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Spatial determinism for a free Z^2-action

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Abstract. We extend the idea of bilateral determinism of a free Z-action by D. Ornstein and B. Weiss to a free Z^2-action. We show that we have a 'stronger' spatial determinism for Z^2-actions: to determine the complete Z^2-name of a point, it is enough to know the name of a fraction of the orbit whose density can be made arbitrarily small. Moreover, for zero-entropy Z^2-actions, we prove that there exists a partition $\tilde{\mathcal{P}}$ such that the $\tilde{\mathcal{P}}$-names of an arbitrarily small one-sided cone determine the points.

1. *Introduction*

If a dynamical system $(X, \mathcal{F}, \mu, T)$ has finite entropy, then it is well known [**7**] that there exists a finite partition $\mathcal{P}$ satisfying $\bigvee_{-\infty}^{\infty} T^i\mathcal{P} = \mathcal{F}$, and hence $h(T) = h(T, \mathcal{P})$. If a process $(X, \mathcal{P}, \mu, T)$ has entropy 0, then it has the property that the tail field of the partition $\mathcal{P}$ is the whole σ-algebra. That is,

$$\bigcap_n \bigvee_{i\le -n} T^{-i}\mathcal{P} = \bigvee_{i=-\infty}^{\infty} T^i\mathcal{P}.$$

Also, the tail field characterizes a Kolmogorov automorphism in the sense that the tail field is trivial if and only if T is a Kolmogorov automorphism.

In the case of a Z^2-process $(X, \mathcal{P}, \mu, \phi)$, the past is defined using the lexicographic order [**3**]. And, the entropy of the system is known to be the conditional entropy of the partition given the past as in the case of a Z-action. The tail field of the partition $\mathcal{P}$ is defined as follows: let $\{(M_k, N_k)\}$ be a sequence going to (∞, ∞). Let

$$E_k = \{(i, j) : j \le 0 \text{ if } i \le -M_k, \text{ or } j \le -N_k \text{ if } i > -M_k\}.$$

If $\tau_0(\mathcal{P})$ denotes the σ-algebra

$$\bigcap_k \left\{ \bigvee_{(i,j)\in E_k} \phi^{-(i,j)}\mathcal{P} \right\},$$

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then the tail field $\tau(\mathcal{P})$ is defined to be $\bigvee_{j=-\infty}^{\infty} \phi^{-(0,j)}\tau_0(\mathcal{P})$. We notice that $\tau_0(\mathcal{P})$ is independent of the sequence $\{(M_k, N_k)\}$ chosen and $\tau(\mathcal{P})$ is invariant under ϕ. It was also shown in the case of a Z^2-action that the tail field is trivial if and only if ϕ is a Kolmogorov automorphism. That is, $\tau(\mathcal{P})$ is trivial if and only if ϕ is of completely positive entropy [**1**].

Ornstein and Weiss [**8**] have proved that every aperiodic transformation is isomorphic to a bilaterally deterministic process: given a generating partition $\mathcal{P}$, there exists a partition $\tilde{\mathcal{P}}$ such that

$$\bigvee_{|i|\geq n} T^{-i}\tilde{\mathcal{P}} = \bigvee_{-\infty}^{\infty} T^{-i}\mathcal{P}$$

for all n. That is, given a dynamical system $(X, \mathcal{F}, \mu, T)$, there exists a partition $\tilde{\mathcal{P}}$ such that the $\tilde{\mathcal{P}}$-name of a point $x = \{x_i\}$ for $|i| \geq n$ determines the $\mathcal{P}$-name for all i. Moreover, $\tilde{\mathcal{P}}$ can be chosen so that $|\mathcal{P} - \tilde{\mathcal{P}}| < \epsilon$ for any given $\epsilon > 0$. We want to generalize the Ornstein and Weiss theorem to Z^2-actions.

In the case of a Z^2-action, we are able to prove a 'stronger' assertion. For an angle $\pi/2 \leq \theta < \pi$, we let

$$G_\theta = \{(i, j) : j \leq 0, \text{ or } 0 < j < |i| \tan(\pi - \theta) \text{ for } i < 0\}.$$

We claim that given a generating partition $\mathcal{P}$ under ϕ and given any $\epsilon > 0$ and any $\pi/2 \leq \theta < \pi$, there exists a partition $\tilde{\mathcal{P}}$ such that:

(1) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$; and

(2) $\bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$.

In fact, we prove more. We construct a partition $\tilde{\mathcal{P}}$ independent of θ: there exists a partition $\tilde{\mathcal{P}}$ such that:

(3) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$; and

(4) $\bigcap_{\pi/2\leq\theta<\pi} \bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$.

It is interesting to compare determinisms between Z-actions and Z^2-actions. We say that this Z^2 analogue of the Ornstein and Weiss theorem is 'stronger' in the sense that the density of G_θ in Z^2 is strictly less than 1, while the set $\{|i| \geq n\}$ has density 1 in Z. We may mention that the above results hold for other regions besides G_θ. See the examples of §2 and Remark 1.

