QUASI-INVARIANT MEASURES CONCENTRATING ON COUNTABLE STRUCTURES

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Abstract. Countable $\mathcal{L}$-structures $\mathcal{N}$ whose isomorphism class supports a permutation invariant probability measure in the logic action have been characterized by Ackerman–Freer–Patel to be precisely those $\mathcal{N}$ which have no algebraicity. Here we characterize those countable $\mathcal{L}$-structures $\mathcal{N}$ whose isomorphism class supports a quasi-invariant probability measure. These turn out to be precisely those $\mathcal{N}$ which are not "highly algebraic"—we say that $\mathcal{N}$ is highly algebraic if outside of every finite F there is some b and a tuple $\bar{a}$ disjoint from b so that b has a finite orbit under the pointwise stabilizer of $\bar{a}$ in $\mathrm{Aut}(\mathcal{N})$. As a byproduct of our proof we show that whenever the isomorphism class of $\mathcal{N}$ admits a quasi-invariant measure, then it admits one with continuous Radon–Nikodym cocycles.

§1. Introduction. The past decade has seen the development of the study of invariant random structures. This field was kindled by the work of Petrov and Vershik [13] on exchangeable random triangle-free graphs. A fundamental result in this area is due to Ackerman, Freer, and Patel [5] who prove that, for a given countable $\mathcal{L}$-structure $\mathcal{M}$, the existence of an invariant random structure almost surely isomorphic to $\mathcal{M}$ is equivalent to $\mathcal{M}$ satisfying a combinatorial property known as no-algebraicity.

Recall that for any first order language $\mathcal{L}$ which contains countably many relation and function symbols, the collection $\mathrm{Str}(\mathcal{L},\mathbb{N})$ of all $\mathcal{L}$-structures whose domain is the natural numbers $\mathbb{N}$ can be endowed with a natural Polish topology (see Section 2.1). For every structure $\mathcal{N}\in\mathrm{Str}(\mathcal{L},\mathbb{N})$, the isomorphism class $[\mathcal{N}]_{\mathrm{iso}}$ of $\mathcal{N}$ is the set:

$$[\mathcal{N}]_{\mathrm{iso}} := \{\mathcal{M}\in\mathrm{Str}(\mathcal{L},\mathbb{N})\colon \mathcal{M} \text{ and } \mathcal{N} \text{ are isomorphic}\}.$$

These classes can be identified with the orbits under the *logic action* $\mathrm{Sym}(\mathbb{N})\curvearrowright\mathrm{Str}(\mathcal{L},\mathbb{N})$ of the Polish group $\mathrm{Sym}(\mathbb{N})$ of all permutations of $\mathbb{N}$ on $\mathrm{Str}(\mathcal{L},\mathbb{N})$ (see Section 2.2).

A *random structure* is just a Borel probability measure on $\mathrm{Str}(\mathcal{L},\mathbb{N})$. A random structure is said to be *invariant* if it is invariant under the logic action. In [5] it was shown that the existence of an invariant random structure supported on $[\mathcal{N}]_{\mathrm{iso}}$ is equivalent to $\mathcal{N}$ having *no-algebraicity*: for any tuple $\bar{a}$, $\mathrm{acl}(\bar{a},\mathcal{N})=\bar{a}$, where $\mathrm{acl}(\bar{a},\mathcal{N})$ refers to the points of $\mathcal{N}$ whose orbit is finite under the pointwise stabilizer

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of $\bar{a}$ (see Section 2.1). This result was followed by an extensive study of invariant random structures [1–3, 8] and invariant random expansions [10].

In the present article we study a generalization of invariant random structures: quasi-invariant random structures. Let $G \curvearrowright X$ be a Borel action of a Polish group on a standard Borel space and recall that a probability measure μ on X is *G-quasi-invariant*, if for every Borel $B \subseteq X$ and every $g \in G$ we have that $\mu(B) = 0 \iff \mu(g \cdot B) = 0$. The class of S_∞-quasi-invariant measures have been studied by Nessonov in [11].

We call a countable structure $\mathcal{N}$ *quasi-invariantly random* if there exists a $\mathrm{Sym}(\mathbb{N})$-quasi-invariant probability measure μ on $\mathrm{Str}(\mathcal{L}, \mathbb{N})$ with $\mu([\mathcal{N}]_{\mathrm{iso}}) = 1$. The main result of this article is the following characterization of quasi-invariantly random structures in terms of the algebraic closure operator, similar in spirit to the main result of [5].

