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SMALL-TIME ALMOST-SURE BEHAVIOUR OF EXTREMAL PROCESSES

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Abstract

An *r*th-order extremal process $\Delta^{(r)} = (\Delta_t^{(r)})_{t\geq 0}$ is a continuous-time analogue of the *r*th partial maximum sequence of a sequence of independent and identically distributed random variables. Studying maxima in continuous time gives rise to the notion of limiting properties of $\Delta_t^{(r)}$ as $t \downarrow 0$. Here we describe aspects of the small-time behaviour of $\Delta^{(r)}$ by characterising its upper and lower classes relative to a nonstochastic nondecreasing function $b_t > 0$ with $\lim_{t\downarrow 0} b_t = 0$. We are then able to give an integral criterion for the almost sure relative stability of $\Delta_t^{(r)}$ as $t \downarrow 0$, $r = 1, 2, \ldots$, or, equivalently, as it turns out, for the almost sure relative stability of $\Delta_t^{(1)}$ as $t \downarrow 0$.

Keywords: Extremal process; upper and lower classes; relative stability; Lévy flight; Lévy dust; fractal property

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1. Introduction

We study the small-time (as $t \downarrow 0$) almost-sure (a.s.) behaviour of an *r*th-order extremal process $\Delta^{(r)} = (\Delta_t^{(r)})_{t\geq 0}$, the continuous-time analogue of the sequence of *r*th largest observations in a sequence of independent and identically distributed (i.i.d.) random variables. Our focus is on deriving criteria for lower and upper classes for $\Delta^{(r)}$ which generalise among others a definitive result of Klass [11], [12] relating to the large-time behaviour of the partial maximum of a sequence of i.i.d. random variables. As an application, we give an integral criterion for the almost sure relative stability at 0 of $(\Delta_t^{(r)})$ as $t \downarrow 0$, i.e. having the property that there exists a nonstochastic nondecreasing function $b_t > 0$ such that $\lim_{t\downarrow 0} \Delta_t^{(r)}/b_t = 1$ a.s. The corresponding weak version: $\Delta_t^{(r)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$ for some nonstochastic nondecreasing function $b_t > 0$, with $\lim_{t\downarrow 0} b_t = 0$, is also characterised.

Our interest in this area was partly motivated by the ideas of Lévy flight and Lévy dust, which occur in various applications areas, in particular, physics. As one interesting example, these kinds of models are being used to describe the self-similar or scale invariant spatial distribution of galaxies in the universe, and their fractal properties. The corresponding extremal processes would then refer to the distances between galaxies, or their order statistics, and the small-time aspect would relate to small-scale or local properties. At the other end of the spectrum, similar processes occur in the study of transmission of light or other particles through thin films at the

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nanoscale. We expect that our results will have application in these areas. Buchmann *et al.* [5] provides some further context to the problems we consider.

The paper is structured as follows. In Section 2 we review some background in Poisson point processes required to define the rth-order extremal process. In Section 3 we derive a characterisation of the lower class of $\Delta^{(r)}$ at 0. In Section 4 we give a version for the upper class, and then combine both lower and upper class results to give a criterion for the almost sure relative stability of $\Delta^{(r)}$. Some examples are given in Section 5, and further motivating background and a brief review of related discrete-time results is given in the concluding Section 6.

2. Poisson point processes and extremal processes

Herein we establish the necessary framework of Poisson point processes and extremal processes which are used throughout the sections that follow. We draw on the methodology of Buchmann *et al.* [5]. Although the corresponding $(\Delta_t)_{t\geq 0}$ in their paper is the jump process of a Lévy process, their point process setup is quite general. (An alternative approach to defining an extremal process as a Markov process in terms of its finite-dimensional distributions is given in [16]. Further details about the structure and properties of extremal processes are given in [6], [15], and [18].)

Let Π be a Borel measure on $(0,\infty)$, locally finite at ∞, thus having a finite-valued tail function $\overline{\Pi}\colon (0,\infty)\to(0,\infty)$, defined by $\overline{\Pi}(x) := \Pi(x,\infty)$, $x>0$. This is a right-continuous and nonincreasing function. Let

$$\mathbb{D} = \sum_{s>0} \delta_{(s,\Delta_s)}$$

be a Poisson point process on $(0,\infty)\times(0,\infty)$ with intensity measure $\mathrm{d}t\otimes\Pi(\mathrm{d}x)$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The measure Π need not be a Lévy measure but as a special case $(\Delta_t)_{t\geq 0}$ could be the jump process $(X_t - X_{t-})_{t\geq 0}$ of a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}$, as in [5].

We assume throughout that $\Pi\{(0,\infty)\} = \overline{\Pi}(0+) = \infty$. The number of points $(\Delta_s)_{0\leq s\leq t}$ exceeding a value $\varepsilon>0$ in time interval $[0,t]$ is distributed as Poisson with parameter $t\overline{\Pi}(\varepsilon)$, which increases without bound as $\varepsilon\downarrow 0$, so in this sense there is an arbitrarily large number of points of Δ in any finite-time interval, a.s. Fix $t>0$ and let $T^{(1)}(\mathbb{D}_t)$ be randomly chosen, independently of $(\Delta_t)_{t\geq 0}$, according to the discrete uniform distribution in the set $\{0\leq s\leq t\colon \Delta_s = \sup_{0\leq u\leq t}\Delta_u\}$, which is a.s. finite. Then define $\Delta_t^{(1)} := \Delta_{T^{(1)}(\mathbb{D}_t)}$ and the *maximum-trimmed point process* on $[0,t]\times(0,\infty)$ by

$${}^{(1)}\mathbb{D}_t := \mathbb{D}_t - \delta_{(T^{(1)}(\mathbb{D}_t),\Delta_t^{(1)})}.$$

Setting $r=2,3,\ldots$, proceed to define iteratively $T^{(r)}(\mathbb{D}_t) := T^{(1)}({}^{(r-1)}\mathbb{D}_t)$ and $\Delta_t^{(r)} := \Delta_{T^{(r)}(\mathbb{D}_t)}$. The *$r$-fold trimmed extremal process* is then given by

$${}^{(r)}\mathbb{D}_t := \mathbb{D}_t - \sum_{i=1}^{r}\delta_{(T^{(i)}(\mathbb{D}_t),\Delta_t^{(i)})}.$$

By this procedure we have specified a sequence of ordered pairs

$$(T_t^{(1)},\Delta_t^{(1)}),\ (T_t^{(2)},\Delta_t^{(2)}),\ (T_t^{(3)},\Delta_t^{(3)}),\ldots\in(0,t]\times(0,\infty),$$

in which $\{\Delta_t^{(1)}\geq\cdots\geq\Delta_t^{(r)}\}$ represent the rth largest order statistics of $\mathbb{D}$, sampled on the time interval $[0,t]$, possibly with ties, located at times $T_t^{(i)} = T^{(i)}(\mathbb{D}_t)$, $i\geq 1$. The process $(\Delta_t^{(r)})_{t\geq 0}$

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is the *r th-order extremal process*, $r = 1, 2, \ldots$. As an extension, the notation $\Delta_I^{(r)}$ will be used to denote the rth largest order statistic of $\mathbb{D}$ sampled on an interval I of the form (a, b) or $(a, b]$, constructed similarly.

Let $(\mathcal{F}_t)_{t>0}$ be the natural filtration of $\Delta = \{\Delta_t : t \geq 0\}$ completed with the null sets of $\mathcal{F}$, so that $\mathcal{F}_0 = \mathcal{F}_{0+}$ (the 'usual' hypothesis). We refer the reader to [5] for more background information on the properties of $(\Delta_t^{(r)})_{t\geq 0}$. In particular, using the Poisson property, we can calculate

$$\mathbb{P}(\Delta_t^{(r)} \leq x) = \mathrm{e}^{-t\overline{\Pi}(x)} \sum_{n=0}^{r-1} \frac{(t\overline{\Pi}(x))^n}{n!}, \qquad t > 0,\ x > 0,\ r = 1, 2, \ldots. \tag{2.1}$$

The $\Delta_t^{(r)}$ are nondecreasing in $t > 0$ for each $r = 1, 2, \ldots$ and nonincreasing in r for each $t > 0$, and since $\Pi\{(0, \infty)\} = \infty$, we have $\Delta_t^{(r)} > 0$ a.s. for all $t > 0$ for each $r = 1, 2, \ldots$, and $\Delta_t^{(r)} \downarrow 0$ a.s. as $t \downarrow 0$ for each $r = 1, 2, \ldots$.

3. Lower classes for $\Delta^{(r)}$ at 0

Throughout, b_t will be a positive nondecreasing nonstochastic function on $(0, \infty)$ with $\lim_{t\downarrow 0} b_t = 0$ and r is a fixed positive integer. We say that b_t is in the *lower class* for $\Delta_t^{(r)}$ at 0 if

$$\mathbb{P}(\Delta_t^{(r)} \leq b_t \text{ i.o. as } t \downarrow 0) = 1, \tag{3.1}$$

where i.o. denotes infinitely often. (Our usage of this terminology varies from some in the (random walk) literature. We use 'lower class' to denote the behaviour in (3.1). 'Upper class' for us will denote the behaviour in (4.1) in Section 4.) If this is not the case then the probability in (3.1) is 0. (This follows from Blumenthal's zero-one law for Markov processes.) In this section we derive a criterion to distinguish between these situations.