Suppose $h(\phi) > 0$. Let $\mathcal{P}$ be a partition such that $h(\phi, \mathcal{P}) > 0$. Since we have

$$h(\phi, \tilde{\mathcal{P}}) = H\left(\tilde{\mathcal{P}} \,\middle|\, \bigvee_{j\leq -1} \phi^{-(i,j)}\tilde{\mathcal{P}} \vee \bigvee_{i\leq -1} \phi^{-(i,0)}\tilde{\mathcal{P}}\right) > 0,$$

the σ-algebra

$$\bigvee_{j\leq -1} \phi^{-(i,j)}\tilde{\mathcal{P}} \vee \bigvee_{i\leq -1} \phi^{-(i,0)}\tilde{\mathcal{P}}$$

can never be the full σ-algebra. In particular,

$$\bigvee_{j\leq 0} \phi^{-(i,j)}\tilde{\mathcal{P}} = \phi^{(0,1)}\left(\bigvee_{j\leq -1} \phi^{-(i,j)}\tilde{\mathcal{P}}\right)$$

is never the full σ-algebra, while

$$\bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F} \quad \text{for an arbitrary } \frac{\pi}{2} \leq \theta < \pi.$$

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We consider zero-entropy Z^2-actions. There are many Z^2-actions which essentially come from Z-actions. Many of these actions will have the property that for some $\vec{v} = (x, y)$ and some $k \geq 0$,

$$\bigvee_{(i,j)\in IP(\vec{v},k)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F},$$

where $IP(\vec{v}, k)$ denotes the double-sided infinite parallelogram in the direction of $\vec{v}$. That is,

$$IP(\vec{v}, k) = \left\{(i, j) : -k + j\frac{x}{y} \leq i \leq k + j\frac{x}{y}\right\}.$$

We note that if $\vec{v} = (p, q)$ is a rational vector and the Z-action $\phi^{(p,q)}$ has finite positive entropy, then the σ-algebra generated by a one-sided infinite parallelogram is not enough to capture the full σ-algebra of the Z^2-action. However, we show in Theorem 4 that for any given one-sided cone, there exists a partition $\tilde{\mathcal{P}}$ such that $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$ and the $\tilde{\mathcal{P}}$-names of the one-sided cone generate the full σ-algebra. This is the property for zero-entropy Z^2-actions analogous to the property (4). If $h(\phi) = 0$, then we may ask for what region D in Z^2 there exists a partition $\tilde{\mathcal{P}}$ such that the $\tilde{\mathcal{P}}$-names of D generate the full σ-algebra. See Remark 3.

We may also mention the following: if ϕ is a K-automorphism, then given $\epsilon > 0$ and a rectangle $[-n, 0] \times [-n, 0]$, there exists k such that

$$\bigvee_{(i,j)\in[-n,0]\times[-n,0]} \phi^{-(i,j)}\tilde{\mathcal{P}}$$

is ϵ-independent of $\bigvee_{(i,j)\in E_k} \phi^{-(i,j)}\tilde{\mathcal{P}}$ [**3**]. Meanwhile, by the property (2) of $\tilde{\mathcal{P}}$, it is easy to see that for any given $\pi/2 \leq \theta < \pi$, the σ-algebra $\bigvee_{(i,j)\in E_{k,\theta}} \phi^{-(i,j)}\tilde{\mathcal{P}}$ completely determines $\bigvee_{(i,j)\in[-n,0]\times[-n,0]} \phi^{-(i,j)}\tilde{\mathcal{P}}$, where

$$E_{k,\theta} = E_k \cup \{(i, j) : 0 < j < |i + M_k| \tan(\pi - \theta)i \leq -M_k\}.$$

It is easy to see that the partition $\tilde{\mathcal{P}}$ satisfies

$$\bigvee_{(i,j)\in B_n^c} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$$

for every n, where

$$B_n = \{(i, j) : 0 \leq |i| \leq n, 0 \leq |j| \leq n\}.$$

This fact can be regarded as a direct generalization of the Ornstein and Weiss theorem [**8**]. By the above property, it is clear that if $(X, \mathcal{P}, \mu, \phi)$ is Bernoulli, then $(X, \tilde{\mathcal{P}}, \mu, \phi)$ is very weak Bernoulli. However it is not weak Bernoulli. (See [**2**] for the definition.)

2. *Construction of* $\tilde{\mathcal{P}}$

2.1. *Positive-entropy case.* We assume that our Z^2-action, $(X, \mathcal{F}, \mu, \phi)$, is a free action. By the definition, for each (i, j), $\phi^{(i,j)}$ is also a free Z-action. Through successive steps, we construct $\tilde{\mathcal{P}}$, a refinement of $\mathcal{P}$, which generates the full σ-algebra under the elements $\phi^{(i,j)}$, where $(i, j) \in G_\theta$ for every $\pi/2 \leq \theta < \pi$.

