{\mathfrak{p}|p},\mathcal{N})$ is a collection of classes

$$\boldsymbol{\kappa} = \{\kappa_n \in \mathrm{Sel}_{\mathrm{Gr}}(K[n],T) \,:\, n \in \mathcal{S}\},$$

such that, whenever q is a rational prime and $n,nq \in \mathcal{S}$,

$$\mathrm{cor}_{K[nq]/K[n]}(\kappa_{nq}) = P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1})\,\kappa_n, \tag{8.1}$$

where $\mathfrak{q}$ is any of the primes of K above q and $P_{\mathfrak{q}}(X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X | T^\vee(1))$.

Similarly, a *'split' Λ-adic anticyclotomic Euler system* for $(T,\{\mathcal{F}^+_{\mathfrak{p}}(T)\}_{\mathfrak{p}|p},\mathcal{N})$ is a collection of classes

$$\boldsymbol{\kappa}_\infty = \{\kappa_{n,\infty} \in \mathrm{Sel}_{\mathrm{Gr}}(K[np^\infty],T) \,:\, n \in \mathcal{S}\}$$

satisfying the previous norm relations. In this case, the classes

$$\kappa_n = \mathrm{pr}_{K[n]}(\kappa_{n,\infty}) \in \mathrm{Sel}_{\mathrm{Gr}}(K[n],T)$$

form an anticyclotomic Euler system in the previous sense, and we say that the Euler system $\boldsymbol{\kappa} = \{\kappa_n\}_n$ extends along the anticyclotomic $\mathbb{Z}_p$-extension.

A (Λ-adic) anticyclotomic Euler system for $(T,\{\mathcal{F}^+_{\mathfrak{p}}(T)\}_{\mathfrak{p}|p})$ is just a (Λ-adic) anticyclotomic Euler system for $(T,\{\mathcal{F}^+_{\mathfrak{p}}(T)\}_{\mathfrak{p}|p},\mathcal{N})$ for some $\mathcal{N}$ as above. We shall usually drop $\{\mathcal{F}^+_{\mathfrak{p}}(T)\}_{\mathfrak{p}|p}$ if there is no risk of confusion.

If $\boldsymbol{\kappa}$ is an anticyclotomic Euler system for T, we define

$$\kappa_0 := \mathrm{cor}_{K[1]/K}(\kappa_1) \in \mathrm{Sel}_{\mathrm{Gr}}(K,T).$$

If it extends along the anticyclotomic $\mathbb{Z}_p$-extension, we similarly define

$$\kappa_\infty := \mathrm{cor}_{K[1]/K}(\kappa_{1,\infty}) \in \mathrm{Sel}_{\mathrm{Gr}}(K_\infty,T),$$

where $\boldsymbol{\kappa}_\infty = \{\kappa_{n,\infty}\}$ is the Λ-adic anticyclotomic Euler system extending $\boldsymbol{\kappa}$.

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When we have an Euler system as above, we will be interested in ensuring that the following orthogonality hypothesis holds.

Hypothesis (HO). For all $n \in \mathcal{S}$ and for all places v of $K[n]$ above p, the local conditions $H^1_{\mathrm{Gr}}(K[n]_v,V)$ and $H^1_{\mathrm{Gr}}(K[n]_{\overline{v}},V)$ are orthogonal complements under the local pairing

$$H^1(K[n]_v,V) \times H^1(K[n]_{\overline{v}},V) \longrightarrow E.$$

Remark 8.2. The condition in hypothesis (HO) holds automatically for all places away from p, by [Rub00, Proposition 1.4.2]. Observe also that if (HO) holds, then for all $n \in \mathcal{S}$ and for all places v of $K[n]$, the local conditions $H^1_{\mathrm{Gr}}(K[n]_v,T)$ and $H^1_{\mathrm{Gr}}(K[n]_{\overline{v}},T)$ are also orthogonal complements under the local pairing

$$H^1(K[n]_v,T) \times H^1(K[n]_{\overline{v}},T) \longrightarrow \mathcal{O},$$

as follows easily from the definitions using [Rub00, Proposition B.2.4] and the commutative diagram

$$\begin{array}{ccc} H^1(K[n]_v,T) \times H^1(K[n]_{\overline{v}},T) & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ H^1(K[n]_v,V) \times H^1(K[n]_{\overline{v}},V) & \longrightarrow & E. \end{array}$$

We assume in the rest of this subsection that hypothesis (HO) holds for our choice of local conditions at p.

Theorem 8.3 [JNS]. *Assume that p splits in K and that Hypothesis (HW) is satisfied, and let $\boldsymbol{\kappa} = \{\kappa_n\}_n$ be an anticyclotomic Euler system for T which extends along the anticyclotomic $\mathbb{Z}_p$-extension. If $\kappa_0 \neq 0$, then the Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K,T)$ has $\mathcal{O}$-rank one.*

Remark 8.4. One can replace the assumptions that p splits in K and the Euler system extends along the anticyclotomic $\mathbb{Z}_p$-extension by the assumption that there exists an element $\gamma \in G_K$ fixing the extension $K(1)^{\circ}(\mu_{p^\infty},(\mathcal{O}_K^{\times})^{1/p^\infty})$ and such that $\gamma - 1$ acts invertibly on V.