Theorem 3.1. *Assume that* $\overline{\Pi}(0+) = \infty$ *and let* r *and* b_t *be as specified.*

(i) *Suppose that* $\liminf_{t\downarrow 0} t\overline{\Pi}(b_t) < \infty$. *Then*

$$\mathbb{P}(\Delta_t^{(r)} \leq b_t \text{ i.o. as } t \downarrow 0) = 1. \tag{3.2}$$

(ii) *Alternatively, suppose that* $\lim_{t\downarrow 0} t\overline{\Pi}(b_t) = \infty$. *Then (3.2) holds if and only if*

$$J_r := \int_0^1 t^{r-1}\overline{\Pi}^r(b_t-)\mathrm{e}^{-t\overline{\Pi}(b_t-)}\,\mathrm{d}t = \infty. \tag{3.3}$$

Remark 3.1. (i) In Theorem 3.1(ii), $t\overline{\Pi}(b_t)$ is large for small t, so $J_r < \infty$ implies that $J_{r-1} < \infty$, $r = 2, 3, \ldots$. Thus, $\Delta_t^{(r)} > b_t$ all but finitely often, a.s., as $t \downarrow 0$, implying that $\Delta_t^{(r-1)} > b_t$ all but finitely often, a.s., as $t \downarrow 0$, as of course it should.

(ii) It is not difficult to show that (3.3) is equivalent, under the assumptions of Theorem 3.1, to

$$\int_0^1 t^{r-1}\overline{\Pi}^r(b_t)\mathrm{e}^{-t\overline{\Pi}(b_t)}\,\mathrm{d}t = \infty.$$

(iii) The monotonicity requirement on b_t can be relaxed somewhat, analogous to the discrete situation, as discussed by Klass [11]. We omit the details.

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Proof of Theorem 3.1. Throughout, assume that $\overline{\Pi}(0+) = \infty$ and let $b_t \downarrow 0$ as $t \downarrow 0$.

(i) Suppose first that $\liminf_{t\downarrow 0} t\overline{\Pi}(b_t) < \infty$. Then there exists $c > 0$ and a sequence $t_k \downarrow 0$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \mathbb{P}(\Delta_{t_k}^{(1)} \le b_{t_k}) = \lim_{k\to\infty} \mathrm{e}^{-t_k\overline{\Pi}(b_{t_k})} = \mathrm{e}^{-c} > 0.$$

This implies

$$\begin{aligned}\mathbb{P}(\Delta_{t_k}^{(1)} \le b_{t_k} \text{ i.o. as } k \to \infty) &= \lim_{N\to\infty} \mathbb{P}\Big(\bigcup_{k\ge N} \{\Delta_{t_k}^{(1)} \le b_{t_k}\}\Big) \\ &\ge \lim_{N\to\infty} \mathbb{P}(\Delta_{t_N}^{(1)} \le b_{t_N}) \\ &= \mathrm{e}^{-c} \\ &> 0.\end{aligned}$$

Hence, by the zero-one law, $\mathbb{P}(\Delta_{t_k}^{(1)} \le b_{t_k} \text{ i.o. as } k \to \infty) = 1$. Then, since $\Delta_t^{(r)} \le \Delta_t^{(1)}$, we have $\mathbb{P}(\Delta_{t_k}^{(r)} \le b_{t_k} \text{ i.o. as } k \to \infty) = 1$ and (3.2) follows.

(ii) From now on assume $\lim_{t\downarrow 0} t\overline{\Pi}(b_t) = \infty$ and without loss of generality that $t\overline{\Pi}(b_t) \ge a_r$ for all $t \in (0, 1]$, where $a_r > 4r^2$ is a constant to be specified later. We will show that (3.2) and (3.3) are equivalent.

Assume that (3.3) holds. Let $t_0 = 1 = b_{t_0}$ and define the sequence t_k recursively by

$$t_{k+1} = t_k - \frac{1}{\overline{\Pi}(b_{t_k})}, \qquad k = 0, 1, 2, \ldots. \tag{3.4}$$

Clearly, $0 < t_{k+1} < t_k$ and $t_k \downarrow 0$ as $k \to \infty$ (because if $t_k \downarrow T > 0$ then we would have $T + 1/\overline{\Pi}(b_{T+}) = T$, which is not possible since $b_{T+} > 0$ and $\overline{\Pi}(b_{T+}) < \infty$).

Using the fact that $f(x) = x^r \mathrm{e}^{-tx}$ is decreasing for $x > r/t$ together with (3.4), we have

$$\begin{aligned}\int_0^1 t^{r-1}\overline{\Pi}^r(b_t)\mathrm{e}^{-t\overline{\Pi}(b_t)}\,\mathrm{d}t &= \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} t^{r-1}\overline{\Pi}^r(b_t)\mathrm{e}^{-t\overline{\Pi}(b_t)}\,\mathrm{d}t \\ &\le \sum_{k=0}^{\infty} \int_{t_{k+1}}^{t_k} t^{r-1}\overline{\Pi}^r(b_{t_k})\mathrm{e}^{-t\overline{\Pi}(b_{t_k})}\,\mathrm{d}t \\ &\le \sum_{k=0}^{\infty} (t_k - t_{k+1})\overline{\Pi}(b_{t_k})(t_k\overline{\Pi}(b_{t_k}))^{r-1}\mathrm{e}^{-t_{k+1}\overline{\Pi}(b_{t_k})} \\ &= \mathrm{e}\sum_{k=0}^{\infty} (t_k\overline{\Pi}(b_{t_k}))^{r-1}\mathrm{e}^{-t_k\overline{\Pi}(b_{t_k})}.\end{aligned}$$

Since (3.3) holds, we conclude that

$$\sum_{k=0}^{\infty} (t_k\overline{\Pi}(b_{t_k}))^{r-1}\mathrm{e}^{-t_k\overline{\Pi}(b_{t_k})} = \infty. \tag{3.5}$$

Recall that $\overline{\Pi}(b_t) \ge 4r^2/t$ for $t \in (0, 1]$. Construct a subsequence $\{s_k\}$ of $\{t_j\}$ as follows. Let $s_1 = t_1$ and assume s_k has been defined. Note that, for any $j \ge 1$, we have $t_{j+1}\overline{\Pi}(b_{t_j}) = t_j\overline{\Pi}(b_{t_j}) - 1 \ge 4r^2 - 1 \ge 1$, so the quantity

$$s_{k+1} := \min\{t_j : t_j\overline{\Pi}(b_{s_k}) \ge 1\}, \qquad k = 0, 1, 2, \ldots,$$

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satisfies $0 < s_{k+1} < s_k$, and we have $s_k \downarrow 0$ as $k \to \infty$. Furthermore,

$$t_j \overline{\Pi}(b_{s_k}) \le 1 \quad \text{for all } t_j < s_{k+1}, \tag{3.6}$$

and

$$s_{k+1} \overline{\Pi}(b_{s_k}) \ge 1. \tag{3.7}$$

Define a sequence of events A_k, $k = 1, 2, \ldots$, by

$$A_k := \{\Delta_{t_i}^{(r)} \le b_{t_i} \text{ for some } t_i \in [s_{k+1}, s_k)\}.$$

Take $k \ge 1$ and $m > k + 2$. Then $s_m < s_{k+2}$ and $b_{s_m} \le b_{s_{k+2}}$. Also define

$$A_k^*(m) := \{\Delta_{(s_m, t_i]}^{(r)} \le b_{t_i} \text{ for some } t_i \in [s_{k+1}, s_k)\}$$

(recall that $\Delta_{(s_m,t_i]}^{(r)}$ is the rth largest of the Δ_s for $s \in (s_m, t_i]$). Then $A_k \subseteq A_k^*(m)$, and $A_k^*(m)$ and A_m are independent. Hence,

$$\mathbb{P}(A_m \cap A_k) \le \mathbb{P}(A_m \cap A_k^*(m)) = \mathbb{P}(A_m)\mathbb{P}(A_k^*(m)).$$

If $A_k^*(m)$ occurs and $\Delta_{s_m}^{(1)} \le b_{s_{k+1}}$ then $\Delta_{t_i}^{(r)} \le b_{t_i}$ for some $t_i \in [s_{k+1}, s_k)$, and so A_k occurs. Thus, we have the inequality

$$\mathbb{P}(A_k^*(m))\mathbb{P}(\Delta_{s_m}^{(1)} \le b_{s_{k+1}}) \le \mathbb{P}(A_k),$$

and we obtain

$$\begin{aligned}
\mathbb{P}(A_m \cap A_k) &\le \mathbb{P}(A_m)\mathbb{P}(A_k^*(m)) \\
&\le \frac{\mathbb{P}(A_m)\mathbb{P}(A_k)}{\mathbb{P}(\Delta_{s_m}^{(1)} \le b_{s_{k+1}})} \\
&= \mathbb{P}(A_m)\mathbb{P}(A_k)\mathrm{e}^{s_m \overline{\Pi}(b_{s_{k+1}})} \\
&\le \mathrm{e}\mathbb{P}(A_m)\mathbb{P}(A_k) \quad \text{(using (3.6), and } m > k + 2).
\end{aligned}$$

Hence, $\mathbb{P}(A_m \cap A_k) \le \mathrm{e}\mathbb{P}(A_m)\mathbb{P}(A_k)$ for all $m > k + 2$.