DEFINITION 1. We say that $\mathcal{N}$ is *highly algebraic* if for every finite $F \subseteq \mathrm{dom}(\mathcal{N})$ there exists a tuple $\bar{a} = (a_0, \dots, a_{n-1})$ in $\mathrm{dom}(\mathcal{N})$ so that $\mathrm{acl}(\bar{a}, \mathcal{N}) \not\subseteq F \cup \{a_0, \dots, a_{n-1}\}$.

THEOREM 2. *A countable structure is quasi-invariantly random if and only if it is not highly algebraic.*

Notice that, when $\mathcal{L}$ contains symbols of arbitrarily large arity, the action of $\mathrm{Sym}(\mathbb{N})$ on $\mathrm{Str}(\mathcal{L}, \mathbb{N})$ is universal (see [6] section 2.7). Hence, up to embedding, Theorem 2 describes all $\mathrm{Sym}(\mathbb{N})$-quasi-invariant actions that admit an orbit of full measure. We also observe that $\mathcal{N}$ is not highly algebraic if and only if $\mathrm{acl}(\emptyset, \mathcal{N})$ is finite and, for all $\bar{a}$ in $\mathrm{dom}(\mathcal{N})$, we have $\mathrm{acl}(\bar{a}, \mathcal{N}) \subseteq \mathrm{acl}(\emptyset, \mathcal{N}) \cup \{a_0, \dots, a_{n-1}\}$. This simply follows from the $\mathrm{Aut}(\mathcal{N})$-invariance of the union of all sets of the form $\mathrm{acl}(\bar{a}, \mathcal{N}) \setminus \{a_0, \dots, a_{n-1}\}$.

Finally, recall that to every quasi-invariant measure μ on a Borel G-space X one can use the Radon–Nikodym derivatives to associate with μ a measurable cocycle $c_\mu \colon G \times X \to \mathbb{R}$. The following comes as a byproduct of our proof.

THEOREM 3. *If the structure $\mathcal{N} \in \mathrm{Str}(\mathcal{L}, \mathbb{N})$ is not highly algebraic, then $[\mathcal{N}]_{\mathrm{iso}}$ is the support of a $\mathrm{Sym}(\mathbb{N})$-quasi-invariant probability measure μ whose cocycle c_μ induces a continuous map $c_\mu(-, \mathcal{N}) \colon \mathrm{Sym}(\mathbb{N}) \to \mathbb{R}$ for each $\mathcal{N} \in \mathrm{Str}(\mathcal{L}, \mathbb{N})$.*

1.1. Outline and sketch of proofs. Section 2 covers the essential definitions and results we need in the proofs. Section 3 is concerned with showing that highly algebraic structures are not quasi-invariantly random; the proof consists of constructing continuum many disjoint translates of any compact subset of the orbit of the structure. Section 4 shows the forward direction of Theorem 2. Here we use invariant measures from [5] to construct quasi-invariant measures concentrated on the orbit of a given a structure that is not highly algebraic. Section 5 collects some useful examples of $\mathcal{L}$-structures in relation to high algebraicity.

§2. Background.

2.1. Countable $\mathcal{L}$-structures. We follow the usual conventions from model theory (see, e.g., [9] for more details). A countable language $\mathcal{L}$ is a collection of function

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symbols and relation symbols. Each symbol has a prespecified arity. An $\mathcal{L}$-structure $\mathcal{M}$ is a set M together with an interpretation of $\mathcal{L}$, i.e., for each relation symbol $R \in \mathcal{L}$ of arity $n(R) \in \mathbb{N}$, a subset $R^{\mathcal{M}}$ of M^n and for each function symbol $f \in \mathcal{L}$ of arity $n(f) \in \mathbb{N}$, a map $f^{\mathcal{M}} \colon M^n \to M$. Given such $\mathcal{L}$ and some countable set N, let $\mathrm{Str}(\mathcal{L}, N)$ be the space of all $\mathcal{L}$-structures $\mathcal{N}$ with $\mathrm{dom}(\mathcal{N}) = N$. As usual, we identify $\mathrm{Str}(\mathcal{L}, N)$ with the Polish space

$$\Big(\prod_{R\in\mathcal{L}} 2^{\overbrace{N\times\cdots\times N}^{n(R)}}\Big)\times\Big(\prod_{f\in\mathcal{L}} N^{\overbrace{N\times\cdots\times N}^{n(f)}}\Big),$$

where the products are taken over all relation and function symbols in $\mathcal{L}$, respectively.