Under the stronger Hypothesis (HS), granted the nontriviality of a Λ-adic anticyclotomic Euler system, the results of [JNS] yield a divisibility towards a corresponding Iwasawa main conjecture.

Theorem 8.5 [JNS]. *Assume that p splits in K and that Hypothesis (HS) is satisfied, and let $\boldsymbol{\kappa}$ be a Λ-adic anticyclotomic Euler system for T.*

(a) *If $\kappa_0 \neq 0$, then $\mathrm{Sel}_{\mathrm{Gr}}(K,A)$ has $\mathcal{O}$-corank one, $\mathrm{Sel}_{\mathrm{Gr}}(K,T)$ has $\mathcal{O}$-rank one and*

$$\mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathrm{Gr}}(K,A)_{/\mathrm{div}}) \leq 2\,\mathrm{length}_{\mathcal{O}}\left(\frac{\mathrm{Sel}_{\mathrm{Gr}}(K,T)}{\mathcal{O}\cdot\kappa_0}\right),$$

where $(-)_{/\mathrm{div}}$ denotes the quotient of $(-)$ by its maximal divisible submodule.

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(b) *If* κ_∞ *is not* Λ_{ac}*-torsion, then* $X_{\mathrm{Gr}}(K_\infty, A)$ *and* $\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T)$ *have both* Λ_{ac}*-rank one, and*

$$\mathrm{Char}_{\Lambda_{\mathrm{ac}}}(X_{\mathrm{Gr}}(K_\infty, A)_{\mathrm{tors}}) \supset \mathrm{Char}_{\Lambda_{\mathrm{ac}}}\left(\frac{\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T)}{\Lambda_{\mathrm{ac}} \cdot \kappa_\infty}\right)^2,$$

where $(-)_{\mathrm{tors}}$ *denotes the maximal* Λ_{ac}*-torsion submodule of* $(-)$.

8.2. Big image results

We now give conditions under which the hypotheses in the general results of Section 8.1 are verified in our setting. To that end, we shall build on [Loe17].

As before, let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D$, let (g,h) be a pair of newforms of weights (l,m) of the same parity, levels (N_g, N_h) and characters (χ_g, χ_h), and let ψ be a Grössencharacter of K of infinity type $(1-k,0)$ for some positive even integer k and of conductor $\mathfrak{f}$. We denote by χ the unique Dirichlet character modulo $N_{K/\mathbb{Q}}(\mathfrak{f})$, such that $\psi((n)) = n^{k-1}\chi(n)$ for integers n coprime to $N_{K/\mathbb{Q}}(\mathfrak{f})$, and we assume that $\chi\varepsilon_K\chi_g\chi_h = 1$.

We now make the further assumptions that:

- neither g nor h are of CM type,
- g is not Galois-conjugate to a twist of h.

As in [Loe17, Section 3.1], we define the open subgroups H_g and H_h of $G_{\mathbb{Q}}$, the quaternion algebras B_g and B_h and the algebraic groups G_g and G_h, and put

$$B = B_g \times B_h, \quad G = G_g \times_{\mathbb{G}_m} G_h.$$

We define H to be the intersection of H_g, H_h and $G_{K(\mathfrak{f})^\circ}$ (note that in *loc. cit.*, H is defined to be the intersection of H_g and H_h, so our H might be a finite index subgroup of his H, but this will not affect the results that follow). We have an adelic representation $\tilde{\rho}_{g,h} : H \to G(\hat{\mathbb{Q}})$, and representations

$$\tilde{\rho}_{g,h,p} : H \longrightarrow G(\mathbb{Q}_p)$$

for every rational prime p, and, by [Loe17, Theorem 3.2.2], $\tilde{\rho}_{g,h,p}(H) = G(\mathbb{Z}_p)$ for all but finitely many p.

Remark 8.6. Note that the representations studied in [Loe17] are the dual to the ones studied in this paper, but as pointed out in [Loe17, Remark 2.1.2], this difference is unimportant when considering the image.

Let L be a finite extension of K containing the Fourier coefficients of g and h and the image of ψ. Let $\mathfrak{P}$ be a prime of L above some rational prime p, and let $E = L_{\mathfrak{P}}$.

Definition 8.7. We say that the prime $\mathfrak{P}$ is *good* if the following conditions hold:

- $p \geq 7$;
- p is unramified in B;

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- p is coprime to $\mathfrak{f}$, N_g and N_h;
- $\tilde{\rho}_{g,h,p}(H) = G(\mathbb{Z}_p)$;
- $E = \mathbb{Q}_p$.

