calculate that $-\frac{2}{\ln 2}W_0\left(-\frac{\ln 2}{2}\right) = 2$ and

$$-\frac{2}{\ln 2}W_{-1}\left(-\frac{\ln 2}{2}\right) = -\frac{4}{\ln 4}W_{-1}\left(-\frac{\ln 4}{4}\right) = 4.$$

Thus, here $\alpha = 4$.

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109.39 Depressing cubics and barycentres

Introduction

The saga chronicling the resolution of cubic equations unravelled when Cardano observed that any cubic polynomial, $p(x) = ax^3 + bx^2 + cx + d$, can be depressed* (see [1] for a captivating summary) by performing the substitution

$$x = y - \frac{b}{3a}. \tag{1}$$

Equations involving depressed cubics, i.e. cubics whose quadratic term is null, are easier to solve both geometrically and algebraically and were already understood by Tartaglia and del Ferro.

When introducing cubic equations, transformation (1) is often presented as a *deus ex machina*. Given the importance of depression in solving equations of arbitrary degree (in the quadratic case, it is equivalent to completing the square) it is surprising that it is generally described as an *ad hoc* algebraic transformation without further ado. Let us explore the mechanism behind this transformation.

* Latin *deprimere*, to press down. The term was coined by Viète.

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Barycentre

Viète's formula tells us that $-\frac{b}{a}$ is the sum of the roots. If we divide this identity by 3 (or more generally by the degree), we obtain the barycentre, or centre of mass, $\mu = -\frac{b}{3a}$ of the roots. From this perspective, what we seek when depressing a polynomial is simply to translate the roots so as to centre them around the origin. This can be done readily by translating each root by μ, i.e. through operation (1).

Example: The roots of the polynomial $x^3 - 3x^2 + x + 5$, are -1 and $2 \pm i$. Their barycentre is 1 and thus by applying the change of coordinates $x \to x - 1$ we obtain the depressed cubic $y^3 - 2y + 4$ whose roots are the latter but shifted so as to be balanced around the origin.

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109.40 On the Euler sequence

As shown in [1], the well-known Euler-Mascheroni constant $\gamma = 0.5772\ldots$ is the limit as $n \to \infty$ of the sequence $\{a_n\}$ with general term given by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n. \tag{1}$$

In this Note, we will examine the related sequence $\{b_n\}$ defined by

$$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \frac{4}{3}\log\left(n + \frac{1}{2}\right) + \frac{1}{6}\log\left(n(n+1)\right) \tag{2}$$

which also tends to γ,

Now, from (1), (2), we have the initial terms

$$a_1 = 1 \text{ and } b_1 = 1 - \frac{4}{3}\log\frac{3}{2} + \frac{1}{6}\log 2 = 0.5749\ldots \tag{3}$$

and the differences of consecutive terms (for $n > 1$)

$$\begin{aligned} a_n - a_{n-1} &= \frac{1}{n} - \log n + \log(n-1) \\ &= \frac{1}{n} + \log\left(1 - \frac{1}{n}\right) = -\frac{1}{2n^2} + O\left(\frac{1}{n^3}\right), \end{aligned} \tag{4}$$

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