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105.47 On a synthetic proof for the conic sections orthoptics

Introduction

An orthoptic is the set of points for which two tangent lines of a given curve meet at a right angle (see [1]).

The orthoptics of the non-degerate conic sections are circles, including also the parabola as limiting case (indeed, the orthoptic of a parabola coincides with its directrix − a circle with infinite radius). A purely axiomatic proof of this statement is quite challenging.

In the following, the author presents his purely axiomatic proof for the ellipse and the hyperbola orthoptics, leaving the simpler case of the parabola as a challenge for the reader. A different and concise synthetic proof for the ellipse case can be found at [2].

The orthoptics of the ellipse and the hyperbola are also known as the director circles or the Fermat-Apollonius circles (see [3]).

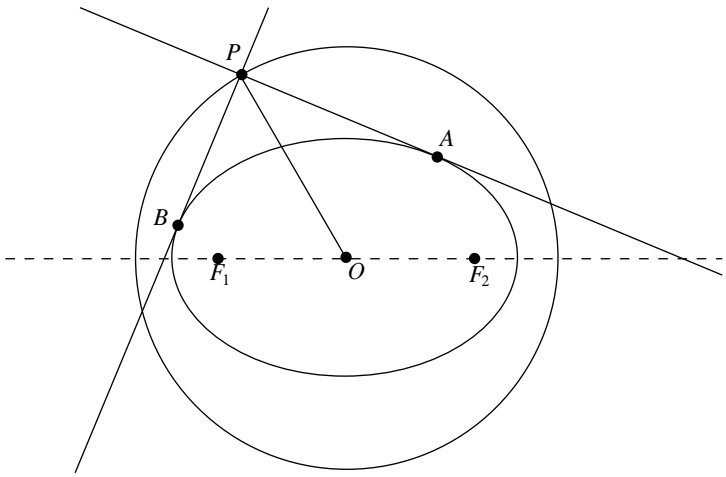


FIGURE 1: The ellipse and its orthoptic

The ellipse orthoptic

Looking at Figure 1, we recall the following:

Theorem: The orthoptic of a given ellipse is a circle, known as the ellipse director circle, centred at the ellipse centre, O, and radius equal to $\sqrt{a^2 + b^2}$, where a is the ellipse semi-major axis and b the semi-minor axis.

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Proof: Looking at Figure 2, let ε be the given ellipse, with centre O and foci F_1 and F_2. Let A and B be on ε with tangent lines to ε at these points meeting at P. Let F_1' and F_1'' be the reflections of F_1 about AP and BP, respectively. We want to show that $AP \perp BP$ if, and only if, $OP = \sqrt{a^2 + b^2}$. This is done in several steps.

Step 1: we show that $AP \perp BP$ if, and only if, P is the midpoint of the line segment $F_1'F_1''$. For, by the reflections, $F_1'P = F_1P = F_1''P$, so that there is a circle γ, centred at P, through F_1', F_1 and F_1''. But, again by the reflections, $F_1F_1' \perp AP$ and $F_1F_1'' \perp BP$ so that $AP \perp BP$ if, and only if, $F_1F_1' \perp F_1F_1''$, and this happens if, and only if, $F_1'F_1''$ is a diameter of γ (by Thales's theorem, see [4], and its converse), that is if, and only if, P is the midpoint of the line segment $F_1'F_1''$.