Whenever the domain N is fixed, we will denote by $a, b, c, \ldots$ the elements of N and by $\bar{a}, \bar{b}, \bar{c} \ldots$ tuples in N. By a *tuple* we mean any finite map $\bar{a} \colon n \to N$, where $n = \{0, \ldots, n-1\}$. Such tuple has length n and we often denote it by $(a_0, \ldots, a_{n-1})$. and we denote by $\{\bar{a}\}$ the associated set $\{a_0, \ldots, a_{n-1}\}$ of its elements. We denote by $\bar{a}\bar{b}$ the concatenation $(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1})$ of the tuples $\bar{a}$ and $\bar{b}$.

Let $\mathcal{N}$ be an $\mathcal{L}$-structure with $\mathrm{dom}(\mathcal{N}) = N$ and let $\bar{a}$ in N. The *quantifier-free type* $\mathrm{qft}(\bar{a}, \mathcal{N})$ of $\bar{a}$ in $\mathcal{N}$ is the collection of all quantifier-free $\mathcal{L}$-formulas $\varphi(\bar{x})$ with $\mathcal{N} \models \varphi(\bar{a})$. We say that $\mathcal{N}$ is *ultrahomogeneous* if $\mathrm{qft}(\bar{a}, \mathcal{N}) = \mathrm{qft}(\bar{b}, \mathcal{N})$ holds if and only if there exists some $g \in \mathrm{Aut}(\mathcal{N})$ with $g\bar{a} = \bar{b}$. A relation $R \in \mathcal{L}$ *captures* $\mathrm{qft}(\bar{a}, \mathcal{N})$ if for every $\bar{b}$ in N we have that $\mathcal{N} \models R(\bar{b})$ if and only if $\mathrm{qft}(\bar{a}, \mathcal{N}) = \mathrm{qft}(\bar{b}, \mathcal{N})$.

Finally, we say that $\mathcal{N}$ is *R-ultrahomogeneous* if it is ultrahomogeneous and for every $\bar{a}$ in N we have that $\mathrm{qft}(\bar{a}, \mathcal{N})$ is captured by some relation R in $\mathcal{L}$. Let

$$\mathrm{Aut}(\mathcal{N})_{\bar{a}} := \{g \in \mathrm{Aut}(\mathcal{N}) \colon g(a_0) = a_0, \ldots, g(a_{n-1}) = a_{n-1}\}$$

be the pointwise stabilizer of $\bar{a}$. The *algebraic closure of* $\bar{a}$ in $\mathcal{N}$ is the set

$$\mathrm{acl}(\bar{a}, \mathcal{N}) := \{b \in \mathrm{dom}(\mathcal{N}) \colon \mathrm{Aut}(\mathcal{N})_{\bar{a}} \cdot b \text{ is finite}\}.$$

The following standard fact is going to be used later on.

LEMMA 4. *For every countable $\mathcal{L}$-structure $\mathcal{N}$ there exists a countable $\mathcal{L}' \supseteq \mathcal{L}$ and some R-ultrahomogeneous $\mathcal{L}'$-structure $\mathcal{N}'$ whose $\mathcal{L}$-reduct $\mathcal{N}' \restriction \mathcal{L}$ is $\mathcal{N}$ and for all $\bar{a}$ we have*

$$\mathrm{Aut}(\mathcal{N})_{\bar{a}} = \mathrm{Aut}(\mathcal{N}')_{\bar{a}}.$$

In particular, $\mathrm{acl}(\bar{a}, \mathcal{N}) = \mathrm{acl}(\bar{a}, \mathcal{N}')$ *holds, for all* $\bar{a}$ *in* $\mathrm{dom}(\mathcal{N})$.

