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A result on hypergeometric functions

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1. *Summary.* The main result of the present paper is contained in the following:

THEOREM 1·1. *For* $\Re s > 0$ *and* $\alpha \neq -1, -2, -3, \ldots$

$$\sum_{n=0}^{\infty} \left(\frac{s}{s+1}\right)^{n+1} {}_2F_1\left[1, n+\alpha+1; \alpha+1; \frac{1}{s+1}\right] \frac{\beta^n}{n!} = e^{\beta}\, {}_1F_1\left[\alpha; \alpha+1; -\frac{\beta}{s+1}\right],$$

where ${}_2F_1[.,.;.;.]$ *and* ${}_1F_1[.;.;.]$ *are as in* (1).

We also deduce the following

COROLLARY 1·1. *For* $\Re s > 0$ *and* $\alpha \neq -1, -2, -3, \ldots$

$$\sum_{m=0}^{\infty} \left(\frac{1}{s+1}\right)^{m} {}_1F_1\left[m+\alpha+1; \alpha+1; \frac{\beta s}{s+1}\right] = \left(\frac{s+1}{s}\right) e^{\beta}\, {}_1F_1\left[\alpha; \alpha+1; -\frac{\beta}{s+1}\right].$$

The result presented in the above theorem seems to be of interest in as much as any infinite series relationship between ${}_2F_1$ and ${}_1F_1$ has not come to the notice of the author so far. The result in the Corollary 1·1 is, of course, a new one for ${}_1F_1$ and only the case $\alpha = 0$ seems to have been covered earlier. An application of these results is indicated by evaluating an integral frequently arising in statistical problems. In particular, for $\Re a^2 > 0$, we show that

$$I = \int_{-\infty}^{\infty} e^{-(ay+b)^2} \operatorname{erf}(y)\, dy = -\frac{\pi^{\frac{1}{2}}}{a} \operatorname{erf}\left\{\frac{b}{(1+a^2)^{\frac{1}{2}}}\right\},$$

where

$$\operatorname{erf}(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^x e^{-t^2}\, dt$$

defines an error function.

2. LEMMA 2·1. *For* $\Re s > 0$ *and* $\alpha \neq -1\ -2, -3, \ldots$

$$\sum_{m=0}^{r} \frac{(\alpha)_m}{m!} \left(\frac{s}{s+1}\right)^{m} = \frac{(\alpha+1)_r}{r!} \left(\frac{s}{s+1}\right)^{r+1} {}_2F_1\left[1, r+\alpha+1; \alpha+1; \frac{1}{s+1}\right],$$

where $(a)_n$ *is the usual Pochhammer's symbol.*

Proof. On using a series expansion, the left-hand side of the lemma could be rewritten as

$$\left(\frac{s}{s+1}\right)^{r+1} \sum_{m=0}^{r} \frac{(\alpha)_m}{m!} \left(1-\frac{1}{s+1}\right)^{-(r-m+1)} = \left(\frac{s}{s+1}\right)^{r+1} \sum_{m=0}^{r} \frac{(\alpha)_m}{m!} \sum_{n=0}^{\infty} \frac{(r-m+1)_n}{n!} \left(\frac{1}{s+1}\right)^{n}$$

$$= \left(\frac{s}{s+1}\right)^{r+1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{s+1}\right)^{n} \sum_{m=0}^{r} \frac{(\alpha)_m}{m!} (r-m+1)_n. \tag{1}$$

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Now, equating the coefficients of t^r on both sides of

$$ {}_1F_0(\alpha;\ ;t)\cdot{}_1F_0(n+1;\ ;t) \equiv {}_1F_0(n+\alpha+1;\ ;t), $$

we get

$$ \sum_{m=0}^{r} \frac{(\alpha)_m}{m!}\frac{(n+1)_{r-m}}{(r-m)!} = \frac{(n+\alpha+1)_r}{r!}, $$

or

$$ \sum_{m=0}^{r} \frac{(\alpha)_m}{m!}(r-m+1)_n = \frac{n!}{r!}\frac{\Gamma(n+\alpha+r+1)}{\Gamma(n+\alpha+1)}. $$

Applying this last result in (1), the lemma is proved after some simple manipulations.