Remark 8.8. Observe that all but the last condition exclude only finitely many primes. The last condition could be somewhat relaxed in some cases, and will be used largely for simplicity. Note also that the above set of conditions holds for a set of primes of positive density.

From now on, we assume that both g and h are ordinary, non-Eisenstein and distinguished with respect to $\mathfrak{P}$.

Lemma 8.9. *Assume that there is at least one prime which divides D but not N_g and one prime which divides D but not N_h. Then, if $\mathfrak{P}$ is a good prime,*

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(H \cap G_{K(p^\infty)^\circ}) = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p).$$

Proof. Let $\mathbb{Q}(\rho_g)$ and $\mathbb{Q}(\rho_h)$ be the Galois extensions of $\mathbb{Q}$ cut out by the representations ρ_g and ρ_h attached to g and h, respectively. These extensions are unramified outside pN_g and pN_h, respectively. Therefore, the condition on D implies that $K \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$ and $K \cap \mathbb{Q}(\rho_h) = \mathbb{Q}$. Moreover, since any Galois extension of $\mathbb{Q}$ contained in K_∞ must itself contain K, we also have $K_\infty \cap \mathbb{Q}(\rho_g) = \mathbb{Q}$ and $K_\infty \cap \mathbb{Q}(\rho_h) = \mathbb{Q}$.

The conditions on $\mathfrak{P}$ imply that

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(H \cap G_{\mathbb{Q}(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p),$$

and, from the remarks in the previous paragraph, it follows that

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(H \cap G_{K_\infty(\mu_{p^\infty})}) = \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p).$$

Finally, since $H \cap G_{K(p^\infty)^\circ}$ is a normal subgroup of $H \cap G_{K_\infty(\mu_{p^\infty})}$ of index dividing $p-1$ and there are no such subgroups in $\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$, the lemma follows. □

Now we are able to give conditions under which the results of [JNS] can be applied to our setting, that is, to the representation $T^{\psi}_{g,h}$ defined above.

Proposition 8.10. *Assume that there is at least one prime which divides D but not N_g and one prime which divides D but not N_h. Let $\mathfrak{P}$ be a good prime. Suppose that there exists $\sigma \in G_{K(p^\infty)^\circ}$, such that $\psi_{\mathfrak{P}}(\sigma) \neq \psi^c_{\mathfrak{P}}(\sigma)$ modulo p. Then, hypotheses (HS) hold for $T^{\psi}_{g,h}$.*

Proof. Since $\psi_{\mathfrak{P}}$ is trivial when restricted to $H \cap G_{K(p^\infty)^\circ}$, condition (1') follows easily from the previous lemma.

To prove condition (2'), we closely follow the proof of [Loe17, Proposition 4.2.1]. Write $\chi_g(\sigma)$ and $\chi_h(\sigma)$ for the images of σ by χ_g and χ_h via the natural identifications

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$\mathrm{Gal}(\mathbb{Q}(\mu_{N_g})/\mathbb{Q}) \cong (\mathbb{Z}/N_g\mathbb{Z})^\times$ and $\mathrm{Gal}(\mathbb{Q}(\mu_{N_h})/\mathbb{Q}) \cong (\mathbb{Z}/N_h\mathbb{Z})^\times$. Then, by the previous lemma, the image of $\sigma H \cap G_{K(p^\infty)^\circ}$ under $\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}}$ contains all the elements of the form

$$\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}\chi_g(\sigma) \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1}\chi_h(\sigma) \end{pmatrix}\right), \quad x,y \in \mathbb{Z}_p^\times.$$

Now choose $x \in \mathbb{Z}_p^\times$, such that $x^{-2}\chi_g(\sigma) \neq 1 \pmod p$ and $x^2\chi_h(\sigma)\psi_{\mathfrak{P}}(\sigma)^{-2} \neq 1 \pmod p$, which is possible since $p \geq 7$, and let $y = x^{-1}\psi_{\mathfrak{P}}(\sigma)$. Choose $\sigma_0 \in \sigma H \cap G_{K(p^\infty)^\circ}$ whose image under $\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}}$ is given by the element above, with the choices of x and y which we have just specified. Then, the eigenvalues of σ_0 acting on $T_{g,h}^\psi$ are 1, $x^{-2}\chi_g(\sigma)$, $x^2\chi_h(\sigma)\psi_{\mathfrak{P}}(\sigma)^{-2}$ and $\psi_{\mathfrak{P}}^c(\sigma)\psi_{\mathfrak{P}}(\sigma)^{-1}$, which proves condition (2').

To check condition (3'), we can argue as in [KLZ17, Remark 11.1.3]. By the previous lemma, we can find an element $\tau_0 \in H \cap G_{K(p^\infty)^\circ}$, such that

$$(\rho_{g,\mathfrak{P}} \times \rho_{h,\mathfrak{P}})(\tau_0) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

so τ_0 acts on $T_{g,h}^\psi$ as multiplication by -1.