Suppose that

$$\sum_{k=0}^{\infty} \mathbb{P}(A_k) = \infty. \tag{3.8}$$

Then, for $n \ge 4$,

$$\begin{aligned}
\sum_{m=2}^{n} \sum_{k=1}^{m-1} \mathbb{P}(A_m \cap A_k) &\le 2 \sum_{m=1}^{n} \mathbb{P}(A_m) + \sum_{m=4}^{n} \sum_{k=1}^{m-3} \mathbb{P}(A_m \cap A_k) \\
&\le 2 \sum_{m=1}^{n} \mathbb{P}(A_m) + \mathrm{e} \sum_{m=1}^{n} \sum_{k=1}^{n} \mathbb{P}(A_m)\mathbb{P}(A_k) \\
&\sim \mathrm{e}\left(\sum_{m=1}^{n} \mathbb{P}(A_m) \right)^2 \quad \text{as } n \to \infty.
\end{aligned}$$

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With this, Spitzer's converse Borel–Cantelli lemma [19, p. 317] for the events $\{A_k\}_{k\in\mathbb{N}}$ allows us to deduce, from (3.8), that $\mathbb{P}(A_k \text{ i.o. as } k \to \infty) = 1$, which implies (3.2).

To complete this part of the proof, it remains to prove (3.8), knowing (3.5). For $k = 1, 2, \ldots$ and $j = 1, 2, \ldots, \ell(k)$, where $\ell(k) = \max\{j\colon js_{k+1} \le s_k\}$, define a sequence of subevents of $\{A_k\}$ by

$$A_{k,j} := \{\Delta_{t_i}^{(r)} \le b_{t_i} \text{ for some } t_i \in [js_{k+1}, ((j+1)s_{k+1}) \wedge s_k)\}.$$

(If there is no t_i in $[js_{k+1}, (j+1)s_{k+1})$ set $A_{k,j} = \varnothing$.) Define another sequence of events by

$$B_{k,j} := \{\Delta^{(r)}_{((j+1)s_{k+1},\,(j+2)s_{k+1}]} > b_{s_k}\}, \qquad j, k, = 1, 2, \ldots.$$

Clearly, $A_{k,j}$ and $B_{k,j}$ are independent for each j and k. In addition, for each k, $B_{k,j}$ and $A_{k,j'}$ are disjoint when $j' \ge j+2$. This is true because, on $B_{k,j}$, we have $\Delta_t^{(r)} > b_{s_k}$ for all $t \ge (j+2)s_{k+1}$. But on $A_{k,j'}$, we have $\Delta_{t_i}^{(r)} \le b_{s_k}$ for some $t_i \ge j's_{k+1}$. This is impossible when $j' \ge j+2$.

Using (2.1), $\mathbb{P}(B_{k,j})$ can be bounded away from 0, because

$$\begin{aligned}
\mathbb{P}(B_{k,j}) &= 1 - \mathbb{P}(\Delta_{s_{k+1}}^{(r)} \le b_{s_k}) \\
&= 1 - \mathrm{e}^{-s_{k+1}\overline{\Pi}(b_{s_k})} \sum_{n=0}^{r-1} \frac{(s_{k+1}\overline{\Pi}(b_{s_k}))^n}{n!} \\
&\overset{(3.7)}{\ge} 1 - \mathrm{e}^{-1} \sum_{n=0}^{r-1} \frac{1}{n!} \\
&=: M_1 \\
&> 0.
\end{aligned} \tag{3.9}$$

In the final inequality, we used the fact that $f(x) = 1 - \mathrm{e}^{-x}\sum_{n=0}^{r-1} x^n/n!$ is an increasing function of x. Equation (3.9) allows us to bound $\mathbb{P}(A_k)$ from below as follows: for each $k = 1, 2, \ldots,$

$$\begin{aligned}
2\mathbb{P}(A_k) &= 2\mathbb{P}\Big(\bigcup_{j=1}^{\ell(k)} A_{k,j}\Big) \\
&\ge \mathbb{P}\Big(\bigcup_{j \text{ even}} (A_{k,j} \cap B_{k,j})\Big) + \mathbb{P}\Big(\bigcup_{j \text{ odd}} (A_{k,j} \cap B_{k,j})\Big) \\
&= \sum_{j=1}^{\ell(k)} \mathbb{P}(A_{k,j} \cap B_{k,j}) \\
&= \sum_{j=1}^{\ell(k)} \mathbb{P}(A_{k,j})\mathbb{P}(B_{k,j}) \\
&\ge M_1 \sum_{j=1}^{\ell(k)} \mathbb{P}(A_{k,j}).
\end{aligned} \tag{3.10}$$

In the second equality we used the disjointness of $B_{k,j}$ and $A_{k,j'}$ for $j' \ge j+2$ and in the third we used the independence of $A_{k,j}$ from $B_{k,j}$.

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Now set

$$C_i = \{\Delta_{t_i}^{(r)} \le b_{t_i}\}, \qquad i = 1, 2, \ldots.$$

By (2.1) and (3.5), we have

$$\begin{aligned}\sum_{i=1}^{\infty} \mathbb{P}(C_i) &= \sum_{i=1}^{\infty} \mathrm{e}^{-t_i \overline{\Pi}(b_{t_i})} \sum_{n=0}^{r-1} \frac{(t_i \overline{\Pi}(b_{t_i}))^n}{n!} \\ &\ge \sum_{i=1}^{\infty} \mathrm{e}^{-t_i \overline{\Pi}(b_{t_i})} \frac{(t_i \overline{\Pi}(b_{t_i}))^{r-1}}{(r-1)!} \\ &= \infty. \end{aligned} \tag{3.11}$$

Let integers $p = p(j,k) < \cdots < q(j,k) = q$ index those t_i occurring in the interval $[js_{k+1}, (j+1)s_{k+1} \wedge s_k)$. (Recall that we restrict attention to nonempty $A_{k,j}$. If there is only one t_i in the interval $[js_{k+1}, (j+1)s_{k+1} \wedge s_k)$, set $p = q$.) Then $js_{k+1} \le t_q < t_{q-1} < \cdots < t_p < (j+1)s_{k+1} \wedge s_k$, and $t_i \in [t_q, t_p]$ for any $p \le i \le q$. Let

$$D_i = \{\text{exactly } r \text{ values of } s \in (t_i, t_{i-1}) \text{ satisfy } \Delta_s > b_{t_p}\}, \qquad i = 1, 2, \ldots.$$

On D_j, for $j \ge i+1$, we have $\Delta_{t_{j-1}}^{(r)} > b_{t_p}$, hence, $\Delta_{t_i}^{(r)} > b_{t_p}$, but on C_i, we have $\Delta_{t_i}^{(r)} \le b_{t_i} \le b_{t_p}$. Thus, D_j is disjoint from C_i for $j \ge i+1$, $i \ge p$. Additionally, C_i is independent of D_i, $i = 1, 2, \ldots$, and

$$\mathbb{P}(D_i) = \frac{((t_{i-1} - t_i)\overline{\Pi}(b_{t_p}))^r \mathrm{e}^{-(t_{i-1}-t_i)\overline{\Pi}(b_{t_p})}}{r!}. \tag{3.12}$$

We can now bound $\mathbb{P}(A_{k,j})$ from below as follows:

$$\begin{aligned} 2\mathbb{P}(A_{k,j}) &\ge \mathbb{P}\Bigg(\bigcup_{i=p+1}^{q} C_i\Bigg) + \mathbb{P}(C_p) \\ &\ge \mathbb{P}\Bigg(\bigcup_{i=p+1}^{q} (C_i \cap D_i)\Bigg) + \mathbb{P}(C_p) \\ &= \sum_{i=p+1}^{q} \mathbb{P}(C_i \cap D_i) + \mathbb{P}(C_p) \\ &= \sum_{i=p+1}^{q} \mathbb{P}(C_i)\mathbb{P}(D_i) + \mathbb{P}(C_p). \end{aligned} \tag{3.13}$$

Here we used the disjointness of D_j from C_i for $j \ge i+1, i \ge p$, together with the independence of C_i and D_i. In the following, hold j, k and $p = p(j,k), q = q(j,k)$, fixed.

If $p = q$, let $m = q$, otherwise set $m = \min\{i : p < i \le q, 4\overline{\Pi}(b_{t_p}) \le \overline{\Pi}(b_{t_i})\}$ with $m = q$ if $4\overline{\Pi}(b_{t_p}) > \overline{\Pi}(b_{t_i})$ for all $p < i \le q$. So $4\overline{\Pi}(b_{t_p}) > \overline{\Pi}(b_{t_i}) \ge \overline{\Pi}(b_{t_{i-1}})$ for $p \le i \le m-1$. We can now estimate $\mathbb{P}(D_i)$ using (3.12). As an upper bound

$$(t_{i-1} - t_i)\overline{\Pi}(b_{t_p}) = \frac{\overline{\Pi}(b_{t_p})}{\overline{\Pi}(b_{t_{i-1}})} \le 1 \quad \text{for } i \ge p+1.$$

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For a lower bound, suppose that $q > p$, so $m > p$, and keep $p < i < m$. Then $4\overline{\Pi}(b_{t_p}) \geq \overline{\Pi}(b_{t_{i-1}})$ and

$$(t_{i-1} - t_i)\overline{\Pi}(b_{t_p}) \geq \tfrac{1}{4}(t_{i-1} - t_i)\overline{\Pi}(b_{t_{i-1}}) = \tfrac{1}{4}.$$

Since $x^r \mathrm{e}^{-x}$ increases on $[0, r]$, we obtain

$$\mathbb{P}(D_i) = \frac{((t_{i-1} - t_i)\overline{\Pi}(b_{t_p}))^r \mathrm{e}^{-(t_{i-1}-t_i)\overline{\Pi}(b_{t_p})}}{r!} \geq \frac{4^{-r}\mathrm{e}^{-1/4}}{r!}.$$

Then as a lower bound for $\mathbb{P}(A_{k,j})$, we deduce that, from (3.13),

$$\mathbb{P}(A_{k,j}) \geq M_2 \sum_{i=p}^{m-1} \mathbb{P}(C_i), \tag{3.14}$$

where $M_2 = 4^{-r}\mathrm{e}^{-1/4}/2r!$. When $m = p = q$, $\mathbb{P}(A_{k,j}) \geq \mathbb{P}(C_p)$ by (3.13).