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THEOREM 1. *For every generating partition $\mathcal{P}$, there exists a partition $\tilde{\mathcal{P}}$ such that:*

(1) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$; *and*

(2) $\bigvee_{(i,j)\in G_\theta} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$ *for any* $\pi/2 \le \theta < \pi$.

Proof. Let $\mathcal{P} = \{\mathcal{P}_0, \ldots, \mathcal{P}_{k-1}\}$ denote the given generating partition under our Z^2-action ϕ. Without confusion, we may also consider $\mathcal{P}$ as a function from X to $\{0, 1, \ldots, k-1\}$ such that $\mathcal{P}(x) = i$ if and only if $x \in \mathcal{P}_i$. Let $\{(p_1, q_1), (p_2, q_2), \ldots\}$ be a sequence of relatively prime pairs of integers. We make a list for each pair to occur infinitely often so that we construct the partition $\tilde{\mathcal{P}}$ with the generating property for any G_θ, $\pi/2 \le \theta < \pi$. We are going to build a sequence of Rokhlin towers $\{R_i\}$ under the Z-actions $\phi^{(p_i,q_i)}$. We choose the ϵ_i such that $\sum \epsilon_i < \infty$. Let $\{n_i\}$ be a sequence going to ∞. We need to put some more conditions on the ϵ_i and n_i, which will be clear later.

For the convenience of the notation, we denote $\phi^{(p_1,q_1)}$ by ϕ_1. Let R_1 denote a Rokhlin tower under ϕ_1 of height n_1^2, so that

$$\mu\left(\bigcup_{j=0}^{n_1^2-1} \phi_1^j F_1\right) > 1 - \epsilon_1,$$

where F_1 denotes the base of the tower R_1. We define $\mathcal{P}^{(1)}$, which is a refinement of $\mathcal{P}$, as follows. If $x \in \bigcup_{j=n_1^2-n_1}^{n_1^2-1} \phi_1^j F_1$ and $x \in \mathcal{P}_i$, then we assign the point to $\mathcal{P}_{i,t}^{(1)}$, $0 \le i < k$. We add the information of the parity check on the bottom n_1 level sets $PC(n_1) = \bigcup_{j=0}^{n_1-1} \phi_1^j F_1$. If $x \in PC(n_1)$ and $x \in \mathcal{P}_i$, then we assign the point to $\mathcal{P}_{i,m}^{(1)}$, where $m = \sum_{j=0}^{n_1-2} \mathcal{P}(\phi_1^{n_1\cdot j} x) \pmod k$. If x belongs to the rest of the tower or the error set and $x \in \mathcal{P}_i$, then we assign x to $\mathcal{P}_i^{(1)}$. The new partition $\mathcal{P}^{(1)}$ is clearly a refinement of $\mathcal{P}$ and it consists of at most $k^2 + 2k$ many atoms

$$\{\mathcal{P}_0^{(1)}, \ldots, \mathcal{P}_{k-1}^{(1)}, \mathcal{P}_{0,t}^{(1)}, \ldots, \mathcal{P}_{k-1,t}^{(1)}, \mathcal{P}_{0,0}^{(1)}, \mathcal{P}_{0,1}^{(1)}, \ldots, \mathcal{P}_{0,k-1}^{(1)},$$
$$\mathcal{P}_{1,0}^{(1)}, \mathcal{P}_{1,1}^{(1)}, \ldots, \mathcal{P}_{k-1,k-2}^{(1)}, \mathcal{P}_{k-1,k-1}^{(1)}\}.$$

Given a point $x \in PC(n_1)$, we note that we add the parity check along the orbit of x under $\phi_1^{n_1}$.

Suppose that we have constructed $\mathcal{P}^{(l-1)}$ using the Rokhlin tower R_{l-1} under $\phi_{l-1} = \phi^{(p_{l-1},q_{l-1})}$. Let R_l denote the Rokhlin tower of height n_l^2 with the base set F_l. We assume that

$$\mu\left(\bigcup_{i=0}^{n_l^2-1} \phi_l^i F_l\right) > 1 - \epsilon_l.$$

We define $\mathcal{P}^{(l)}$, which is a refinement of $\mathcal{P}$ and differs from $\mathcal{P}^{(l-1)}$ arbitrarily small by choosing n_l sufficiently large. We note that the orbits of a point under ϕ_{l-1} and under ϕ_l could be completely different.

As we did on R_1, if x belongs to the top n_l level sets of R_l and $x \in \mathcal{P}_i$, then we assign x to $\mathcal{P}_{i,t}^{(l)}$. If $x \in PC(n_l) = \bigcup_{j=0}^{n_l-1} \phi_l^j F_l$ and $x \in \mathcal{P}_i$, then we assign x to $\mathcal{P}_{i,m}^{(l)}$,

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where $m = \sum_{j=0}^{n_l-2} \mathcal{P}(\phi_l^{n_l \cdot j} x) \pmod k$. If

$$x \in \bigcup_{j=n_l}^{2n_l-1} \phi_l^j F_l \cup \bigcup_{j=n_l^2-2n_l}^{n_l^2-n_l-1} \phi_l^j F_l$$

and $x \in \mathcal{P}_i$, then we assign x to $\mathcal{P}_i^{(l)}$. If x belongs to the rest of the tower or the error set, then we assign x to the same atom of $\mathcal{P}^{(l-1)}$. For almost every $x \in \bigcup_{j=3n_l}^{n_l(n_l-2)-1} \phi_l^j(F_l)$, if we know the $\mathcal{P}^{(l)}$-name of the point along the tower R_l except for n_l consecutive places containing x, then we can reconstruct the missing n_l consecutive names from the parity check on $PC(n_l)$.