Finally, condition (4') follows from the assumption that g and h are non-Eisenstein and p-distinguished. □

Remark 8.11. If we are just interested in ensuring that hypotheses (HW) hold for $T_{g,h}^\psi$, we can relax some of the assumptions above. For example, we do not need to require g and h to be non-Eisenstein, and we can require that there exist $\sigma \in G_{K(1)^\circ(\mu_{p^\infty})}$, such that $\psi_{\mathfrak{P}}(\sigma) \neq \psi_{\mathfrak{P}}^c(\sigma)$, without requiring this inequality to hold modulo p.

9. Proof of Theorems B, C and D

Let the setting be as in the Introduction. In particular, $g \in S_l(N_g,\chi_g)$ and $h \in S_m(N_h,\chi_h)$ are newforms of weights $l \geq m \geq 2$ of the same parity, $K/\mathbb{Q}$ is an imaginary quadratic field of discriminant $-D < 0$, ψ is a Grössencharacter for K of infinity type $(1-k,0)$ for some even integer $k \geq 2$ and we consider the G_K-representation

$$V_{g,h}^\psi = V_g \otimes_E V_h(\psi_{\mathfrak{P}}^{-1})(1-c),$$

where $c = (k+l+m-2)/2$.

Lemma 9.1. *The Bloch–Kato Selmer group of* $V_{g,h}^\psi$ *is given by*

$$\mathrm{Sel}(K,V_{g,h}^\psi) \cong \begin{cases} \mathrm{Sel}_{\mathrm{bal}}(K,V_{g,h}^\psi) & \text{if } l-m < k < l+m, \\ \mathrm{Sel}_{\mathcal{F}}(K,V_{g,h}^\psi) & \text{if } k \geq l+m. \end{cases}$$

Proof. Note that by Shapiro's lemma $H^1(K,V_{g,h}^\psi) \cong H^1(\mathbb{Q},V_{fgh})$, where $f = \theta_\psi$ is the theta series of ψ, and V_{fgh} is the specialisation of the big Galois representation $\mathbb{V}_{\mathbf{fgh}}^\dagger$ in (7.1) to weights (k,l,m). One immediately checks that the Hodge–Tate weights of the $G_{\mathbb{Q}_p}$-subrepresentation $\mathscr{F}^2V_{fgh} \subset V_{fgh}$ (respectively, $V_{fgh}^f \subset V_{fgh}$) are all < 0 (with the p-adic

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cyclotomic character ϵ_{cyc} having Hodge–Tate weight -1) if and only if $l-m<k<l+m$ (respectively, $k\geq l+m$). The result follows. □

Here, we collect a set of hypotheses for our later reference. For any nonzero $m\in\mathbb{Z}$, $\mathrm{prime}(m)$ denotes the set of primes that divide m, and $\mathrm{prime}^c(m)$ its complement.

Hypotheses 9.2.

(h1) g and h are ordinary at p, non-Eisenstein and p-distinguished.

(h2) p splits in K,

(h3) p does not divide the class number of K,

(h4) $\psi_{\mathfrak{P}}|_{G_{K(p^\infty)^\circ}}\neq\psi^c_{\mathfrak{P}}|_{G_{K(p^\infty)^\circ}}$ modulo p,

(h5) neither g nor h are of CM type,

(h6) g is not Galois-conjugate to a twist of h.

(h7) $\mathrm{prime}(D)\cap\mathrm{prime}^c(N_g)\neq\emptyset$, and $\mathrm{prime}(D)\cap\mathrm{prime}^c(N_h)\neq\emptyset$,

(h8) $\mathfrak{P}$ is a good prime in the sense of Definition 8.7.

9.1. Proof of Theorem B

Let $\kappa_{\psi,g,h,1,\infty}\in H^1_{\mathrm{Iw}}(K[p^\infty],T^\psi_{g,h})$ be the Iwasawa cohomology class of conductor $n=1$ from Theorem 6.5, and set

$$\kappa_{\psi,g,h}=\kappa_{\psi,g,h,1}\in H^1(K,T^\psi_{g,h}),\tag{9.1}$$

where $\kappa_{\psi,g,h,1}=\mathrm{pr}_K(\kappa_{\psi,g,h,1,\infty})$.

If $l-m<k<l+m$, the next result recovers Theorem B in the Introduction. Note, however, that the result does not require these inequalities to hold.

Theorem 9.3. *Assume hypotheses (h1)–(h8). Then the following implication holds:*

$$\kappa_{\psi,g,h}\neq 0\quad\Longrightarrow\quad\dim_E\mathrm{Sel}_{\mathrm{bal}}(K,V^\psi_{g,h})=1.$$

In particular, if $l-m<k<l+m$ and $\kappa_{\psi,g,h}\neq 0$, then the Bloch–Kato Selmer group $\mathrm{Sel}(K,V^\psi_{g,h})$ is one-dimensional.