The next task is to estimate the summation term in (3.14). For brevity in this calculation, write $y_i = \overline{\Pi}(b_{t_i})$, $i \geq 1$. Define $\mathcal{E}_0 := \{i : p \leq i \leq q, 4y_p > y_i\}$ and, for $\ell \geq 1$, let $\mathcal{E}_\ell := \{i : m \leq i \leq q, 4^\ell y_p \leq y_i \leq 4^{\ell+1} y_p\}$. Then, since $(t_i - t_{i+1})y_i = 1$,

$$|\mathcal{E}_\ell| \leq 1 + 2 \sum_{\{i \in \mathcal{E}_\ell,\, i+1 \in \mathcal{E}_\ell\}} (t_i - t_{i+1})y_i \leq 1 + 2 \cdot 4^{\ell+1} y_p \sum_{\{i \in \mathcal{E}_\ell,\, i+1 \in \mathcal{E}_\ell\}} (t_i - t_{i+1}) \leq 4^{\ell+2} t_p y_p.$$

We have $\frac{1}{2}t_p \leq \frac{1}{2}(j+1)s_{k+1} \leq j s_{k+1} \leq t_i$ and $y_i \geq 4^\ell y_p$ when $i \in \mathcal{E}_\ell$ and $\ell \geq 1$. Thus, $t_i y_i \geq 4^\ell t_p y_p/2 > r - 1$ and since $x^{r-1}\mathrm{e}^{-x}$ decreases for $x > r - 1$, we can calculate

$$\begin{aligned}
\sum_{i \in \mathcal{E}_\ell} \mathbb{P}(C_i) &\leq r \sum_{i \in \mathcal{E}_\ell} (t_i y_i)^{r-1} \mathrm{e}^{-t_i y_i} \\
&\leq r \left(\frac{4^\ell t_p y_p}{2} \right)^{r-1} \mathrm{e}^{-4^\ell t_p y_p/2} |\mathcal{E}_\ell| \\
&\leq 32r \left(\frac{4^\ell t_p y_p}{2} \right)^{r} \mathrm{e}^{-4^\ell t_p y_p/2}.
\end{aligned} \tag{3.15}$$

In order to sum this expression over ℓ, we use the following estimate, true for any $r \geq 1$, $c > 0$, $a > 1$, with $ca > r$:

$$\begin{aligned}
c^r \sum_{\ell \geq 2} a^{\ell r} \mathrm{e}^{-ca^\ell} &\leq c^r \sum_{\ell \geq 2} \int_{\ell-1}^{\ell} a^{xr} \mathrm{e}^{-ca^x} \, \mathrm{d}x \\
&\leq \frac{1}{\log a} \int_{ca}^{\infty} y^{r-1} \mathrm{e}^{-y} \, \mathrm{d}y \\
&\leq \frac{r(ca)^{r-1}\mathrm{e}^{-ca}}{\log a}.
\end{aligned}$$

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Applying this with $a = 4$ and $c = \frac{1}{2}t_p y_p$ we have, from (3.15),

$$\begin{aligned}\sum_{\ell=1}^{\infty}\sum_{i\in\mathcal{E}_\ell}\mathbb{P}(C_i) &\le 32r(2t_p y_p)^r \mathrm{e}^{-2t_p y_p} + \frac{32r^2(2t_p y_p)^{r-1}\mathrm{e}^{-2t_p y_p}}{\log 4}\\ &\le 64r^2 2^r (t_p y_p)^r \mathrm{e}^{-2t_p y_p}\\ &\le \frac{(t_p y_p)^{r-1}\mathrm{e}^{-t_p y_p}}{2(r-1)!}\\ &\le \tfrac{1}{2}\mathbb{P}(C_p)\end{aligned}$$

(provided we choose $t_p y_p = t_p \overline{\Pi}(b_{t_p}) > a_r$ large enough so that $128r^2 2^r (r-1)!\, t_p y_p \le \mathrm{e}^{t_p y_p}$). Hence,

$$\sum_{i=p}^{q}\mathbb{P}(C_i) = \sum_{i=p}^{m-1}\mathbb{P}(C_i) + \sum_{\ell=1}^{\infty}\sum_{i\in\mathcal{E}_\ell}\mathbb{P}(C_i) \le \sum_{i=p}^{m-1}\mathbb{P}(C_i) + \frac{\mathbb{P}(C_p)}{2},$$

yielding

$$\sum_{i=p}^{m-1}\mathbb{P}(C_i) \ge \frac{1}{2}\sum_{i=p}^{q}\mathbb{P}(C_i),$$

and, from (3.14),

$$\mathbb{P}(A_{k,j}) \ge \frac{M_2}{2}\sum_{i=p}^{q}\mathbb{P}(C_i).$$

Finally, (3.8) follows from (3.11) and

$$\begin{aligned}2\sum_{k=1}^{\infty}\mathbb{P}(A_k) &\overset{(3.10)}{\ge} M_1\sum_{k=1}^{\infty}\sum_{j=1}^{\ell(k)}\mathbb{P}(A_{k,j}).\\ &\ge M_3\sum_{k=1}^{\infty}\sum_{j=1}^{\ell(k)}\sum_{i=p}^{q}\mathbb{P}(C_i) \quad (\text{with } M_3 := \tfrac{1}{2}M_1M_2)\\ &= M_3\sum_{i=1}^{\infty}\mathbb{P}(C_i)\\ &= \infty.\end{aligned}$$

This proves the forward direction of the theorem. For the converse, assume that

$$\int_{(0,1]}(t\overline{\Pi}(b_{t-}))^r \mathrm{e}^{-t\overline{\Pi}(b_{t-})}\frac{\mathrm{d}t}{t} < \infty \quad \text{for some } r = 1, 2, \ldots. \tag{3.16}$$

We will show that

$$\mathbb{P}(\Delta_t^{(r)} \le b_t \text{ i.o. as } t \downarrow 0) = 0.$$

To do this, define a sequence of random times $\{\tau_n\}_{n=0,1,2,\ldots}$ by

$$\tau_0 = 1, \qquad \tau_{n+1} = \inf\{0 < t < \tau_n \colon \Delta_t = \Delta_{\tau_n-}^{(r)}\}, \qquad n = 0, 1, 2, \ldots,$$

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where we use the notation $\Delta_{t-}^{(r)} = \sup_{0<s<t} \Delta_s^{(r)}$. Since $\Delta_t^{(r)} > 0$ a.s. for all $t > 0$ and $\Delta_t^{(r)} \downarrow 0$ a.s. as $t \downarrow 0$, we have $0 < \tau_n \downarrow 0$ as $n \to \infty$.

For $0 < t \le 1$, let $n_0(t)$ be the integer such that $\tau_{n_0(t)} \le t < \tau_{n_0(t)-1}$. Then

$$\begin{aligned}\mathbb{P}(\Delta_t^{(r)} \le b_t \text{ i.o. as } t \downarrow 0) &= \lim_{t\downarrow 0} \mathbb{P}\Bigg(\bigcup_{0<s<t} \{\Delta_s^{(r)} \le b_s\}\Bigg) \\ &= \lim_{t\downarrow 0} \mathbb{P}\Bigg(\bigcup_{n\ge n_0(t)} \bigcup_{\tau_{n+1}\le s<\tau_n} \{\Delta_s^{(r)} \le b_s\}\Bigg) \\ &\le \lim_{t\downarrow 0} \mathbb{P}\Bigg(\bigcup_{n\ge n_0(t)} \{\Delta_{\tau_n-}^{(r)} \le b_{\tau_n-}\}\Bigg).\end{aligned}$$

Here we used the fact that $\tau_{n+1} \le s < \tau_n$ implies $\Delta_s^{(r)} = \Delta_{\tau_{n+1}}^{(r)} = \Delta_{\tau_n-}^{(r)}$ and $b_s \le b_{\tau_n-}$. Hence,

$$\mathbb{P}(\Delta_t^{(r)} \le b_t \text{ i.o. as } t \downarrow 0) \le \lim_{t\downarrow 0} \mathbb{P}(\Delta_{s-}^{(r)} \le b_{s-} \text{ for some } s \in (0,t] \text{ with } \Delta_s > \Delta_{s-}^{(r)}).$$