3. *Proof of the theorem.* Replacing ${}_2F_1$, involved in the theorem, by the finite series obtained in the above lemma, and rearranging the terms of the resulting double series, we get

$$ \begin{aligned} &\sum_{n=0}^{\infty}\left(\frac{s}{s+1}\right)^n {}_2F_1\left[1, n+\alpha+1; \alpha+1; \frac{1}{s+1}\right]\frac{\beta^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{\beta^n}{(\alpha+1)_n}\sum_{m=0}^{n}\frac{(\alpha)_m}{m!}\left(\frac{s}{s+1}\right)^m \\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}\left(\frac{s}{s+1}\right)^m\sum_{n=m}^{\infty}\frac{\beta^n}{(\alpha+1)_n} \\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}\left(\frac{\beta s}{s+1}\right)^m\sum_{r=0}^{\infty}\frac{\beta^r}{(\alpha+1)_{r+m}} \\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\alpha+1)_m m!}\left(\frac{\beta s}{s+1}\right)^m\sum_{r=0}^{\infty}\frac{(1)_r}{(m+\alpha+1)_r}\frac{\beta^r}{r!} \\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\alpha+1)_m m!}\left(\frac{\beta s}{s+1}\right)^m {}_1F_1[1; m+\alpha+1; \beta] \\ &= e^{\beta}\sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\alpha+1)_m m!}\left(\frac{\beta s}{s+1}\right)^m {}_1F_1[m+\alpha; m+\alpha+1; -\beta], \end{aligned} $$

by virtue of Kummer's transformation. Applying now the multiplication theorem ((2), Formula 1, p. 283),

$$ {}_1F_1[a; c; \lambda x] = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n n!}(\lambda-1)^n x^n {}_1F_1[a+n; c+n; x], $$

and taking $a=\alpha$, $c=\alpha+1$, $\lambda = 1/(s+1)$, we get

$$ \sum_{m=0}^{\infty}\frac{(\alpha)_m}{(\alpha+1)_m m!}\left(\frac{\beta s}{s+1}\right)^m {}_1F_1[m+\alpha; m+\alpha+1; -\beta] = {}_1F_1\left[\alpha; \alpha+1; -\frac{\beta}{s+1}\right]. $$

Hence, the theorem is established.

The Corollary 1·1 is easily obtained by expressing ${}_2F_1$, involved in the theorem, in terms of the usual infinite series and subsequently changing the order of summation in the resulting double series.

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4. *An application.* To evaluate the integral I, we expand e^{-2aby} involved therein, in power series, and then interchange the order of summation and integration. Thus,

$$\begin{aligned} I &= -e^{-b^2} \sum_{r=0}^{\infty} \frac{(2ab)^{2r+1}}{(2r+1)!} \int_{-\infty}^{\infty} e^{-a^2y^2} y^{2r+1} \operatorname{erf}(y)\, dy \\ &= -e^{-b^2} \sum_{r=0}^{\infty} \frac{(2ab)^{2r+1}}{\Gamma(2r+2)} \frac{2}{\pi^{\frac{1}{2}}} \int_0^{\infty} y^{2r+1} e^{-a^2y^2} \gamma(\tfrac{1}{2}, y^2)\, dy \\ &= -\frac{e^{-b^2}}{\pi^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{(2ab)^{2r+1}}{\Gamma(2r+2)} \int_0^{\infty} t^r e^{-a^2t} \gamma(\tfrac{1}{2}, t)\, dt \\ &= -\frac{2}{\pi^{\frac{1}{2}}} e^{-b^2} \sum_{r=0}^{\infty} \frac{(2ab)^{2r+1}}{\Gamma(2r+2)} \frac{\Gamma(r+\frac{3}{2})}{(1+a^2)^{r+\frac{3}{2}}} {}_2F_1\left[1, r+\tfrac{3}{2}; \tfrac{3}{2}; \frac{1}{1+a^2}\right], \end{aligned}$$

by expressing $\operatorname{erf}(y)$ in terms of the incomplete gamma function $\gamma(n, y)$ ((3), formula 1, p. 147) and subsequently using the Laplace transform ((3), formula 7, p. 138) for the function $t^r . \gamma(\frac{1}{2}, t)$. Now, we apply Legendre's duplication formula for gamma functions and then apply the theorem proved in the present paper to obtain

$$\begin{aligned} I &= -\frac{2b\, e^{-b^2}}{a(1+a^2)^{\frac{1}{2}}} \sum_{r=0}^{\infty} \left(\frac{a^2}{a^2+1}\right)^{r+1} {}_2F_1\left[1, r+\tfrac{3}{2}; \tfrac{3}{2}; \frac{1}{1+a^2}\right] \frac{(b^2)^r}{r!} \\ &= -\frac{2b}{a(1+a^2)^{\frac{1}{2}}} {}_1F_1\left[\tfrac{1}{2}; \tfrac{3}{2}; -\frac{b^2}{1+a^2}\right] \\ &= -\frac{\pi^{\frac{1}{2}}}{a} \operatorname{erf}\left\{\frac{b}{(1+a^2)^{\frac{1}{2}}}\right\}, \end{aligned}$$

since

$$\operatorname{erf}(x) = \frac{2}{\pi^{\frac{1}{2}}} x\, {}_1F_1[\tfrac{1}{2}; \tfrac{3}{2}; -x^2],$$

and thus, the required result is proved.

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