PROOF. This follows from the usual "orbit completion" argument which can be found in [9, Theorem 4.1.4] or [6, Section 1.5]. In short, for every $n \in 1, 2, \ldots$, and for every orbit $O \subseteq \mathrm{dom}(\mathcal{N})^n$ in the diagonal action $g \cdot (a_0, \ldots, a_{n-1}) = (g \cdot a_0, \ldots, g \cdot a_{n-1})$ of $\mathrm{Aut}(\mathcal{N})$ on $\mathrm{dom}(\mathcal{N})^n$, we introduce a n-ary relation symbol $R^{n,O}$ and we set $\mathcal{N}' \models R^{n,O}(\bar{a})$ if and only if $\bar{a} \in O$. ⊣

2.2. Logic action. Let $\mathrm{Sym}(N)$ be the Polish group of all bijections $g \colon N \to N$. The group operation is the operation of composition $(g \cdot h)(n) = g(h(n))$ and the

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topology is that of pointwise convergence. The *logic action* is the continuous action $\mathrm{Sym}(N) \curvearrowright \mathrm{Str}(\mathcal{L}, N)$ given by the *push-forward* operation:

$$\begin{aligned}&(g, \mathcal{M}) \mapsto g_*\mathcal{M}, \text{ where}\\ &g_*\mathcal{M} \models R(a_0, \dots, a_{n-1}) \iff \mathcal{M} \models R(g^{-1}(a_0), \dots, g^{-1}(a_{n-1})), \text{ and}\\ &g_*\mathcal{M} \models \big(b = f(a_0, \dots, a_{n-1})\big) \iff \mathcal{M} \models \big(g^{-1}(b) = f(g^{-1}(a_0), \dots, g^{-1}(a_{n-1}))\big).\end{aligned}$$

§3. Highly algebraic structures are not quasi-invariantly random. We fix some $\mathcal{N} \in \mathrm{Str}(\mathcal{L}, \mathbb{N})$ and we assume throughout this section that $\mathcal{N}$ is highly algebraic and R-ultrahomogeneous. We also set $X := [\mathcal{N}]_{\mathrm{iso}}$ to be the isomorphism class of $\mathcal{N}$ in $\mathrm{Str}(\mathcal{L}, \mathbb{N})$. Notice that the ultrahomogeneity of $\mathcal{N}$ implies that X is a G_δ subset of $\mathrm{Str}(\mathcal{L}, \mathbb{N})$ and hence a Polish space. The main goal of this section is to prove the following.

LEMMA 5. *If $K \subseteq X$ is compact, then there is a family $(g_\alpha : \alpha \in 2^{\mathbb{N}})$ of permutations $g_\alpha \in \mathrm{Sym}(\mathbb{N})$ so that for all $\alpha, \beta \in 2^{\mathbb{N}}$ with $\alpha \neq \beta$ we have that $g_\alpha K \cap g_\beta K = \emptyset$.*

This lemma directly implies one of the directions of Theorem 2.

PROOF OF ($\Rightarrow$) DIRECTION OF THEOREM 2 FROM LEMMA 5. Let $\mathcal{N}^* \in \mathrm{Str}(\mathcal{L}^*, \mathbb{N})$ be some highly algebraic structure and assume towards a contradiction that there exists a $\mathrm{Sym}(\mathbb{N})$-quasi-invariant probability measure μ on $\mathrm{Str}(\mathcal{L}^*, \mathbb{N})$ so that $\mu([\mathcal{N}^*]_{\mathrm{iso}}) = 1$. Let $\mathcal{L} := (\mathcal{L}^*)'$ and $\mathcal{N} := (\mathcal{N}^*)'$ be the language and the structure provided by Lemma 4 and set $X := [\mathcal{N}]_{\mathrm{iso}}$.

Consider now the map $\pi : X \to \mathrm{Str}(\mathcal{L}, \mathbb{N})$, which implements the $\mathcal{L}$-reduct $\mathcal{M} \mapsto \mathcal{M} \restriction \mathcal{L}$. Then π is clearly a $\mathrm{Sym}(\mathbb{N})$-equivariant Borel isomorphism and hence we can transfer μ to a $\mathrm{Sym}(\mathbb{N})$-quasi-invariant probability measure on X by pushing it forward via π^{-1}. Since $\mathcal{N}$ is ultrahomogeneous, we have that X is a Polish space, and Lemma 5 applies. But probability measures on Polish spaces are regular and hence, there exists a compact $K \subseteq X$ so that $\mu(K) > 0$. By quasi-invariance of μ we have that $\mu(g_\alpha K) > 0$, for all $\alpha \in 2^{\mathbb{N}}$, where $(g_\alpha : \alpha \in 2^{\mathbb{N}})$ is as in Lemma 5. Hence there is some $\varepsilon > 0$ so that $\{\alpha \in 2^{\mathbb{N}} : \mu(g_\alpha K) > \varepsilon\}$ is uncountable. But since $g_\alpha K \cap g_\beta K = \emptyset$ when $\alpha \neq \beta$, σ-additivity of μ contradicts that μ is a probability measure. ⊣

The proof of Lemma 5 occupies the rest of this section. Interestingly, there are two distinct ways for a structure $\mathcal{N}$ to exhibit high algebraicity. Each way contributes to the proof of Lemma 5 through a different argument. In Lemma 9 below we establish this dichotomy-in-behaviour, before we conclude with the proof of Lemma 5.