We note that $\mathcal{P}^{(l)}$ differs from $\mathcal{P}^{(l-1)}$ on a set of measure at most $4/n_l$. Hence, the set

$$E_l = \{x : \mathcal{P}^{(k)}(x) \neq \mathcal{P}^{(l)}(x) \text{ for some } k > l\}$$

has measure at most $\alpha_l = \sum_{k=l+1}^{\infty}(4/n_k)$. If we let

$$\tilde{E}_l = \bigcup_{j=0}^{n_l^2-1} \{\phi_l^j(x) : \text{there exists no } 0 \leq k \leq n_l^2 - 1 \text{ such that } \phi_l^k(x) \in E_l\},$$

then we have that $\mu(\tilde{E}_l^c) < n_l^2\alpha_l + \epsilon_l$. If $x \in \tilde{E}_l$, then the $\mathcal{P}^{(k)}$-name of x throughout the tower R_l remains the same for all $k \geq l$. Through successive steps we choose the n_l sufficiently large and the ϵ_l sufficiently small so that $\sum_l(n_l^2\alpha_l + \epsilon_l) < \infty$. Then for almost every x there exist only finitely many of the l such that $x \in \tilde{E}_l^c$. Therefore, $\tilde{\mathcal{P}} = \lim \mathcal{P}^{(l)}$ is well defined and consists of

$$\{\tilde{\mathcal{P}}_0, \ldots, \tilde{\mathcal{P}}_{k-1}, \tilde{\mathcal{P}}_{0,t}, \ldots, \tilde{\mathcal{P}}_{k-1,t}, \tilde{\mathcal{P}}_{0,0}, \tilde{\mathcal{P}}_{0,1}, \ldots, \tilde{\mathcal{P}}_{0,k-1}, \ldots, \tilde{\mathcal{P}}_{1,0}, \tilde{\mathcal{P}}_{1,1}, \ldots, \tilde{\mathcal{P}}_{k-1,k-2}, \tilde{\mathcal{P}}_{k-1,k-1}\}.$$

It is now enough to show that for almost every $x \in X$, $\{\tilde{\mathcal{P}}(\phi^{(i,j)}x) : (i, j) \in G_\theta\}$ determines $\mathcal{P}(\phi^{(i,j)}x)$ for all $(i, j) \in Z^2$. For a given θ, we choose n_θ such that

$$0 < \frac{p_{n_\theta}}{q_{n_\theta}} < \frac{\tan(\pi - \theta)}{2}.$$

Suppose $(i_0, j_0) \in G_\theta^c$. Notice that the line L through (i_0, j_0) with slope $-p_{n_\theta}/q_{n_\theta}$ hits only finitely many lattice points outside the region G_θ. Let $g(i_0, j_0)$ denote the number of lattice points on the line outside G_θ. Let

$$y = \phi^{(i_0,j_0)}x \quad \text{and} \quad H_l = \tilde{E}_l \cap \left(\bigcup_{i=3n_l}^{n_l(n_l-2)-1} \phi_l^i F_l\right).$$

Since $\mu(H_l^c) < n_l^2\alpha_l + (5/n_l) + \epsilon_l$, by our choice of the n_l, there exists $l_0(y)$ satisfying $y \in H_l$ for all $l \geq l_0$. We choose $l \geq l_0$ such that $p_l/q_l = p_{n_\theta}/q_{n_\theta}$ and $n_l \geq g(i_0, j_0)$. By our construction, we can determine the $\mathcal{P}^{(l)}$-name and hence the $\tilde{\mathcal{P}}$-name of $x(i, j)$ for all $(i, j) \in G_\theta^c \cap L$. We repeat this for each $(i, j) \in G_\theta^c$ to recover the complete $\mathcal{P}$-name of x. □

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Besides the G_θ, there are many regions of smaller 'density' whose names generate the full σ-algebra. We have the following theorem.

THEOREM 2. *Let $B \subset Z^2$ be a region such that there exists a rational direction including ∞ for all $(i, j) \in B^c$ so that the line L through (i, j) in the direction intersects only finitely many lattice points outside B. Then we can construct $\tilde{\mathcal{P}}$ such that:*

(1) $|\mathcal{P} \Delta \tilde{\mathcal{P}}| < \epsilon$; *and*

(2) $\bigvee_{(i,j)\in B} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$.