To compute the probability on the right-hand side we apply the compensation formula [13, Theorem 4.4, p. 95] as follows. Let

$$E_t = \{\Delta_{s-}^{(r)} \le b_{s-} \text{ for some } s \in (0,t] \text{ with } \Delta_s > \Delta_{s-}^{(r)}\}, \qquad t > 0,$$

and let

$$N_t = \int_{(0,t]\times(0,\infty)} \mathbf{1}_{\{\Delta_{s-}^{(r)} \le b_{s-},\, x > \Delta_{s-}^{(r)}\}}\, \mathbb{D}(\mathrm{d}s, \mathrm{d}x)$$

be the number of points (s, Δ_s) which satisfy $\Delta_{s-}^{(r)} \le b_{s-}$ and $\Delta_s > \Delta_{s-}^{(r)}$ for $s \in (0,t]$. Then, we have, for $0 < t < 1$,

$$\begin{aligned}\mathbb{P}(E_t) &= \mathbb{P}(N_t \ge 1) \le \mathbb{E}[N_t] \quad \text{(by Markov's inequality)} \\ &= \mathbb{E}\Bigg[\int_{(0,t]\times(0,\infty)} \mathbf{1}_{\{\Delta_{s-}^{(r)} \le b_{s-},\, x > \Delta_{s-}^{(r)}\}}\, \mathbb{D}(\mathrm{d}s, \mathrm{d}x)\Bigg] \\ &= \int_{(0,t]\times(0,\infty)} \mathbb{E}\Big[\mathbf{1}_{\{\Delta_{s-}^{(r)} \le b_{s-},\, x > \Delta_{s-}^{(r)}\}}\Big]\, \mathrm{d}s\Pi(\mathrm{d}x) \\ &= \int_{(0,t]\times(0,\infty)} \mathbb{P}(\Delta_{s-}^{(r)} \le b_{s-},\ x > \Delta_{s-}^{(r)})\, \mathrm{d}s\Pi(\mathrm{d}x) \\ &= \int_0^t \Bigg(\int_{0<x\le b_{s-}} \mathbb{P}(\Delta_{s-}^{(r)} < x)\Pi(\mathrm{d}x) + \mathbb{P}(\Delta_{s-}^{(r)} \le b_{s-})\overline{\Pi}(b_{s-})\Bigg)\, \mathrm{d}s \\ &=: \int_0^t (A(s) + B(s))\, \mathrm{d}s. \end{aligned} \tag{3.17}$$

(In the third equality we used the compensation formula.)

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We consider A and B separately. Keep $s > 0$ small enough for $s\overline{\Pi}(b_{s-}) \geq r$ and write

$$\begin{aligned} A(s) &= \int_{0<x\leq b_{s-}} \mathbb{P}(\Delta_{s-}^{(r)} < x)\Pi(\mathrm{d}x) \\ &\overset{(2.1)}{=} \int_{0<x\leq b_{s-}} \sum_{i=0}^{r-1} \frac{(s\overline{\Pi}(x-))^i}{i!} \mathrm{e}^{-s\overline{\Pi}(x-)}\Pi(\mathrm{d}x) \\ &\leq r \int_{0<x\leq b_{s-}} (s\overline{\Pi}(x-))^{r-1}\mathrm{e}^{-s\overline{\Pi}(x-)}\Pi(\mathrm{d}x), \end{aligned}$$

in which we note that $s\overline{\Pi}(x-) \geq s\overline{\Pi}(b_{s-}) \geq 1$. Define $x = \overline{\Pi}^{\rightarrow}(y) := \sup\{z > 0\colon \overline{\Pi}(z) > y\}$, $y > 0$, and use the change of variable formula [4, p. 301] to write the last integral as

$$\begin{aligned} \int_{\overline{\Pi}(b_{s-})}^{\infty} (s\overline{\Pi}(\overline{\Pi}^{\rightarrow}(y)-))^{r-1}\mathrm{e}^{-s\overline{\Pi}(\overline{\Pi}^{\rightarrow}(y)-)}\mathrm{d}y &\leq \int_{\overline{\Pi}(b_{s-})}^{\infty} (sy)^{r-1}\mathrm{e}^{-sy}\mathrm{d}y \\ &= s^{-1}\int_{s\overline{\Pi}(b_{s-})}^{\infty} y^{r-1}\mathrm{e}^{-y}\mathrm{d}y. \end{aligned} \tag{3.18}$$

In the last inequality, we used $\overline{\Pi}(\overline{\Pi}^{\rightarrow}(y)-) \geq y$ and the fact that $y^{r-1}\mathrm{e}^{-y}$ is decreasing for $y > r-1$, noting that when $0 < x \leq b_{s-}$, $s\overline{\Pi}(x-) \geq s\overline{\Pi}(b_{s-}) \geq r$ by assumption. Estimating the incomplete gamma function, from (3.18), we obtain, for small $s > 0$,

$$A(s) \leq rs^{-1}(s\overline{\Pi}(b_{s-}))^{r-1}\mathrm{e}^{-s\overline{\Pi}(b_{s-})}.$$

Easier is

$$B(s) = \overline{\Pi}(b_{s-})\mathrm{e}^{-s\overline{\Pi}(b_{s-})} \sum_{i=0}^{r-1} \frac{(s\overline{\Pi}(b_{s-}))^i}{i!} \leq rs^{-1}(s\overline{\Pi}(b_{s-}))^r\mathrm{e}^{-s\overline{\Pi}(b_{s-})}.$$

So we have, for small $s > 0$,

$$A(s) + B(s) \leq 2rs^{-1}(s\overline{\Pi}(b_{s-}))^r\mathrm{e}^{-s\overline{\Pi}(b_{s-})}.$$

The last expression is integrable over $0 < s \leq t$ by (3.16), so the right-hand side of (3.17) tends to 0 as $t \downarrow 0$. Thus, $\lim_{t\downarrow 0}\mathbb{P}(E_t) = 0$ and this means that $\mathbb{P}(\Delta_t^{(r)} \leq b_t \text{ i.o. as } t \downarrow 0) = 0$. This completes the proof of Theorem 3.1. □

4. Upper classes for $\Delta_t^{(r)}$ and relative stability

We say that b_t is in the upper class for $\Delta^{(r)}$ at 0 if

$$\mathbb{P}(\Delta_t^{(r)} > b_t \text{ i.o. as } t \downarrow 0) = 1. \tag{4.1}$$

Otherwise, $\mathbb{P}(\Delta_t^{(r)} > b_t \text{ i.o. as } t \downarrow 0) = 0$. Recall we assume throughout that $\overline{\Pi}(0+) = \infty$, the nonstochastic function b_t is positive and nondecreasing on $(0, \infty)$ with $\lim_{t\downarrow 0} b_t = 0$, and r is a fixed positive integer. In this section we characterise the upper class for $\Delta_t^{(r)}$ at 0 relative to b_t. We often write $b(\cdot)$ for $b_\cdot$ from now on.

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Proposition 4.1. *The following are equivalent:*

$$\mathbb{P}(\Delta_t^{(r)} > b_t \text{ i.o. as } t \downarrow 0) = 0, \qquad \int_0^1 \overline{\Pi}^r(b_t)\,\mathrm{d}t^r < \infty,$$
$$\sum_{n\geq 0}(2^{-n}\overline{\Pi}(b(2^{-n})))^r < \infty. \tag{4.2}$$

If any of these hold then $\lim_{t\downarrow 0} t\overline{\Pi}(b_t) = 0$.

Proof. This is in Lemma 3.1 of [14]. □

We will also need the following differential formula:

$$\frac{|\mathrm{d}\overline{\Pi}^r(x)|}{\mathrm{d}\Pi(x)} = r\overline{\Pi}^{r-1}(x)\,\mathbf{1}_{\{\Delta\overline{\Pi}(x)=0\}} + \frac{\overline{\Pi}^r(x-) - \overline{\Pi}^r(x)}{\Delta\overline{\Pi}(x)}\,\mathbf{1}_{\{\Delta\overline{\Pi}(x)\neq 0\}}$$
$$= \sum_{k=0}^{r-1}\overline{\Pi}^k(x)\overline{\Pi}^{r-1-k}(x-), \qquad x > 0,$$

where $\Delta\overline{\Pi}(x) := \overline{\Pi}(x-) - \overline{\Pi}(x)$. (For an analogous formula, see [9, Equation (3.3), p. 437].) Consequently, we have the inequalities

$$\overline{\Pi}^{r-1}(x-) \leq \frac{|\mathrm{d}\overline{\Pi}^r(x)|}{\mathrm{d}\Pi(x)} \leq r\overline{\Pi}^{r-1}(x-). \tag{4.3}$$

The right-continuous inverse of the nonincreasing function $\overline{\Pi}$ is defined by

$$\overline{\Pi}^{\leftarrow}(x) = \inf\{y > 0 \colon \overline{\Pi}(y) \leq x\}, \qquad x > 0.$$

By convention, the inf of the empty set is taken as ∞. The following properties of the inverse function will be used: for each $x, y > 0$, $\overline{\Pi}^{\leftarrow}(x) \leq y$ if and only if $\overline{\Pi}(y) \leq x$, $\overline{\Pi}(\overline{\Pi}^{\leftarrow}(x)) \leq x \leq \overline{\Pi}(\overline{\Pi}^{\leftarrow}(x)-)$, and $\overline{\Pi}^{\leftarrow}(\overline{\Pi}(x)) \leq x$.