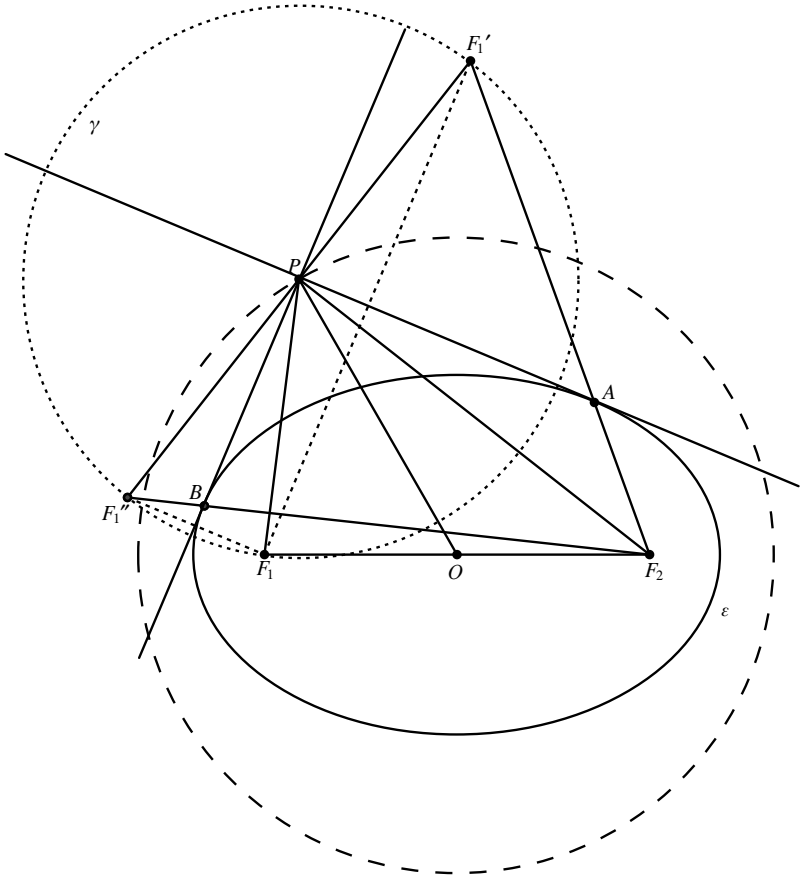


FIGURE 2: Proof diagram for the ellipse orthoptic

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Step 2: we show that P is the midpoint of the line segment $F_1'F_1''$ if, and only if, $F_1'P \perp F_2P$. Thanks to the ellipse focusing property (see [5]), the line segments F_2F_1' and F_2F_1'' contain the tangent points A and B respectively, and have the same length: $2a$. It follows that triangle $F_1'F_2F_1''$ is isosceles. Since $F_1'P = F_1''P$, P is on the perpendicular bisector of $F_1'F_1''$, that is, on the median of $F_1'F_2F_1''$, and so $F_1'P \perp F_2P$ if, and only if, P is on the foot of this median, as required.

Step 3: we show that $F_1'P \perp F_2P$ if, and only if, $OP = \sqrt{a^2 + b^2}$. For, OP is a median of triangle F_1PF_2, so that, applying the median theorem (see [6]): $F_1P^2 + F_2P^2 = 2\left(OP^2 + OF_1^2\right)$. Then, $F_1P = F_1'P$, by the reflection, and also $OF_1 = \sqrt{a^2 - b^2}$, so that $OP^2 = \frac{1}{2}\left(F_1'P^2 + F_2P^2\right) - \left(a^2 - b^2\right)$. But, $F_1'F_2 = 2a$, as in Step 2, so by Pythagoras applied to triangle F_2PF_1' we have $F_1'P \perp F_2P$ if, and only if, $F_1'P^2 + F_2P^2 = 4a^2$, or $OP^2 = 2a^2 - \left(a^2 - b^2\right) = a^2 + b^2$.