Proof. It is clear from our construction of $\tilde{\mathcal{P}}$ as in Theorem 1. □

Notice that our partition $\tilde{\mathcal{P}}$ satisfies the following.

Example 1. For any given $a > 0$, we let $B_a = \{(i, j) : j \geq ai^2 \text{ or } j \leq -ai^2\}$. The $\tilde{\mathcal{P}}$-name of a point in B_a determines the point completely. Hence, we have

$$\bigcap_{a>0} \bigvee_{(i,j)\in B_a} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}.$$

Example 2. Given any $a > 0$ and $b > 0$, the region

$$B_{a,b} = \{(i, j) : j \geq ai^2 + b \text{ or } j \leq -ai^2 - b\}$$

also satisfies the condition of Theorem 2.

Example 3. Let $C(\vec{v}, \theta)$ denote the double-sided cone in the direction of $\vec{v}$ with the angle θ. That is,

$$C(\vec{v}, \theta) = \left\{(x, y) : \tan(\alpha - \theta) \leq \frac{y}{x} \leq \tan(\alpha + \theta), -\infty < x, y < \infty\right\},$$

where α is the angle between $\vec{v}$ and the x coordinate. Since it satisfies the condition of Theorem 2 for any angle θ, we have

$$\bigvee_{(i,j)\in C(\vec{v},\theta)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}.$$

In particular, if we know the $\tilde{\mathcal{P}}$-name of a point in the first and the third quadrants, then we know the full name of the point. It is clear that as θ goes to 0, the density of $C(\vec{v}, \theta)$ in Z^2 goes to 0, and hence it can be made arbitrarily small. Moreover, we note that, for example, $C_a(\vec{i}, \theta)$ also satisfies the condition of Theorem 2, where

$$C_a(\vec{i}, \theta) = \left\{(x, y) : -\tan\theta \leq \frac{y}{|x| - a} \leq \tan\theta, |x| \geq a\right\}$$

is a double-sided cone whose vertices are apart.

Remark 1. Let $B = \{(i, j) : j \leq 0 \text{ or } j \leq \log|i| \text{ for } i < 0\}$. It is not clear yet if there exists a partition $\tilde{\mathcal{P}}$ such that

$$\bigvee_{(i,j)\in B} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}.$$

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2.2. *Zero-entropy case.* Recall that a zero-entropy Z-action has the property that if $\mathcal{P}$ is a generating partition, then the tail field of the partition $\mathcal{P}$ is the whole σ-algebra. We will prove the analogous result for zero-entropy Z^2-actions, which is stronger than Theorem 2. That is, we construct $\tilde{\mathcal{P}}$ so that the one-sided cone $\tilde{\mathcal{P}}$-names generate the full σ-algebra. It is known that if $h(\phi) = 0$, then, for any generating partition $\mathcal{P}$, we have

$$\bigvee_{j \le -1} \phi^{-(i,j)}\mathcal{P} \vee \bigvee_{i \le -1} \phi^{-(i,0)}\mathcal{P} = \mathcal{F}.$$

We need the following lemma, which provides us with the rough estimates of the number of elements in $\bigvee_{0 \le i,j < n} \phi^{-(i,j)}\mathcal{P}$.

LEMMA 1. *Let $h(\phi) = 0$. Given a generating partition $\mathcal{P}$ and an $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $n \ge N(\epsilon)$, there exists a collection $\mathcal{E}_n$, which consists of atoms of $\bigvee_{0 \le i,j < n} \phi^{-(i,j)}\mathcal{P}$, where:*

(1) $|\mathcal{E}_n| < 2^{\epsilon n^2}$;

(2) $\mu(E_n) < \epsilon$, *where* $E_n = \bigcup_{C \notin \mathcal{E}_n} C$.

The above lemma follows from the Shannon–McMillan–Breiman theorem [**10**] of Z^2-actions.

PROPOSITION 3. *Let $h(\phi) = 0$ and let $\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$ be a generating partition under ϕ. There exists a partition $\tilde{\mathcal{P}}$ satisfying the following: for a given $\epsilon > 0$ and a given θ:*

(1) $|\mathcal{P} \Delta \tilde{\mathcal{P}}| < \epsilon$;

(2) $\bigvee_{(i,j) \in OC(\vec{v},\theta)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$, *where $OC(\vec{v}, \theta)$ denotes the one-sided cone in the direction $\vec{v} = (-1, -1)$ with the angle θ.*

Proof. We will describe the construction of $\tilde{\mathcal{P}}$ motivated by an idea of Thouvenot [**11**]. We choose $\{\epsilon_l\}$ sufficiently small and $\{n_l\}$ sufficiently large so that:

(i) $\epsilon_l < 1/2^{2l}$;

(ii) $n_l \gg 2^{2l}$ and n_l is a multiple of 2^l;

(iii) $\epsilon_l \ge \sum_{k \ge l+1} \epsilon_k$;

(iv) a Rokhlin tower R_l of size $[0, n_l) \times [0, n_l)$ has fewer than $2^{\epsilon_l {n_l}^2}$ names with an error set of measure less than ϵ_l.