Proof. By Proposition 6.6, the classes $\kappa_{\psi,g,h,n}:=\mathrm{pr}_{K[n]}(\kappa_{\psi,g,h,n,\infty})$ land in $\mathrm{Sel}_{\mathrm{bal}}(K[n],T^\psi_{g,h})$, and by Theorem 6.5, they form an anticyclotomic Euler system for $V^\psi_{g,h}$. Therefore, the result follows from Theorem 8.3 and Proposition 8.10. □

Remark 9.4. If $k=2$ and $l=m\geq 2$, working with the classes $\kappa_{\psi,g,h,n}$ from Theorem 4.6, rather than those from Theorem 6.5 as above, hypotheses (h2)–(h3) in Theorem 9.3 can be replaced by the assumption that there exists an element $\gamma\in G_K$ satisfying the conditions in Remark 8.4. Further, (h1) and (h4) can be relaxed as discussed in Remark 8.11.

9.2. Proof of Theorem C

Recall that $\theta_\psi\in S_k(N_\psi,\chi\varepsilon_K)$ is the theta series attached to ψ, and put $N=\mathrm{lcm}(N_\psi,N_g,N_h)$.

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The next theorem, establishing cases of the Bloch–Kato conjecture for $V_{g,h}^{\psi}$ in analytic rank zero, recovers Theorem C in the Introduction.

Theorem 9.5. *Assume hypotheses (h1)–(h8), and, in addition, that:*

- $\varepsilon_\ell(\theta_\psi, g, h) = +1$ *for all primes* $\ell \mid N$,
- $\gcd(N_\psi, N_g, N_h)$ *is squarefree.*

If $k \geq l + m$, *then the following implication holds:*

$$L(V_{g,h}^{\psi}, 0) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}(K, V_{g,h}^{\psi}) = 0.$$

Proof. We continue to denote by $\kappa_{\psi,g,h}$ the image of the class in (9.1) under the isomorphism

$$H^1(K, V_{g,h}^{\psi}) \cong H^1(\mathbb{Q}, V_{fgh})$$

coming from Shapiro's lemma. If $k \geq l + m$, the central value $L(V_{g,h}^{\psi}, 0)$ is in the range of interpolation of the triple product p-adic L-function of Theorem 7.1, and so by Proposition 7.3 and Theorem 7.4, its nonvanishing implies that the image of $\kappa_{\psi,g,h}$ under the natural map

$$\mathrm{res}_p : \mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q}, V_{fgh}) \longrightarrow H^1(\mathbb{Q}_p, V_f^{gh})$$

is nonzero. In particular, $\kappa_{\psi,g,h} \neq 0$, and therefore by Theorem 9.3, the balanced Selmer group $\mathrm{Sel}_{\mathrm{bal}}(K, V_{g,h}^{\psi}) = \mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q}, V_{fgh})$ is one-dimensional.

From the exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{Q}, V_{fgh}) \longrightarrow \mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q}, V_{fgh}) \xrightarrow{\mathrm{res}_p} H^1(\mathbb{Q}_p, V_f^{gh}) \longrightarrow \mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow \mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow 0$$

coming from global duality (adopting notations similar to those in Theorem 7.15), we thus see that $\mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{Q}, V_{fgh}) = 0$ and that $\mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{Q}, V_{fgh}) = \mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q}, V_{fgh})$. Together with the exact sequence

$$\mathrm{Sel}_{\mathcal{F}\cup+}(\mathbb{Q}, V_{fgh}) \xrightarrow{\mathrm{res}_p} H^1(\mathbb{Q}_p, V_f^{gh}) \longrightarrow \mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow \mathrm{Sel}_{\mathcal{F}\cap+}(\mathbb{Q}, V_{fgh})^\vee \longrightarrow 0,$$

it follows that $\mathrm{Sel}_{\mathcal{F}}(\mathbb{Q}, V_{fgh}) = 0$, and combined with Lemma 9.1, this concludes the proof. □

Refining the proof of Theorem 9.5, we can further bound the size of the Bloch–Kato Selmer group for the discrete module $A_{g,h}^{\psi} = V_{g,h}^{\psi}/T_{g,h}^{\psi}$ in terms of L-values. For the statement, let $\mathbf{f}$ be the Hida family associated to ψ as in Section 6, so that $\mathbf{f}_k$ is the ordinary p-stabilisation of θ_ψ, and, keeping with the notations in Theorem 7.1, put $\alpha_k = \psi(\overline{\mathfrak{p}})$ and $\beta_k = \psi(\mathfrak{p})$. Let also $\varepsilon_\ell(\theta_\psi, g, h) = \varepsilon_\ell(V_{fgh})$ denote the epsilon factor associated to $V_{fgh}|_{G_{\mathbb{Q}_\ell}}$, where $f = \theta_\psi$.

