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The distance to upper triangular operators

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We show how a generalization of a distance formula of W. Arveson(1) may be derived from an elementary but ingenious special case, due to S. Parrot(2), involving 2×2 matrices.

To state the result we need a little notation. A set $\mathscr{P}$ of self-adjoint projections on a Hilbert space H is called a *nest of projections*, or simply a *nest*, if it is totally ordered under the usual operator ordering. This means precisely that the corresponding set of ranges is totally ordered under inclusion. Let $\mathscr{P}$ and $\mathscr{Q}$ be two order isomorphic nests acting on Hilbert spaces H and K respectively, and let α be the implementing isomorphism, $\alpha: \mathscr{P} \to \mathscr{Q}$. The *upper triangular operators* with respect to the ordered pair $\{\mathscr{P}, \mathscr{Q}\}$ is the collection $\mathscr{T}(\mathscr{P}, \mathscr{Q})$ of bounded linear operators T from H to K such that

$$\alpha(P)^{\perp} TP = 0 \quad (P \text{ in } \mathscr{P}).$$

In general this collection may depend on α.

THEOREM 1. *If X is a bounded linear operator from H to K then*

$$d(X, \mathscr{T}(\mathscr{P}, \mathscr{Q})) = \sup_{P \in \mathscr{P}} \|\alpha(P)^{\perp} XP\|.$$

Here $d(\ ,\ )$ denotes the distance induced by the operator norm. The theorem, which was proved in (1) for $\mathscr{P} = \mathscr{Q}$, $\alpha =$ identity, by different methods to those below, will be made to depend, ultimately, on the following

THEOREM 2. (S. Parrot(2).) *Let* $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$ *be an operator matrix, determined by an operator X and decompositions of the domain space and range space. Let $\mathscr{S}$ be the collection of operators of the form* $\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$ *with respect to the same decomposition. Then*

$$d(X, \mathscr{S}) = \max\left\{\left\|\begin{bmatrix} X_1 & 0 \\ X_3 & 0 \end{bmatrix}\right\|, \left\|\begin{bmatrix} 0 & 0 \\ X_3 & X_4 \end{bmatrix}\right\|\right\}.$$

We now show how a little induction can be used to prove the general finite dimensional variant of Theorem 2 corresponding to arbitrary finite decompositions.

Let $\mathscr{P} = \{P_1, P_2, \ldots, P_n\}$ and $\mathscr{Q} = \{Q_1, Q_2, \ldots, Q_n\}$ be two nests, each containing distinct projections, with $P_1 \neq 0$, $P_n = I$, $Q_1 \neq 0$ and $Q_n = I$. Let $\mathscr{S}(\mathscr{P}, \mathscr{Q})$ be the collection of strictly upper triangular operators S (acting from H to K) such that

$$Q_{i-1}^{\perp} SP_i = 0 \quad (i = 1, 2, \ldots, n),$$

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where Q_0 is taken to be 0. Such operators have operator matrices of the form

$$\begin{bmatrix} 0 & S_{12} & S_{13} & \dots & S_{1n} \\ & 0 & S_{23} & \dots & S_{2n} \\ & & 0 & & \\ & & & & \\ & & & & S_{n-1n} \\ & & & & 0 \end{bmatrix}.$$

LEMMA. *If X is a bounded linear operator from H to K then*

$$d(X, \mathscr{S}(\mathscr{P}, \mathscr{Q})) = \max_{1 \leqslant i \leqslant n} \|Q_{i-1}^{\perp} X P_i\|.$$

Proof. We know, from Theorem 2, that the lemma holds for the case $n = 2$. Suppose then that the lemma is true for $n-1$ where $n > 3$. We write $X+S$ for S in $\mathscr{S}(\mathscr{P}, \mathscr{Q})$ as the operator matrix

$$\left[\begin{array}{c|ccc} X_{11} & X_{12}+S_{12} & X_{13}+S_{13} & \dots \\ \hline X_{21} & X_{22} & X_{23}+S_{23} & \dots \\ \vdots & & & \\ X_{n1} & & & X_{nn} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where, of course, B and D depend on S. Note first that it is a consequence of the $n = 2$ case that

$$\inf_S \|X+S\| = \inf_S \max\left\{\left\|\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\right\|, \left\|\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right\|\right\}. \tag{1}$$

Since A and C do not depend on S this is equal to

$$\max\left\{\|Q_0^{\perp} X P_1\|, \inf_S \left\|\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right\|\right\}. \tag{2}$$

But it follows from the induction hypothesis that

$$\inf_S \left\|\begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}\right\| = \inf_S \|[C \quad D]\| = \max_{2 \leqslant i \leqslant n} \|Q_{i-1}^{\perp} X P_i\|. \tag{3}$$

Combining (1)–(3) we see that the lemma holds for n and so is true in general.