We describe the induction step. Let B_l denote the base of the Rokhlin tower. We partition the Rokhlin tower into columns according to their $\mathcal{P}$-names. By a column, we mean a subset of a Rokhlin tower,

$$C_{l,t} = \bigcup_{0 \le i,j < n_l} \phi^{(i,j)} B_{l,t},$$

so that

$$\phi^{(i,j)} B_{l,t} \in P_{t(i,j)},$$

where $t(i, j) = 0, 1, \ldots, k-1$. And, $B_{l,t}$ denotes the base of the tth column of the Rokhlin tower R_l, $t = 1, \ldots, u_l$, where $u_l < 2^{\epsilon_l n_l^2}$. We encode each of the column names on the left bottom square of size $[0, n_l/2^l) \times [0, n_l/2^l)$ as follows.

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We define $\mathcal{P}^{(l)}$ by changing the partition

$$\mathcal{P}^{(l-1)} = \{P_r^{(l-1)},\ P_{r,s}^{(l-1)} : 0 \le r \le k-1,\ s = 0 \text{ or } 1\},$$

if necessary, to distinguish the $B_{l,t}$ by their $\mathcal{P}^{(l)}$-names on

$$BR_l = \bigcup_{0 \le i,j < n_l/2^l} \phi^{(i,j)} B_l.$$

That is, each of the $[0, n_l/2^l) \times [0, n_l/2^l)$-$\mathcal{P}^{(l)}$-names of $B_{l,t}$ uniquely determines the $[0, n_l) \times [0, n_l)$-$\mathcal{P}$-name of $B_{l,t}$. This is possible, since using $s = 0$ or 1, there are $2^{(n_l/2^l)^2}$ possible names, which is greater than $2^{\epsilon_l n_l^2}$ by the condition (i). We may need to change parts of $P_r^{(l-1)}$ into $P_{r,s}^{(l)}$ for each r and also parts of $P_{r,s}^{(l-1)}$ into $P_{r,s'}^{(l)}$, where $r = 0, 1, \ldots, k-1$, and s or $s' = 0$ or 1 and $s \ne s'$. On the rest of the tower or on the error set, we define $\mathcal{P}^{(l)}$ to be the same as $\mathcal{P}^{(l-1)}$. Note that we have changed $\mathcal{P}^{(l-1)}$ negligibly little to get $\mathcal{P}^{(l)}$. Hence, $\lim \mathcal{P}^{(l)} = \tilde{\mathcal{P}}$ exists. Note that $\tilde{\mathcal{P}}$ also consists of $3k$ atoms $\{P_0, \ldots, P_{k-1}, P_{r,s}\}_{\substack{r=0,\ldots,k-1 \\ s=0,1}}$ and the set $\bigcup_{\substack{r=0,\ldots,k-1 \\ s=0,1}} P_{r,s}$ has small measure.

We call a subcolumn a subset $C_{l,t}^o$ of a column of R_l,

$$C_{l,t}^o = \bigcup_{0 \le i,j < n_l} \phi^{(i,j)} B_{l,t}^o,$$

where $B_{l,t}^o$ is a subset of $B_{l,t}$. For a given θ, let $A_l = \bigcup_{(i,j) \in I_l} \phi^{(i,j)} B_l$, where

$$I_l = \left[n_l \left(\tan\left(\frac{\pi}{4} - \theta \right) - \frac{1}{2^l} \right), n_l \right) \times \left[n_l \left(\tan\left(\frac{\pi}{4} - \theta \right) - \frac{1}{2^l} \right), n_l \right).$$

Note that I_l is a square of size $n_l(1 - \tan(\pi/4 - \theta) + 1/2^l)$ at the upper right corner of the square $[0, n_l) \times [0, n_l)$ and, if $(i, j) \in I_l$, then the one-sided cone $OC(\vec{v}, \theta)$ starting at (i, j) contains the square $[0, n_l/2^l) \times [0, n_l/2^l)$. Hence, if $x \in A_l$, then the one-sided cone name starting at x determines the name of the column $C_{l,t}$ containing x, because the cone contains the $[0, n_l/2^l) \times [0, n_l/2^l)$-$\mathcal{P}^{(l)}$-name.

We need to show that $\mu(\bigcup_{l \ge k} A_l) = 1$ for all k. We note that I_l takes up greater than a fixed fraction of $[0, n_l) \times [0, n_l)$ for each l. We choose n_l sufficiently large compared with n_{l-1} so that:

(v) the measure of R_{l-1} in R_l which intersects the boundary of R_l or which intersects both A_l and $R_l \backslash A_l$ is less than ϵ_l;

(vi) for each column of R_l, the frequency of R_{l-1} in A_l is within $2\epsilon_l$ of the frequency of R_{l-1} in $R_l \backslash A_l$ (the ergodic theorem). Therefore, the frequencies of A_{l-1} in A_l and in $R_l \backslash A_l$ are within $2\epsilon_l$ of each other.

