more special case $d = 0$ and $e = 1$, the curve γ is the Viviani curve (see [6]). This shows that the Viviani curve can also be defined as the intersection of a sphere with a cone of revolution.

When $\frac{c}{d} = e$, the sphere is tangent to the cone at its vertex and $a = b$, so the limaçon is a cardioid (see [2, pp. 4-7], [3, pp. 34-43]).

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103.42 The Feuerbach hyperbola revisited

For a scalene triangle ABC, it may be shown by means of areal coordinates that the Nagel point E and the incentre I lie on the circumscribing Feuerbach rectangular hyperbola ([1]).

In this Note we will use a Cartesian framework to locate the excentres I_A, I_B, I_C and also the Bevan point.

The Cartesian framework

Taking the asymptotes of the hyperbola $xy = 1$ for our coordinate axes, we assign the coordinates $A\left(a, \frac{1}{a}\right)$, $B\left(b, \frac{1}{b}\right)$, $C\left(c, \frac{1}{c}\right)$, $E\left(e, \frac{1}{e}\right)$ and $I\left(i, \frac{1}{i}\right)$. The orthocentre $H\left(h, \frac{1}{h}\right)$ also lies on $xy = 1$ subject to the condition $abch = -1$. The parameters e, h, i and two of a, b, c may be taken as positive.

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The origin $F(0,0)$ is the Feuerbach point where the nine-points circle and incircle touch.

Now the trisection EGI on the Nagel line gives the centroid coordinates $G\left(\frac{e}{3}+\frac{2i}{3}, \frac{1}{3e}+\frac{2}{3i}\right)$ which leads to the nine-points centre $N\left(\frac{e}{4}+\frac{h}{4}+\frac{i}{2}, \frac{1}{4e}+\frac{1}{4h}+\frac{1}{2i}\right)$ via the quadrisection GNH on the Euler line.

Hence since $\frac{FN}{FI}=\frac{R}{2r}$ (for circumradius R and inradius r), we have

$$\frac{e+h}{4}=\left(\frac{R}{2r}-\frac{1}{2}\right)i \qquad \text{and} \qquad \frac{1}{4}\left(\frac{1}{e}+\frac{1}{h}\right)=\left(\frac{R}{2r}-\frac{1}{2}\right)\frac{1}{i},$$

whence $eh=i^2$ by division.

We next find coordinates for the three excentres.

The excentres I_A, I_B, I_C

Angle chasing reveals that both $\angle BII_A$ and $\angle CII_A$ are right angles so

$$II_A^2=BI^2+BI_A^2=CI^2+CI_A^2.$$

Thus for $I_A(X, Y)$ we have

$$\begin{aligned}(X-i)^2+\left(Y-\frac{1}{i}\right)^2&=(b-i)^2+\left(\frac{1}{b}-\frac{1}{i}\right)^2+(X-b)^2+\left(Y-\frac{1}{b}\right)^2\\&=(c-i)^2+\left(\frac{1}{c}-\frac{1}{i}\right)^2+(X-c)^2+\left(Y-\frac{1}{c}\right)^2,\end{aligned}$$

leading to the simultaneous linear equations

$$X-\frac{Y}{ib}=b-\frac{1}{ib^2}, \qquad X-\frac{Y}{ic}=c-\frac{1}{ic^2}.$$

Hence

$$I_A\left(\frac{1}{ibc}+b+c, ibc+\frac{1}{b}+\frac{1}{c}\right),$$

$$I_B\left(\frac{1}{ica}+c+a, ica+\frac{1}{c}+\frac{1}{a}\right),$$

$$I_C\left(\frac{1}{iab}+a+b, iab+\frac{1}{a}+\frac{1}{b}\right),$$

are the coordinates of the three excentres, where the points $\left(\frac{-1}{ibc}, -ibc\right)$, $\left(\frac{-1}{ica}, -ica\right)$, $\left(\frac{-1}{iab}, -iab\right)$ are the orthocentres of the triangles partitioned by the incentre $I\left(i, \frac{1}{i}\right)$.

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The excentral triangle $I_AI_BI_C$

Now since $\angle BII_A$, $\angle CII_A$, etc. are right angles, I is the orthocentre $H'\left(i, \frac{1}{i}\right)$ of the excentral triangle.

Furthermore, since the centroid of the host triangle is $G\left(\frac{e}{3}+\frac{2i}{3}, \frac{1}{3e}+\frac{2}{3i}\right)$, it follows that the centroid of the excentral triangle is $G'\left(\frac{2e}{3}-\frac{2h}{3}+i, \frac{2}{3e}-\frac{2}{3h}+\frac{1}{i}\right)$ and that its circumcentre is the Bevan point $O'\left(e-h+i, \frac{1}{e}-\frac{1}{h}+\frac{1}{i}\right)$.

Conclusion

This Cartesian framework may be used to examine properties of the Bevan point ([2]) and also to solve the Euler disc problem ([1] again).

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103.43 An inequality arising from the inarc centres of a triangle

The inarc circles and the inequality

An inarc circle is a circle tangent to two sides of a triangle and internally to the circumcircle of the triangle. The terminology is not unique and they are also called mixtilinear incircles [1], but this name obviously doesn't say much. It is known [2], that the midpoint of the touching points of the inarc circle and the sides of the triangle is the incentre I of the triangle. To find the inarc centre opposite vertex A, let the line through the incentre I perpendicular to AI meet the side AB at the point X. Then the inarc centre L_A is the intersection of the line through X perpendicular to AB and the line AI, see Figure 1.

We denote by ρ_A, ρ_B, ρ_C the radii and by L_A, L_B, L_C the centres of the inarc circles of a triangle ABC with sides a, b, c, inradius r, circumradius R and semiperimeter s. From [2, p. 293], it is straightforward that the inarc radius is

$$\rho_A = \frac{r}{\cos^2 \frac{1}{2}A}. \tag{1}$$

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