This lemma immediately gives Theorem 1 in the case of finite nests, by thinking of $d(X, \mathscr{T}(\mathscr{P}, \mathscr{Q}))$ as the distance of a trivial modification of X to $\mathscr{S}(\mathscr{P}', \mathscr{Q}')$ where the length of $\mathscr{P}'$ is greater than the length of $\mathscr{P}$ by 1, and $\mathscr{P}$ is a subnest of $\mathscr{P}'$. Illustrating this for the case $n = 2$, we have

$$\inf_S \left\|\begin{bmatrix} X_{11}+S_{11} & X_{12}+S_{12} \\ X_{21} & X_{22}+S_{22} \end{bmatrix}\right\| = \inf_S \left\|\begin{bmatrix} 0 & X_{11}+S_{11} & X_{12}+S_{12} \\ 0 & X_{21} & X_{22}+S_{22} \\ 0 & 0 & 0 \end{bmatrix}\right\|.$$

Proof of Theorem 1. The proof now proceeds in an exactly analogous fashion to (1). Let $\mathscr{T} = \mathscr{T}(\mathscr{P}, \mathscr{Q})$ and for any finite subnest $\mathscr{F}$ of $\mathscr{P}$ let $\mathscr{T}_{\mathscr{F}} = \mathscr{T}(\mathscr{F}, \alpha(\mathscr{F}))$. It follows, trivially, that

$$\mathscr{T} = \cap \{\mathscr{T}_{\mathscr{F}}; \mathscr{F} \text{ finite subnest of } \mathscr{P}\}. \tag{4}$$

Also we have

$$d(X, \mathscr{T}_{\mathscr{F}}) = \max_{P \in \mathscr{F}} \|\alpha(P)^{\perp} X P\|.$$

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Thus, it will be sufficient to show that

$$d(X,\mathscr{T}) = \sup\{d(X,\mathscr{T}_{\mathscr{F}});\ \mathscr{F} \text{ a finite subnest of } \mathscr{P}\}. \tag{5}$$

Since $\mathscr{T}$ is always contained in $\mathscr{T}_{\mathscr{F}}$ it follows that the left-hand side of (5) dominates the right-hand side. Suppose then that σ is the supremum in (5), and fix $\epsilon > 0$. For each finite subnest $\mathscr{F}$ we may choose $T_{\mathscr{F}}$ so that $\|X - T_{\mathscr{F}}\| \leqslant \sigma + \epsilon$. In fact the set $\mathscr{C}_{\mathscr{F}}$ of such $T_{\mathscr{F}}$ is a non void subset of the space of all operators from H to K, which is compact in the weak operator topology. But a little reflection shows that the collection of the sets $\mathscr{C}_{\mathscr{F}}$ has the finite intersection property and so there is an operator T in the intersection of the sets $\mathscr{C}_{\mathscr{F}}$, with $\|X - T\| \leqslant \sigma + \epsilon$. Since, by (4), T is in $\mathscr{T}$ it follows that $d(X,\mathscr{T}) \leqslant \sigma$, completing the proof.

Perhaps it should be pointed out that even for finite nests the upper triangular operators may not look like the usual well-known case and do not, in general, form an algebra. For instance we may have a collection of operators of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}. \tag{6}$$

In this case the distance of a matrix $X = (X_{ij})_{i,j=1}^{4}$ to this collection is the maximum of the norms of the rectangular submatrices of X below the 'diagonal' of (6).

One might guess that something similar is valid for a collection defined by a completely arbitrary arrangement of zeros, rather than the subtriangular configuration of (6), and that the distance in this case is the maximum of the norms of the corresponding rectangular submatrices. Is this so?

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