Proposition 4.1 combined with Theorem 3.1 gives our next result, Theorem 4.1, regarding the almost sure relative stability of $(\Delta_t^{(r)})$ as $t \downarrow 0$. As background, we mention that the relative stability of the order statistics from a sample $\{\xi_n \colon n \geq 0\}$ of i.i.d. random variables has been studied by a number of authors through the years. The sequence $(\xi_n^{(r)})$ of rth largest values among $(\xi_i)_{1\leq i\leq n}$ is said to be *relatively stable* if there exists a nonrandom sequence $b_n > 0$ such that $\xi_n^{(r)}/b_n \to 1$ as $n \to \infty$, either in probability or almost surely. These cases are discussed in [7] and [10], among others. Integral criteria for the almost sure relative stability were given in [2], [9], and [18]. See also [20]. (Some further background is given in Section 6.) We can analogously study the relative stability of $\Delta_t^{(r)}$ as $t \downarrow 0$, and begin by stating necessary and sufficient conditions for the almost sure version of this in the next theorem.

Theorem 4.1. *Assume that* $\overline{\Pi}(0+) = \infty$ *and fix* $r = 1, 2, \ldots$. *There exists a nonstochastic nondecreasing function* $b_t > 0$ *such that* $\lim_{t\downarrow 0}\Delta_t^{(r)}/b_t = 1$ *a.s. if and only if*

$$\int_0^1 \left(\frac{\overline{\Pi}(x-)}{\overline{\Pi}((1-\varepsilon)x-)}\right)^r \frac{\Pi(\mathrm{d}x)}{\overline{\Pi}(x-)} < \infty \quad \text{for all } \varepsilon \in (0,1). \tag{4.4}$$

If this is the case then b_t *is slowly varying at* 0 *and can be chosen such that* $b_t = \overline{\Pi}^{\leftarrow}(1/t)$, $t > 0$.

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Furthermore, if (4.4) holds for some $r > 1$ then it holds for $r = 1, 2, \ldots$; thus, $\Delta_t^{(r)}$ is relatively stable a.s. as $t \downarrow 0$ if and only if $\Delta_t^{(1)}$ is relatively stable a.s. as $t \downarrow 0$.

Before proving Theorem 4.1 we need to characterise relative stability in probability.

Proposition 4.2. *There exists a nonstochastic nondecreasing function $b_t > 0$ such that*

$$\frac{\Delta_t^{(r)}}{b_t} \xrightarrow{\mathbb{P}} 1 \quad \text{as } t \downarrow 0$$

if and only if

$$\lim_{x \downarrow 0} \frac{\overline{\Pi}((1+\varepsilon)x)}{\overline{\Pi}(x)} = 0 \quad \text{for all } \varepsilon > 0. \tag{4.5}$$

If these hold then $b_t \sim \overline{\Pi}^{\leftarrow}(1/t)$ and b_t is slowly varying as $t \downarrow 0$.

Remark 4.1. Since condition (4.5) does not depend on r, we see that $\Delta_t^{(r)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$ for any $r = 1, 2, \ldots$ if and only if $\Delta_t^{(1)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$; thus, relative stability in probability is obtained regardless of which (fixed) value of r is specified. This is perhaps not so surprising, as the same occurs for the relative stability in probability of the order statistics from an i.i.d. sample (see [10], and, for a similar phenomenon, [20]), but it is more surprising that the same holds for almost sure relative stability, This was established in [9] for discrete time. Theorem 4.1 shows that the same is obtained in our present continuous-time setup. We draw on the methods of [9] in that part of the proof of Theorem 4.1.

Proof of Proposition 4.2. For $b_t > 0$ and $\varepsilon > 0$, use (2.1) to write

$$\mathbb{P}(\Delta_t^{(r)} > (1+\varepsilon)b_t) = 1 - \mathrm{e}^{-t\overline{\Pi}((1+\varepsilon)b_t)} \sum_{i=0}^{r-1} \frac{(t\overline{\Pi}((1+\varepsilon)b_t))^i}{i!}. \tag{4.6}$$

(i) Assume that $\Delta_t^{(r)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$, where $b_t > 0$ and $\lim_{t\downarrow 0} b_t = 0$. Then, for $\varepsilon > 0$, we have $\mathbb{P}(\Delta_t^{(r)} > (1+\varepsilon)b_t) \to 0$ as $t \downarrow 0$. Take a subsequence through which $t\overline{\Pi}((1+\varepsilon)b_t)$ tends to a limit $a \in [0, \infty]$ as $t \downarrow 0$. By (4.6), $a \in (0, \infty]$ is impossible so we have $a = 0$. This is true for all subsequences so we have $\lim_{t\downarrow 0} t\overline{\Pi}((1+\varepsilon)b_t) = 0$. A similar argument gives $\lim_{t\downarrow 0} t\overline{\Pi}((1-\varepsilon)b_t) = \infty$ for $\varepsilon \in (0, 1)$. Substitute these back in (4.6) with $r = 1$ to deduce that $\Delta_t^{(1)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$. Then, from

$$\frac{\Delta_{2t}^{(1)}}{b_{2t}} \le \frac{\Delta_{2t}^{(1)}}{b_t} = \frac{\max(\Delta_t^{(1)}, \sup_{t<s\le 2t} \Delta_s)}{b_t} \xrightarrow{\mathbb{P}} 1 \quad \text{as } t \downarrow 0,$$

we obtain $\Delta_{2t}^{(1)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$, and, hence, $b_{2t} \sim b_t$, as $t \downarrow 0$. Since b_t is nondecreasing, this means that b_t is slowly varying as $t \downarrow 0$. Given $\delta \in (0, 1)$, set $\eta = \delta/(2+\delta)$, so that $(1+\delta)(1-\eta) = 1+\eta$. Then choose $x > 0$ small enough and $t = t(x)$ such that

$$(1-\eta)b(2t) \le b\left(\tfrac{1}{2}t\right) \le b(t-) \le x \le b(t+) \le b(2t),$$

implying

$$(1+\eta)b(2t) = (1+\delta)(1-\eta)b(2t) \le (1+\delta)b\left(\tfrac{1}{2}t\right) \le (1+\delta)x.$$

Then

$$\frac{\overline{\Pi}((1+\delta)x)}{\overline{\Pi}((1-\delta)x)} \le \frac{t\overline{\Pi}((1+\eta)b(2t))}{t\overline{\Pi}((1-\delta)b(2t))}.$$

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The right-hand side tends to 0 because $\Delta_{2t}^{(r)}/b_{2t} \xrightarrow{\mathbb{P}} 1$, and, hence, (replacing $(1-\delta)x$ by x and $(1+\delta)/(1-\delta)$ by $1+\varepsilon$), we see that (4.5) holds.

(ii) Conversely, assume that (4.5) holds and choose $b_t := \overline{\Pi}^{\leftarrow}(1/t)$. Then b_t is nondecreasing and $\overline{\Pi}(b_t) \le 1/t \le \overline{\Pi}(b_t-)$. By (4.5), we have $t\overline{\Pi}((1+\varepsilon)b_t) = o(t\overline{\Pi}(b_t)) = o(1)$, so by (4.6), $\mathbb{P}(\Delta_t^{(r)} > (1+\varepsilon)b_t) \to 0$ as $t \downarrow 0$; while also

$$t\overline{\Pi}((1-\varepsilon)b_t) \ge \frac{\overline{\Pi}((1-\varepsilon)b_t)}{\overline{\Pi}(b_t-)} \ge \frac{\overline{\Pi}((1-\varepsilon)b_t)}{\overline{\Pi}((1-\varepsilon/2)b_t)} \to \infty,$$

so by (2.1), $\mathbb{P}(\Delta_t^{(r)} < (1-\varepsilon)b_t) \to 0$ as $t \downarrow 0$. These give $\Delta_t^{(r)}/b_t \xrightarrow{\mathbb{P}} 1$ as $t \downarrow 0$ with $b_t = \overline{\Pi}^{\leftarrow}(1/t)$. □

Proof of Theorem 4.1. (i) We will first show that (4.4) implies

$$\mathbb{P}(\Delta_t^{(r)} > (1+\varepsilon)b_t \text{ i.o. as } t \downarrow 0) = 0 \quad \text{for all } \varepsilon > 0, \tag{4.7}$$

with b_t defined as $\overline{\Pi}^{\leftarrow}(1/t)$, $t > 0$. Assume (4.4) and define b_t in this way. Then $t\overline{\Pi}(b_t) \le 1 \le t\overline{\Pi}(b_t-)$. Take $\varepsilon > 0$ and use the right-hand side of (4.3) to estimate

$$\begin{aligned}
&\sum_{n\ge1}(2^{-n}\overline{\Pi}((1+\varepsilon)b(2^{-n})-))^r \\
&\quad\le r\sum_{n\ge1}2^{-nr}\int_{[(1+\varepsilon)b(2^{-n}),\,\infty)}\overline{\Pi}^{r-1}(x-)\Pi(\mathrm{d}x) \\
&\quad\le r\sum_{n\ge1}2^{-nr}\sum_{j=1}^{n}\int_{[(1+\varepsilon)b(2^{-j}),\,(1+\varepsilon)b(2^{-j+1}))}\overline{\Pi}^{r-1}(x-)\Pi(\mathrm{d}x) + c_r.
\end{aligned}$$