We denote by $R_{l-1}^{(l)}$ the subset of R_{l-1} which intersects neither the boundary of R_l nor the boundary of A_l. By (v), we have $\mu(R_{l-1}^{(l)}) > \mu(R_{l-1}) - \epsilon_l$. Let

$$\tilde{R}_{l-1} = \bigcap_{k \ge l} {R_{l-1}}^{(k)} \quad \text{and} \quad \tilde{A}_{l-1} = \tilde{R}_{l-1} \cap A_{l-1}.$$

We note that

$$\mu(\tilde{R}_{l-1}) > \mu(R_{l-1}) - \sum_{k \ge l} \epsilon_k > \mu(R_{l-1}) - \epsilon_{l-1} \ge 1 - 2\epsilon_{l-1}.$$

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To show that almost every $x \in X$ belongs to $\tilde{A}_l$ infinitely often, we need that $\mu(\bigcap_{l\geq m} \tilde{A}_l^c) = 0$ for all m. This is true by the 'almost' independence of the A_l within the error of the ergodic theorem. Since there is an error set E_l of measure less than ϵ_l, we note that $\mu(\tilde{A}_l^c) \leq \alpha/(1-2\epsilon_l)$, where $\alpha = (1-(|I_l|/n_l^2)) < 1$. And, we may assume that the frequency of $\tilde{A}_l^c$ in $\tilde{A}_{l+1}^c$ is at most $\mu(\tilde{A}_l^c)\cdot(1+2\epsilon_{l+1})$. Hence,

$$\mu\left(\bigcap_{l\geq m}^{n} \tilde{A}_l^c\right) \leq \frac{\alpha^{m-n+1}}{1-2\epsilon_m} \prod_{k=m+1}^{n} \frac{(1+2\epsilon_k)}{(1-2\epsilon_k)} \leq \frac{\alpha^{m-n+1}}{1-2\epsilon_m} \prod_{k=m+1}^{n} (1+5\epsilon_k).$$

Since $\sum_{k\geq l+1}^{\infty} \epsilon_k < \epsilon_l$, we have

$$\mu\left(\bigcap_{l\geq m}^{n} \tilde{A}_l^c\right) < \frac{\alpha^{n-m+1}}{1-2\epsilon_m} \epsilon^{6\epsilon_m} \to 0$$

as $n \to \infty$ for all m.

For a given positive m, we let F_l^m denote the set $\{(i, j) : (i, j)$ is at least distance m away from the boundary of the square $[0, n_l) \times [0, n_l)\}$. Since for each m, the set

$$R_l^m = \bigcup_{(i,j)\in F_l^m} \phi^{(i,j)} B_l$$

has measure greater than $(1-\epsilon_l)(1-(4m/n_l))$, almost every x belongs to R_l^m except for only finitely many of the l. That is, a larger and larger square centered at x is contained in $\tilde{A}_l$ as $l \to \infty$ for almost every x. Hence, we have the property (2). □

Remark 2. Note that for each l we may use the $(0, 0)$ coordinate as a marker to indicate the start of $BR_{(0,0)}$. Then $\tilde{\mathcal{P}}$ may consist of $\{\mathcal{P}_0, \ldots, \mathcal{P}_{k-1}, \mathcal{P}_{r,s}\}_{\substack{r=0,\ldots,k-1 \\ s=c,0,1}}$, where $\mathcal{P}_{r,c}$ indicates a marker.

THEOREM 4. *If $h(\phi) = 0$, then there exists a partition $\tilde{\mathcal{P}}$ such that for a given $\epsilon > 0$, we have the following.*

(1) $|\mathcal{P}\Delta\tilde{\mathcal{P}}| < \epsilon$.

(2) $\bigvee_{(i,j)\in OC(\vec{v},\theta)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$ *for every* $\vec{v}$ *and every angle* θ.

Proof. Given $\epsilon > 0$, we choose l_0 such that $(1/2^{l_0}) \leq (\epsilon/4)$. We start from $l \geq l_0$. In the Rokhlin tower R_l, we consider the squares of size $(n_l/2^l) \times (n_l/2^l)$ which are located at $(0, (n_l/2^l)j)$, $((n_l/2^l)j, 0)$, $(n_l - (n_l/2^l), (n_l/2^l)j)$ or $((n_l/2^l)j, n_l - (n_l/2^l))$ for $j = 0, \ldots, 2^l - 1$. These squares meet the boundary of R_l. We define $\mathcal{P}^{(l)}$ different from $\mathcal{P}^{(l-1)}$ only on the boundary squares. As in Proposition 3, each of the $\mathcal{P}^{(l)}$-names of BR_l uniquely determines the $\mathcal{P}$-name of the column of the Rokhlin tower $C_{l,t}$ for $t = 1, \ldots, u_l$. After we redefine the partition $\mathcal{P}^{(l-1)}$ into $\mathcal{P}^{(l)} = \{\mathcal{P}_0^{(l)}, \ldots, \mathcal{P}_{r,s}^{(l)}\}_{\substack{r=0,\ldots,k-1 \\ s=0,1}}$ on BR_l, for each column $C_{l,t}$ we copy the $\mathcal{P}^{(l)}$-name of

$$BR_{l,t} = \bigcup_{0\leq i,j<n_l/2^l} \phi^{(i,j)} B_{l,t}$$

on each of the boundary squares of $C_{l,t}$.