Theorem 9.6. *Assume hypotheses (h1)–(h8), and, in addition, that:*

- $\varepsilon_\ell(\theta_\psi, g, h) = +1$ *for all primes* $\ell \mid N$,
- $\gcd(N_\psi, N_g, N_h)$ *is squarefree,*

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- $H^1(\mathbb{Q}_p, T_f^{gh})$ *is torsion-free,*
- $H^1_{\mathcal{L}}(\mathbb{Q}_p, T_{fgh})$ *is torsion-free for* $\mathcal{L} \in \{\mathrm{bal}, \mathcal{F}, \mathcal{F}\cap+, \mathcal{F}\cup+\}$.

If $k \geq l+m$ *and* $L(V_{g,h}^{\psi}, 0) \neq 0$*, then the* $\mathcal{O}$*-module* $\mathrm{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})$ *is finite and*

$$\mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})) \leq 2\, v_{\mathfrak{P}}\left(\frac{(l-2)!(m-2)!}{(k-c-1)!} \cdot \frac{\mathcal{E}_1(\mathbf{f}_k)}{\mathcal{E}(\mathbf{f}_k, g, h)} \cdot \mathscr{L}_p^{\xi}(\breve{\mathbf{f}}, \breve{\underline{g}}, \breve{\underline{h}})(k) \right),$$

where $\mathcal{E}_1(\mathbf{f}_k) = \left(1 - \frac{\beta_k}{p\alpha_k}\right)$, $\mathcal{E}(\mathbf{f}_k, g, h) = \left(1 - \frac{\alpha_k\alpha_g\alpha_h}{p^c}\right)\left(1 - \frac{\beta_k\beta_g\alpha_h}{p^c}\right)\left(1 - \frac{\beta_k\alpha_g\beta_h}{p^c}\right)\left(1 - \frac{\beta_k\beta_g\beta_h}{p^c}\right)$ *and* $c = (k+l+m-2)/2$.

Proof. As in the proof of Theorem 9.5, if $k \geq l+m$ and $L(V_{g,h}^{\psi}, 0) \neq 0$, then the class $\kappa_{\psi,g,h}$ is nonzero. Since by Theorem 6.5 this is the bottom class of an anticyclotomic Euler system for V_{gh}^{ψ}, from Theorem 8.5 and Proposition 8.10, we deduce that $\mathrm{Sel}_{\mathrm{bal}}(K, A_{g,h}^{\psi})$ has $\mathcal{O}$-corank one, with

$$\mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathrm{bal}}(K, A_{g,h}^{\psi})_{/\mathrm{div}}) \leq 2\,\mathrm{length}_{\mathcal{O}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(K, T_{g,h}^{\psi})}{\mathcal{O} \cdot \kappa_{\psi,g,h}} \right). \tag{9.2}$$

By the exact sequence (7.8) specialised to weight k, it follows that $\mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^{\psi})$ has also $\mathcal{O}$-corank one. Thus, both $\mathrm{Sel}_{\mathrm{bal}}(K, T_{g,h}^{\psi}) \subset \mathrm{Sel}_{\mathcal{F}\cup+}(K, T_{g,h}^{\psi})$ have $\mathcal{O}$-rank one, and therefore

$$\mathrm{Sel}_{\mathrm{bal}}(K, T_{g,h}^{\psi}) = \mathrm{Sel}_{\mathcal{F}\cup+}(K, T_{g,h}^{\psi}), \tag{9.3}$$

since their quotient is $\mathcal{O}$-torsion free. Moreover, letting $\pi \in \mathcal{O}$ be a uniformiser, as in the proof of Lemma 7.13, we find that

$$\mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^{\psi})[\pi^i] \cong E/\mathcal{O}[\pi^i] \oplus \mathrm{Sel}_{\mathcal{F}\cap+}(K, A_{g,h}^{\psi})[\pi^i]$$

for all i, and hence $\mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}\cup+}(K, A_{g,h}^{\psi})_{/\mathrm{div}}) = \mathrm{length}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}\cap+}(K, A_{g,h}^{\psi}))$.