Here, $c_r := r2^{-r}\overline{\Pi}^r((1+\varepsilon)b(1)-)/(1-2^{-r})$ and we interpret intervals of the form $[a,a)$ or $(a,a]$ as empty. Interchange the order of summations in the last expression and perform the inner summation to obtain an upper bound of the form

$$\frac{r}{1-2^{-r}}\sum_{j\ge1}2^{-jr}\int_{[(1+\varepsilon)b(2^{-j}),\,(1+\varepsilon)b(2^{-j+1}))}\overline{\Pi}^{r-1}(x-)\Pi(\mathrm{d}x) + c_r. \tag{4.8}$$

When $(1+\varepsilon)b(2^{-j}) \le x$, we have

$$b(2^{-j}) \le \frac{x}{1+\varepsilon\wedge1} < \frac{x}{1+(\varepsilon\wedge1)/2} = (1-\delta)x,$$

where $\delta = \delta(\varepsilon) = (\varepsilon\wedge1)/(2+\varepsilon\wedge1) \in (0,1)$. So (since $t\overline{\Pi}(b_t) \le 1$)

$$2^{-jr} \le \frac{1}{\overline{\Pi}^r(b(2^{-j}))} \le \frac{1}{\overline{\Pi}^r((1-\delta)x-)}.$$

Substitute this into (4.8) to obtain the upper bound

$$\frac{r}{1-2^{-r}}\sum_{j\ge1}\int_{[(1+\varepsilon)b(2^{-j}),\,(1+\varepsilon)b(2^{-j+1}))}\left(\frac{\overline{\Pi}(x-)}{\overline{\Pi}((1-\delta)x-)}\right)^r\frac{\Pi(\mathrm{d}x)}{\overline{\Pi}(x-)} + c_r.$$

This is finite by (4.4). This working shows that (4.2) holds with $b(2^{-n})$ replaced by $(1+\varepsilon)b(2^{-n})$, and (4.2) in that form implies (4.7) by Proposition 4.1.

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(ii) Next we show that if (4.4) holds and $b_t = \overline{\Pi}^{\leftarrow}(1/t)$, then

$$\mathbb{P}(\Delta_t^{(r)} \le (1-\varepsilon)b_t \text{ i.o. as } t \downarrow 0) = 0 \quad \text{for all } \varepsilon \in (0,1). \tag{4.9}$$

This together with (4.7) will give $\lim_{t\downarrow 0} \Delta_t^{(r)}/b_t = 1$ a.s.

Assume (4.4) and $b_t = \overline{\Pi}^{\leftarrow}(1/t)$. Then, as shown in part (i), we have

$$\sum_{n\ge 1} (2^{-n}\overline{\Pi}((1+\varepsilon)b(2^{-n})-))^r < \infty \quad \text{for all } \varepsilon > 0. \tag{4.10}$$

This in turn implies that

$$\lim_{x\downarrow 0} \frac{\overline{\Pi}(x)}{\overline{\Pi}((1-\varepsilon)x)} = 0 \quad \text{for all } \varepsilon \in (0,1). \tag{4.11}$$

To see this, note that (4.10) implies that

$$\lim_{n\to\infty} 2^{-n}\overline{\Pi}((1+\varepsilon)b(2^{-n})-) = 0 \quad \text{for all } \varepsilon > 0.$$

Using

$$2^{n-1} \le \overline{\Pi}(\overline{\Pi}^{\leftarrow}(2^{n-1})-) = \overline{\Pi}(b(2^{-n+1})-),$$

we obtain

$$\lim_{n\to\infty} \frac{\overline{\Pi}((1+\varepsilon)b(2^{-n})-)}{\overline{\Pi}(b(2^{-n+1})-)} = 0.$$

Then, given $x > 0$, choose $n = n(x)$ so that $b(2^{-n}) \le x \le b(2^{-n+1})$ and apply this relation to obtain

$$\lim_{x\downarrow 0} \frac{\overline{\Pi}((1+\varepsilon)x-)}{\overline{\Pi}(x-)} = 0 \quad \text{for all } \varepsilon \in (0,1),$$

which is equivalent to (4.11).

Since $\overline{\Pi}(b_t-) \ge 1/t$, (4.11) implies that $\lim_{t\downarrow 0} t\overline{\Pi}((1-\varepsilon)b_t-) = \infty$ for all $\varepsilon \in (0,1)$, hence, $\lim_{t\downarrow 0} t\overline{\Pi}((1-2\varepsilon)b_t) = \infty$ for all $\varepsilon \in (0, \frac{1}{2})$. Referring to Theorem 3.1, we can thus prove (4.9) by showing that

$$\int_0^1 t^{r-1}\overline{\Pi}^r((1-2\varepsilon)b_t-)\mathrm{e}^{-t\overline{\Pi}((1-2\varepsilon)b_t-)}\,\mathrm{d}t < \infty \quad \text{for all } \varepsilon \in (0, \tfrac{1}{2}). \tag{4.12}$$

To this end, note that, for $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\int_0^1 t^{r-1}\overline{\Pi}^r((1-\varepsilon)b_t-)\mathrm{e}^{-t\overline{\Pi}((1-\varepsilon)b_t-)}\,\mathrm{d}t \\
&\quad = \int_0^1 \frac{t^{-r-1}}{\overline{\Pi}^r((1-\varepsilon)b_t-)}(t\overline{\Pi}((1-\varepsilon)b_t-))^{2r}\mathrm{e}^{-t\overline{\Pi}((1-\varepsilon)b_t-)}\,\mathrm{d}t \\
&\quad \le C\int_0^1 \frac{t^{-r-1}}{\overline{\Pi}^r((1-\varepsilon)b_t-)}\mathrm{d}t,
\end{aligned} \tag{4.13}$$

where $C := \sup_{x>0} x^{2r}\mathrm{e}^{-x}$. Let $\delta = \delta(\varepsilon) = \varepsilon/(1-\varepsilon)$, so $1+\delta = 1/(1-\varepsilon)$, and let A_n denote the intervals

$$A_n = \left\{ t > 0 \colon \frac{1}{\overline{\Pi}((1+\delta)b(2^{-n-1}))} \le t < \frac{1}{\overline{\Pi}((1+\delta)b(2^{-n}))} \right\}, \qquad n = 1, 2, \ldots.$$

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For $t \in A_n$ we have $1/t > \overline{\Pi}((1+\delta)b(2^{-n}))$, and so

$$b_t = \overline{\Pi}^{\leftarrow}\left(\frac{1}{t}\right) \le \overline{\Pi}^{\leftarrow}(\overline{\Pi}((1+\delta)b(2^{-n}))) \le (1+\delta)b(2^{-n}).$$

Hence, $(1-\varepsilon)b_t \le b(2^{-n})$ and $\overline{\Pi}((1-\varepsilon)b_t-) \ge \overline{\Pi}(b(2^{-n})-) \ge 2^n$. Thus, the integral on the right-hand side of (4.13) does not exceed

$$C\sum_{n\ge 0} 2^{-nr}\int_{A_n} t^{-r-1}\,\mathrm{d}t \le C\sum_{n\ge 0} 2^{-nr}\overline{\Pi}^r((1+\delta)b(2^{-n-1})),$$

and this is finite by (4.10). This implies (4.12), so (4.9) is proved.

(iii) Conversely, assume that $\lim_{t\downarrow 0}\Delta_t^{(r)}/b_t = 1$ a.s. for a nonstochastic nondecreasing function $b_t > 0$. Then the same convergence holds in probability, so by Proposition 4.2, we have $b_t \sim \overline{\Pi}^{\leftarrow}(1/t)$, thus, without loss of generality we may take $b_t = \overline{\Pi}^{\leftarrow}(1/t)$. Also take $\eta > 0$. Then (4.7) holds with ε replaced by η, so (4.2) holds with b_t replaced by $(1+\eta)b_t$. Use the left-hand side of (4.3) and a similar interchange of the order of summations as in (4.8) to write

$$\sum_{n\ge 1}(2^{-n}\overline{\Pi}((1+\eta)b(2^{-n})))^r \ge a_r\sum_{j\ge 1}2^{(-j+1)r}\int_{((1+\eta)b(2^{-j}),\,(1+\eta)b(2^{-j+1})]}\overline{\Pi}^{r-1}(x-)\Pi(\mathrm{d}x), \tag{4.14}$$

where $a_r := 2^{-r}/(1-2^{-r})$. When $x \le (1+\eta)b(2^{-j+1})$ and $\varepsilon = \eta/(1+\eta)$, we have

$$\overline{\Pi}((1-\varepsilon)x-) = \overline{\Pi}\left(\frac{x}{1+\eta}-\right) \ge \overline{\Pi}(b(2^{-j+1})-),$$

so

$$2^{-j+1} \ge \frac{1}{\overline{\Pi}(b(2^{-j+1})-)} \ge \frac{1}{\overline{\Pi}((1-\varepsilon)x-)}.$$

Thus, a lower bound for the right-hand side of (4.14) is, apart from constants,

$$\sum_{j\ge 1}\int_{((1+\eta)b(2^{-j}),\,(1+\eta)b(2^{-j+1})]}\left(\frac{\overline{\Pi}(x-)}{\overline{\Pi}((1-\varepsilon)x-)}\right)^r\frac{\Pi(\mathrm{d}x)}{\overline{\Pi}(x-)}.$$

Thus, (4.4) holds.