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We note that we have

$$|\mathcal{P}^{(l-1)}\Delta\mathcal{P}^{(l)}| \le \frac{(n_l/2^l)^2\ (4\cdot 2^l)}{{n_l}^2} = \frac{4}{2^l}.$$

Let $\tilde{\mathcal{P}} = \lim \mathcal{P}^{(l)}$; we have

$$|\mathcal{P}\Delta\tilde{\mathcal{P}}| \le \sum_{l\ge l_0+1}^{\infty}\left(\frac{4}{2^l}\right) = 4\cdot\frac{1}{2^{l_0}} < \epsilon.$$

To prove the second property, it is enough to consider the vector $\vec{v} = (x, y)$, where $y > 0$. Let θ_0 denote the angle such that $\tan\theta_0 = y/x$. We assume $0 \le \theta_0 < \pi/2$. For the case of $\pi/2 \le \theta_0 \le \pi$, we can use similar arguments. For any given θ, we let

$$\tau(\theta) = \min\{|\tan(\pi/2 - \theta_0 + \theta) - \tan(\pi/2 - \theta_0)|, |\tan(\pi/2 - \theta_0 - \theta) - \tan(\pi/2 - \theta_0)|,$$
$$|\tan(\theta_0 + \theta) - \tan\theta_0|, |\tan(\theta_0 - \theta) - \tan\theta_0|\}.$$

There exists $\tilde{l}(\theta)$ such that if $l \ge \tilde{l}$, then $(1 - (1/2^l))\tau > 2/2^l$. Then the cone $C(\vec{v}, \theta)$ starting at $(0, 0)$ contains a boundary square of size $n_l/2^l \times n_l/2^l$ for $l \ge \tilde{l}$. The $\mathcal{P}^{(l)}$-name of this boundary square determines the $\mathcal{P}$-name of the column of R_l. Moreover, for a given θ and $0 < c < 1$, there exists $\tilde{l}(\theta, c)$ such that if $l \ge \tilde{l}$, then $(1 - c - (1/2^l)) \cdot n_l \cdot \tau > 2 \cdot n_l/2^l$. Any cone $C(\vec{v}, \theta)$ starting at $(i, j) \in I_l = [0, cn_l) \times [0, cn_l)$ contains at least one boundary square. Hence, the $\mathcal{P}^{(l)}$-name of the cone $C(\vec{v}, \theta)$ determines the $\mathcal{P}$-name of the column of R_l. As in Proposition 3, it is not hard to see that almost every x belongs to $A_l = \bigcup_{(i,j)\in I_l} \phi^{(i,j)}B_l$ infinitely often.

For a given m, we let

$$R_l^m = \bigcup_{m\le i,\, j<n_l-m} \phi^{(i,j)}B_l.$$

We note that there exists $l'(m)$ such that $x \in R_l^m$ for all $l \ge l'(m)$. For each m, we have $x \in R_l^m \cap A_l$ infinitely often; hence, we have

$$\bigvee_{(i,j)\in OC(\vec{v},\theta)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}$$

for every θ. Since this holds for every $\vec{v}$, we have the property (2). □

Remark 3. If a system $(X, \mathcal{P}, \mu, \phi)$ has the property

$$\lim_{n\to\infty}\frac{1}{n^2}H\left(\bigvee_{(i,j)\in R_n}\phi^{-(i,j)}\mathcal{P}\right) = 0$$

and

$$\lim_{n\to\infty}\frac{1}{n^{\frac{3}{2}}}H\left(\bigvee_{(i,j)\in R_n}\phi^{-(i,j)}\mathcal{P}\right) = \infty,$$

where $R_n = \{(i, j) : 0 \le i, j \le n\}$, then it is clear that there exists no $\tilde{\mathcal{P}}$ with the property that for some $\vec{v} = (x, y)$ and some $k \ge 0$,

$$\bigvee_{(i,j)\in IP(\vec{v},k)} \phi^{-(i,j)}\tilde{\mathcal{P}} = \mathcal{F}.$$

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However, the above Theorem 4 says that a one-sided cone is enough to capture the full σ-algebra. We believe that this is related to the 'complexity' of zero-entropy Z^2-actions. (See [**4–6**].) It is not clear yet if it is possible to refine or strengthen Proposition 3 and Theorem 4 even for the class of systems of low complexity.

Remark 4. It is not hard to see that the above constructions of $\tilde{\mathcal{P}}$ in Theorems 1, 2 and 4 can be generalized to Z^n-actions for both positive and zero entropy. However, it is not clear how to extend them to general amenable groups [**9, 10**].

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