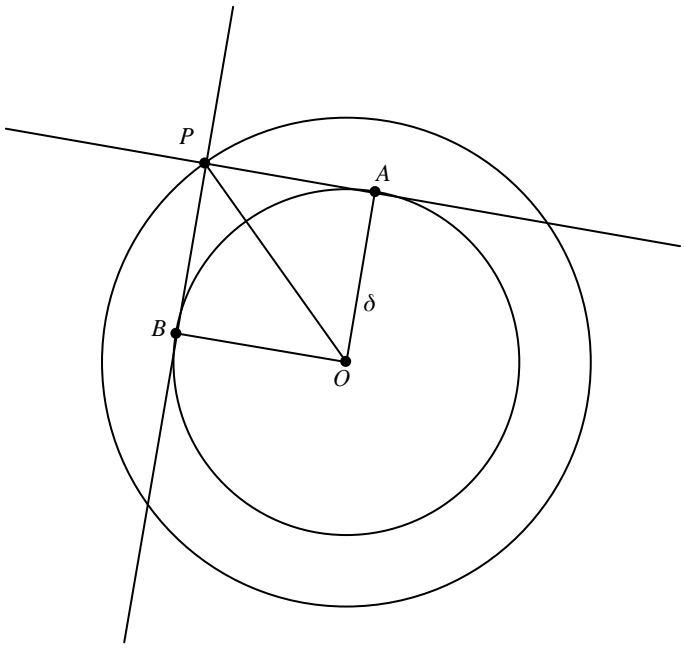


FIGURE 3: The circle orthoptic

Discussion

Interestingly, the presented proof shows that the ellipse orthoptic coincides with the geometric locus of a point such that the sum of the squares of whose distances from two given points (the ellipse focii) is equal to a given constant.

Thanks to the symmetry of the problem, the proof could have been developed reflecting the focus F_2 about AP and BP, instead of F_1.

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The orthoptic of a given circle is a circle. Its centre coincides with the centre of the given circle, and its radius is equal to $\delta\sqrt{2}$, where δ is the radius of the given circle. This statement can be deduced analysing the limiting case of the ellipse when the focii coalesce in the centre O. The situation is represented in Figure 3. In this particular configuration, the quadrilateral $APBO$ is a square.

The hyperbola orthoptic

The orthoptic of the hyperbola is represented in Figure 4. Using the same notation of the previous sections (a represents the hyperbola semi-major axis, and b the semi-minor axis), we state the following:

Theorem: For $a > b$, the orthoptic of a given hyperbola is a circle, known as the hyperbola director circle, centered at the hyperbola centre, O, and radius equal to $\sqrt{a^2 - b^2}$.

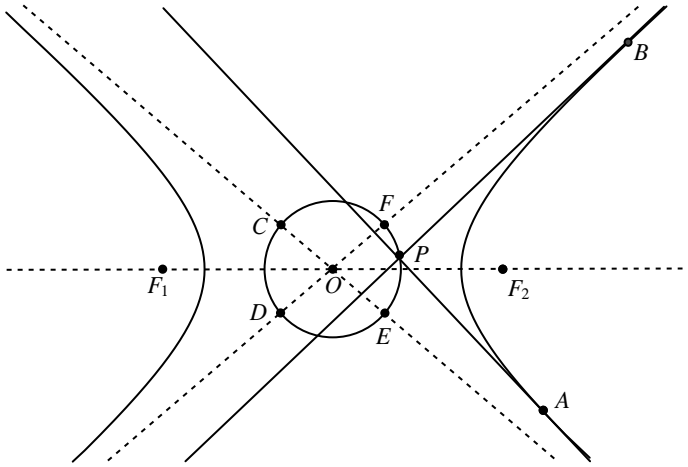


FIGURE 4. The hyperbola and its orthoptic

Proof: Assuming $a > b$ and looking at Figure 5, let ϕ be the given hyperbola, with centre O and foci F_1 and F_2. Let A and B be on ϕ with tangent lines to ϕ at these points meeting at P. Let F_1' and F_1'' be the reflections of F_1 about AP and BP, respectively.

From this point onwards, the proof for the ellipse orthoptic can be repeated step by step, taking care of swapping the word *ellipse* with the word *hyperbola*, replacing b^2 by $-b^2$ throughout and also, in Step 2, noticing that now, applying the hyperbola focusing property (see [5]), point A does *not* lie between F_1' and F_2 and, $F_1'F_2 = F_1'A - F_2A = F_1A - F_2A = 2a$. We also need $a \geqslant b$ so that $\sqrt{a^2 - b^2}$ is real.

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Discussion

For the proof diagram of the Theorem (Figure 5), we have considered the two tangent points, A and B, on the very same hyperbola branch. The proof would have proceeded exactly in the same way if we would have considered these two points on the different branches of the hyperbola. Like in the ellipse case, the same proof could have been developed reflecting the focus F_2 about AP and BP, instead of F_1.