It remains to show that if (4.4) holds for some $r > 1$ then it holds for $r = 1, 2, \ldots$. This part of the proof uses key results proved in [9]. So assume that (4.4) holds with an $r > 1$ and argue as follows. The integral in (4.4) is equal to

$$\begin{aligned}\int_{0<x<1}\frac{\overline{\Pi}^{r-1}(x-)\Pi(\mathrm{d}x)}{\overline{\Pi}^r((1-\varepsilon)x-)} &= \int_{0<x<1}\overline{\Pi}^{r-1}(x-)\int_{0<z<x}\mathrm{d}\overline{\Pi}^{-r}((1-\varepsilon)z)\Pi(\mathrm{d}x)\\ &= \int_{0<z<1}\left(\int_{z<x<1}\overline{\Pi}^{r-1}(x-)\Pi(\mathrm{d}x)\right)\mathrm{d}\overline{\Pi}^{-r}((1-\varepsilon)z)\\ &\asymp \int_{0<z<1}\overline{\Pi}^r(z)\,\mathrm{d}\overline{\Pi}^{-r}((1-\varepsilon)z)\\ &= \int_{0<y<1-\varepsilon}\overline{\Pi}^r((1+\delta)y)\mathrm{d}\overline{\Pi}^{-r}(y),\end{aligned} \tag{4.15}$$

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where $1+\delta = 1/(1-\varepsilon)$ and the symbol '$\asymp$' connecting two quantities means that they are bounded above and below by constant multiples of each other (we used the inequalities (4.3) to obtain this in (4.15)). Now choose $x_0 > 0$ so that $\overline{\Pi}(\mathrm{e}^{-x_0}) > 1$, and $\delta > 0$ so small that $\log(1+\delta) < x_0$, and define

$$\bar{F}(x) = \frac{1}{\overline{\Pi}(\mathrm{e}^{-x}-)}\,\mathbf{1}_{\{x\ge x_0\}} + \mathbf{1}_{\{x<x_0\}}\,. \tag{4.16}$$

Then $\bar{F}$ is right continuous, nonincreasing on $[0,\infty)$ with $\bar{F}(0+) = 1$ and $\bar{F}(\infty) = 0$, so we can regard it as the tail of a cumulative distribution function on $[0,\infty)$. Let $\overline{\Pi}_-(x) := \overline{\Pi}(x-)$, $x > 0$. Decomposing Π into continuous and discrete components, convergence or otherwise of the last integral in (4.15) (for all $\delta > 0$) is seen to be equivalent to that of the same integral with $\overline{\Pi}$ replaced by $\overline{\Pi}_-$ (for all $\delta > 0$). Substituting $\bar{F}$ as in (4.16) in this modified integral shows that the convergence of the last integral in (4.15) is equivalent to that of

$$\int_{0<y<\mathrm{e}^{-x_0}} \frac{\mathrm{d}\bar{F}^r(-\log y)}{\bar{F}^r(-\log y - c)} = \int_{x>x_0} \frac{|\mathrm{d}\bar{F}^r(x)|}{\bar{F}^r(x-c)} \asymp \int_{x>x_0} \frac{\bar{F}^{r-1}(x-)|\mathrm{d}\bar{F}(x)|}{\bar{F}^r(x-c)}.$$

Here we set $c := \log(1+\delta) \in (0, x_0)$ and make use of analogous inequalities as in (4.3), for $\bar{F}$. In the denominator of the last integral, $\bar{F}(x-c) \le \bar{F}((x-c)-) \le \bar{F}(x-c/2)$, so we can replace $\bar{F}(x-c)$ by $\bar{F}((x-c)-)$, then the modified integral is $I_r(c)$ in the notation of [9, Equation (2.6)]. By Equations (2.19) and (2.20) of [9], we have $I_r(\varepsilon) < \infty$ for every $\varepsilon > 0$ if and only if $I_1(\varepsilon) < \infty$ for every $\varepsilon > 0$. Referring back to (4.15) this means that (4.4) holds for $r = 1$, hence, for $r = 1, 2, \ldots$, as required. □

5. Examples

We conclude with some examples to illustrate the possibilities in Theorems 3.1 and 4.1. We use the functions $\ell_2(t) = \log|\log t|$, $\ell_{k+1}(t) = \log \ell_k(t)$, $k = 2, 3, \ldots$ when $0 < t \le \delta$ for a sufficiently small $\delta \in (0,1)$.

Example 5.1. Let $b_t \equiv t$ and take $\eta \ge 0$. Define Π_η so that

$$t\overline{\Pi}_\eta(b_t) = t\overline{\Pi}_\eta(t) = \ell_2(t) + 2\ell_3(t) + (1+\eta)\ell_4(t).$$

Then

$$\mathrm{e}^{-t\overline{\Pi}_\eta(b_t)} = \frac{1}{|\log t|(\log|\log t|)^2(\log\log|\log t|)^{1+\eta}},$$

so the integral in (3.3), taken over $(0,\delta)$, converges or diverges with

$$\int_0^\delta \frac{(\log|\log t|)^{r-1}}{|\log t|\log|\log t|(\log\log|\log t|)^{1+\eta}}\frac{\mathrm{d}t}{t}.$$

When $r = 1$, this integral converges for $\eta > 0$ and diverges for $\eta = 0$. But when $r = 2, 3, \ldots$ it diverges for all $\eta \ge 0$. Thus, for an extremal process corresponding to Π_η with $\eta > 0$, $\Delta_t^{(1)} > t$ all but finitely often, a.s., as $t \downarrow 0$, whereas $\Delta_t^{(2)} \le t$ infinitely often, a.s., as $t \downarrow 0$.

Example 5.2. Here is an example where $\Delta_t^{(r)}$ is relatively stable in probability but not almost surely. Consider the function

$$f(x) := \mathrm{e}^{|\log x|\ell_2(x)}, \qquad 0 < x \le \delta.$$

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A calculation shows that, for $0 < x \le \delta$ and $0 < \lambda < 1$,

$$\begin{aligned}\log f(x\lambda) &= |\log(x\lambda)| \log(|\log(x\lambda)|) \\ &= (|\log x| + |\log \lambda|)\left(\log|\log x| + \log\left(1 + \frac{|\log\lambda|}{|\log x|}\right)\right) \\ &= \log f(x) + |\log\lambda|\ell_2(x) + O(1) \quad \text{as } x \downarrow 0.\end{aligned}$$

Hence,

$$\frac{f(x\lambda)}{f(x)} = \mathrm{e}^{|\log\lambda|\ell_2(x)+O(1)} \asymp |\log x|^{|\log\lambda|} \quad \text{as } x \downarrow 0.$$

It follows that $\lim_{x\downarrow 0} f(x\lambda)/f(x) = \infty$ for each $0 < \lambda < 1$, so if we define Π such that $\overline{\Pi}(x) = f(x)$, then $\overline{\Pi}$ is rapidly varying at 0 and the corresponding $\Delta_t^{(r)}$ is relatively stable in probability at 0.

On the other hand, $f'(x) < 0$ for $0 < x \le \delta$, and

$$|f'(x)| \sim \frac{f(x)\log|\log x|}{x} \quad \text{as } x \downarrow 0,$$

so, considering the integral in (4.4) with $\overline{\Pi}(x) = f(x)$, $r = 1$, and $1 - \varepsilon = \lambda \in (0, 1)$,

$$\begin{aligned}\int_{(0,\delta]} \frac{\Pi(\mathrm{d}x)}{\overline{\Pi}(x\lambda)} &= \int_{(0,\delta]} \left(\frac{|f'(x)|}{f(x\lambda)}\right)\mathrm{d}x \\ &\asymp \int_{(0,\delta]} \left(\frac{f(x)\log|\log x|}{f(x)|\log x|^{|\log\lambda|}}\right)\frac{\mathrm{d}x}{x} \\ &= \int_{(0,\delta]} \left(\frac{\log|\log x|}{|\log x|^{|\log\lambda|}}\right)\frac{\mathrm{d}x}{x}.\end{aligned}$$

This integral diverges when $|\log\lambda| \le 1$, i.e. $\lambda > \mathrm{e}^{-1}$, so $\Delta_t^{(1)}$ is not a.s. relatively stable at 0.

6. Further background and connections

We conclude with some further historical background concerning the discrete case. Barndorff-Nielsen [1] gave a lower class criterion relative to a given nonstochastic sequence for the partial maximum of a sequence of i.i.d. random variables, but needed to impose an unpleasant side condition. (Resnick and Rubinovitch [17] mentioned continuous-time, small-, and large-time, versions of this result.) Klass [11], [12] removed the side condition, thus giving clean necessary and sufficient conditions for the partial maximum of a sequence of i.i.d. random variables to be in the lower class (according to our usage of the term).

Godbole [8] reproved Klass's result by martingale methods, avoiding the need for the delicate (and clever) choices of subsequences in Klass's papers. These authors restricted themselves to the $r = 1$ case. Harry Kesten in some unpublished notes to the first author in the 1990s outlined an extension of Klass's result to rth-order statistics. Our Theorem 3.1 is a continuous-time version of these discrete-time results, proved for general $r = 1, 2, \ldots$, making specific use of the Poisson property of the jumps in the continuous-time setup as well as a modification of Klass's subsequence method. Godbole's martingale methods do not seem to transfer easily to our situation because time 'runs in reverse' as we let $t \downarrow 0$.

Upper class criteria are much easier to establish and are quoted, for example, in [1] and the earlier literature.

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