The finiteness of $\mathrm{Sel}_{\mathcal{F}}(K, A_{g,h}^{\psi})$ with the stated bound on its $\mathcal{O}$-length thus follows from (9.2) by the same argument as in the proof of Theorem 7.15, noting that by Theorem 7.4 and the same calculation as in [BSV22, Section 8.5] (see especially, the equality following [*op. cit.*, (189)]), the map

$$\xi_k \cdot \left\langle \exp_p^*(-), \eta_{\breve{f}} \otimes \omega_{\breve{g}} \otimes \omega_{\breve{h}} \right\rangle,$$

where $f = \theta_{\psi}$ and ξ_k is the weight k specialisation of the congruence ideal generator $\xi \in \Lambda_{\mathbf{f}}$, gives an isomorphism $H^1(\mathbb{Q}_p, T_f^{gh}) \to \mathcal{O}$ taking $\kappa_{\psi,g,h}$ to

$$\frac{(l-2)! \cdot (m-2)!}{(k-c-1)!} \cdot \frac{\mathcal{E}_0(\mathbf{f}_k) \cdot \mathcal{E}_1(\mathbf{f}_k)}{\mathcal{E}(\mathbf{f}_k, g, h)} \cdot \mathscr{L}_p^{\xi}(\breve{\mathbf{f}}, \breve{\underline{g}}, \breve{\underline{h}})(k),$$

where $\mathcal{E}_0(\mathbf{f}_k) = \left(1 - \frac{\beta_k}{\alpha_k}\right)$ is a p-adic unit.

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More precisely, under the freeness assumption in the statement, the weight k specialisations of (7.8) and (7.9) yield the exact sequences

$$0 \longrightarrow \mathrm{coker}(\mathrm{res}_p) \longrightarrow \mathrm{Sel}_{\mathcal{F}\cup+}(K, A^{\psi}_{g,h})^{\vee} \longrightarrow \mathrm{Sel}_{\mathrm{bal}}(K, A^{\psi}_{g,h})^{\vee} \longrightarrow 0, \tag{9.4}$$

$$0 \longrightarrow \mathrm{coker}(\mathrm{res}_p) \longrightarrow \mathrm{Sel}_{\mathcal{F}}(K, A^{\psi}_{g,h})^{\vee} \longrightarrow \mathrm{Sel}_{\mathcal{F}\cap+}(K, A^{\psi}_{g,h})^{\vee} \longrightarrow 0,$$

where the two terms $\mathrm{coker}(\mathrm{res}_p)$ are equal in light of (9.3). Thus, we find

$$\begin{aligned}
\mathrm{lt}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}}(K, A^{\psi}_{g,h})) = \mathrm{lt}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}}(K, A^{\psi}_{g,h})^{\vee}) &= \mathrm{lt}_{\mathcal{O}}(\mathrm{Sel}_{\mathcal{F}\cap+}(K, A^{\psi}_{g,h})^{\vee}) + \mathrm{lt}_{\mathcal{O}}(\mathrm{coker}(\mathrm{res}_p)) \\
&= \mathrm{lt}_{\mathcal{O}}((\mathrm{Sel}_{\mathcal{F}\cup+}(K, A^{\psi}_{g,h})_{/\mathrm{div}})^{\vee}) + \mathrm{lt}_{\mathcal{O}}(\mathrm{coker}(\mathrm{res}_p)) \\
&= \mathrm{lt}_{\mathcal{O}}((\mathrm{Sel}_{\mathrm{bal}}(K, A^{\psi}_{g,h})_{/\mathrm{div}})^{\vee}) + 2\,\mathrm{lt}_{\mathcal{O}}(\mathrm{coker}(\mathrm{res}_p)) \\
&= \mathrm{lt}_{\mathcal{O}}(\mathrm{Sel}_{\mathrm{bal}}(K, A^{\psi}_{g,h})_{/\mathrm{div}}) + 2\,\mathrm{lt}_{\mathcal{O}}(\mathrm{coker}(\mathrm{res}_p)),
\end{aligned}$$

where the third equality follows from (9.4) and Lemma 9.7 below, concluding the proof. □

Lemma 9.7. *Let* $0 \to A \xrightarrow{j} B \to C \to 0$ *be an exact sequence of finitely generated* $\mathcal{O}$*-modules, and assume that* A *is finite. Then* $B_{\mathrm{tors}}/j(A) \cong C_{\mathrm{tors}}$.

In particular, if B', C' *are cofinitely generated* $\mathcal{O}$*-modules, and we have an exact sequence* $0 \to A \xrightarrow{j} (B')^{\vee} \to (C')^{\vee} \to 0$ *with* A *finite, then*

$$(B'_{/\mathrm{div}})^{\vee}/j(A) \cong (C'_{/\mathrm{div}})^{\vee},$$

and so $\mathrm{lt}_{\mathcal{O}}((B'_{/\mathrm{div}})^{\vee}) = \mathrm{lt}_{\mathcal{O}}(A) + \mathrm{lt}_{\mathcal{O}}((C'_{/\mathrm{div}})^{\vee})$.