The asymptotes intersections with the director circle represent the limiting cases for which one of the two tangent lines coincides with one of the asymptotes. In addition, the asymptotes represent the borders delimiting the regions of points on the director circle through which a pair of tangent lines touch only one branch of the hyperbola (arcs CD and EF), from the regions of points through which the pair of tangent lines touch the two different branches of the hyperbola (arcs DE and FC) in the semi-plane opposite to the one containing the point on the director circle.

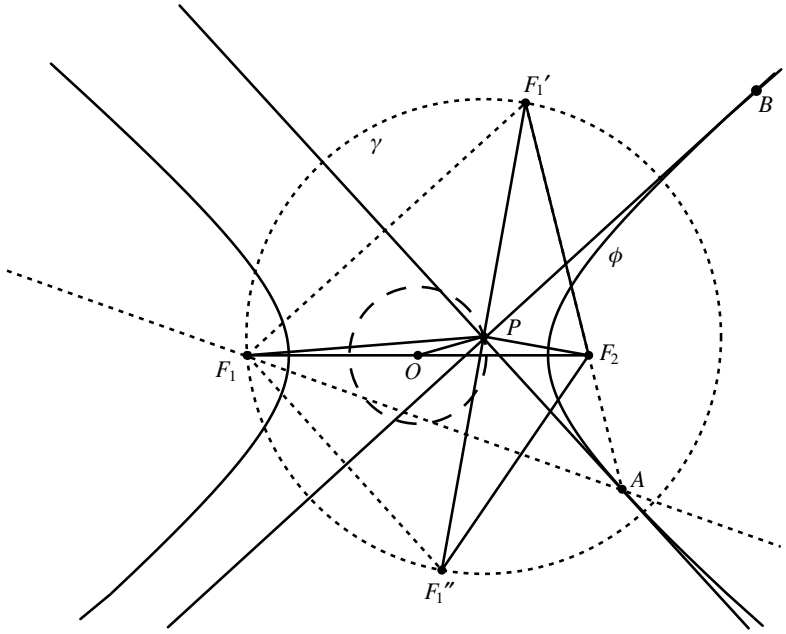


FIGURE 5: Proof diagram for the hyperbola orthoptic. Note the represented isosceles triangle $F_1'F_2F_1''$ and the right-angled triangle $F_1'F_1F_1''$ as the key elements for the proof development.

When $\angle EOF \geqslant 90°$, there is no point in the right and left quadrants for which the tangents to the respective branch of the hyperbola form a right angle. This is because any angle here will be greater than $\angle EOF$ (see Figure 6). It is then understood that the hyperbola orthoptic exists when the asymptotes $\angle EOF$ does not exceed the right angle. Such a configuration corresponds to the condition $a \geqslant b$.

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For $a = b$, the hyperbola is rectangular and $\angle EOF$ is a right angle. The director circle degenerates in point O. In this limiting case, the asymptotes are the only hyperbola tangent lines meeting each other at O at a right angle, and the hyperbola at infinity (see Figure 6).

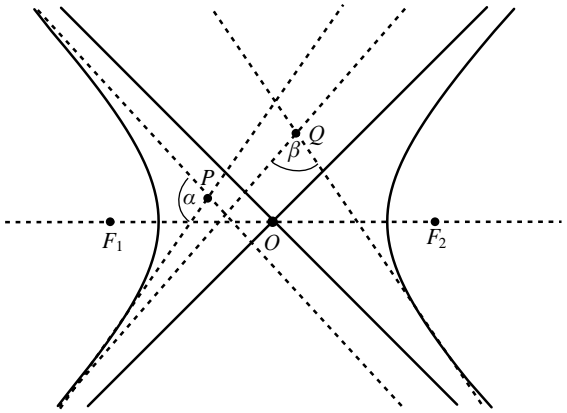


FIGURE 6: In the case of a rectangular hyperbola ($a = b$) the orthoptic coalesces in point O. Any pair of tangent lines, except the asymptotes, intersects at angles different than the right angle.

The parabola orthoptic

The following theorem is left as a challenge for the reader:

Theorem: The orthoptic of a given parabola coincides with its directrix.

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