Proof. Writing $B \cong \mathcal{O}^r \oplus B_{\mathrm{tors}}$, $C \cong \mathcal{O}^s \oplus C_{\mathrm{tors}}$, we have, by the finiteness of A, $r = s$ and $j(A) \subset B_{\mathrm{tors}}$, so

$$\mathcal{O}^r \oplus C_{\mathrm{tors}} \cong C \cong B/j(A) \cong \mathcal{O}^r \oplus (B_{\mathrm{tors}}/j(A)),$$

which implies the result. □

Remark 9.8. The condition that $H^1(\mathbb{Q}_p, T_f^{gh})$ is torsion-free is equivalent to the vanishing of $H^0(\mathbb{Q}_p, A_f^{gh})$, which is satisfied if $k+2 \neq l+m$ modulo $2(p-1)$ or if $\chi_f(p)\alpha_g\alpha_h/\alpha_k \neq 1$ modulo p. Similarly, the last condition in the statement of Theorem 9.6 can be recast in terms of the vanishing of the corresponding 0-th cohomology groups.

Remark 9.9. By Theorem 7.1, the nonvanishing of $L(V^{\psi}_{g,h}, 0)$ implies that $\mathscr{L}_p^{\xi}(\underline{\breve{\mathbf{f}}}, \underline{\breve{g}}, \underline{\breve{h}})(x) \neq 0$, so the upper bound provided by Theorem 9.6 is nontrivial. Moreover, by the interpolation formula in Theorem 7.1, this upper bound can be expressed in terms of the central L-value $L(V^{\psi}_{g,h}, 0)$, thus giving a result towards the Tamagawa number conjecture of [BK90].

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9.3. Proof of Theorem D

As before, let $\mathbf{f}$ be the Hida family attached to ψ as in Section 6. Let $\kappa_{\psi,g,h,1,\infty}$ be the Λ-adic class of conductor $n=1$ constructed in Theorem 6.5, and set

$$\kappa_{\psi,g,h,\infty} := \kappa_{\psi,g,h,1,\infty} \in H^1_{\mathrm{Iw}}(K_\infty, T^\psi_{g,h}).$$

As noted before the proof of Proposition 6.6, under the Shapiro isomorphism

$$H^1(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh}) \cong H^1(K, \Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1}) \hat{\otimes}_{\mathcal{O}} T^\psi_{g,h}) \cong H^1_{\mathrm{Iw}}(K_\infty, T^\psi_{g,h}),$$

the balanced Selmer group $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{Q}, \mathbb{V}^\dagger_{\mathbf{f}gh})$ of Section 7.3 is identified with the Greenberg Selmer group $\mathrm{Sel}_{\mathrm{Gr}}(K_\infty, T^\psi_{g,h})$ of Section 8.1 attached to G_{K_v}-invariant subspaces $\mathcal{F}^+_v(V^\psi_{g,h}) \subset V^\psi_{g,h}$ in (4.1) at the primes $v \mid p$. Moreover, under this isomorphism, the class $\kappa(\mathbf{f},g,h)$ in Section 7.2 corresponds to the class $\kappa_{\psi,g,h,\infty}$.

The next result, establishing one of the divisibilities predicted by the Iwasawa main conjectures from Section 7.3, recovers Theorem D in the Introduction.

Theorem 9.10. *Assume hypotheses (h1)–(h8), and, in addition, that:*

- $\varepsilon_\ell(\theta_\psi, g, h) = +1$ *for all primes* $\ell \mid N$,
- $\gcd(N_\psi, N_g, N_h)$ *is squarefree.*

If $\kappa(\mathbf{f},g,h)$ *is not* $\Lambda_{\mathbf{f}}$*-torsion, then the following hold:*

(a) *The modules* $\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ *and* $X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh})$ *have both* $\Lambda_{\mathbf{f}}$*-rank one and*

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathrm{bal}}(\mathbb{A}^\dagger_{\mathbf{f}gh})_{\mathrm{tors}}) \supset \mathrm{Char}_{\Lambda_{\mathbf{f}}}\left(\frac{\mathrm{Sel}_{\mathrm{bal}}(\mathbb{V}^\dagger_{\mathbf{f}gh})}{\Lambda_{\mathbf{f}} \cdot \kappa(\mathbf{f},g,h)}\right)^2.$$

(b) *The modules* $\mathrm{Sel}_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ *and* $X_{\mathcal{F}}(\mathbb{V}^\dagger_{\mathbf{f}gh})$ *are both* $\Lambda_{\mathbf{f}}$*-torsion and*

$$\mathrm{Char}_{\Lambda_{\mathbf{f}}}(X_{\mathcal{F}}(\mathbb{A}^\dagger_{\mathbf{f}gh})) \supset (L_p(\mathbf{f},g,h))$$

in $\Lambda_{\mathbf{f}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Proof. The nontriviality assumption on $\kappa(\mathbf{f},g,h)$ implies that $\kappa_{\psi,g,h,\infty}$ is not $\Lambda_{\mathbf{f}}$-torsion. Since, by Theorem 6.5, the class $\kappa_{\psi,g,h,\infty}$ is the bottom class of a Λ-adic Euler system for $V^\psi_{g,h}$, part (a) follows from Theorem 8.5 and Proposition 8.10. By Theorem 7.15, part (b) of the theorem follows from part (a), so this concludes the proof. □

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