3.1. Highly algebraic structures. Below we will make extensive use of the Neumann's lemma [12, Lemma 2.3]. Rephrased as in [7, Theorem 6.2], it states the following.

LEMMA 6 (Neumann). *Let H be a group acting on a countable set Ω with no finite orbit. Let Γ and Δ be finite subsets of Ω, then there is $h \in H$ such that $h \cdot \Gamma \cap \Delta = \varnothing$.*

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LEMMA 7. *For any $\mathcal{L}$-structure $\mathcal{M}$, the following statements are equivalent*:
(1) *$\mathcal{M}$ is highly algebraic*;
(2) *for all $\bar{c}$ there exist $\bar{a}$, b so that $\{\bar{a}\},\{b\},\{\bar{c}\}$ are pairwise disjoint and $b \in \mathrm{acl}(\bar{a}\bar{c},\mathcal{M})$*;
(3) *for all $\bar{c}$ there exist $\bar{a}$, b so that $\{\bar{a}\},\{b\},\{\bar{c}\}$ are pairwise disjoint and $b \in \mathrm{acl}(\bar{a},\mathcal{M})$*.

PROOF. It is straightforward to see the equivalence between (1) and (2). It is also clear that (3) implies (2). We prove that (2) implies (3) as well. Assume (2) holds and let $\bar{c}$ be given.

We may assume that $\mathrm{acl}(\emptyset,\mathcal{M})$ is finite since, otherwise, (3) would follow by simply setting $\bar{a} := \emptyset$. Let now $\bar{e}$ be a finite tuple enumerating $\mathrm{acl}(\emptyset,\mathcal{M})$. By (2), and by shrinking $\bar{c}$ if necessary, we get $\bar{a}, b$ so that $\{\bar{a}\},\{b\},\{\bar{c}\},\{\bar{e}\}$ are pairwise disjoint and $b \in \mathrm{acl}(\bar{a}\bar{c}\bar{e})$.

CLAIM 8. *There exists $g \in \mathrm{Aut}(\mathcal{M})_{\bar{e}}$ so that the tuple $g(\bar{a}b\bar{c})$ is disjoint from $\bar{c}\bar{e}$*.

PROOF OF CLAIM. Every $d \in \mathrm{dom}(\mathcal{M}) \setminus \{\bar{e}\}$ has an infinite orbit under $\mathrm{Aut}(\mathcal{M})_{\bar{e}}$. Indeed, since $\bar{e}$ enumerates $\mathrm{acl}(\emptyset,\mathcal{M})$, there exist $h_1,\dots,h_k \in \mathrm{Aut}(\mathcal{M})$ so that every $g \in \mathrm{Aut}(\mathcal{M})$ is of the form $g = h_i h$ for some $h \in \mathrm{Aut}(\mathcal{M})_{\bar{e}}$ and $i \leq k$. Hence, if the set $E = \{gd : g \in \mathrm{Aut}(\mathcal{M})_{\bar{e}}\}$ was finite, then so would be the set $\{gd : g \in \mathrm{Aut}(\mathcal{M})\} = \bigcup_i h_i E$. Since $\{\bar{a}b\bar{c}\} \subseteq \mathrm{dom}(\mathcal{M}) \setminus \{\bar{e}\}$, by Lemma 6 there exists a g as in the statement of the claim. ⊣

Fix g as in the claim and set $\bar{a}' := g(\bar{a}\bar{c})$ and $b' := g(b)$. By the claim, and since $\{\bar{a}\},\{b\}$, and $\{\bar{c}\}$ were pairwise disjoint, so are $\{\bar{a}'\},\{b'\}$, and $\{\bar{c}\}$. But then we have that

$$b \in \mathrm{acl}(\bar{a}\bar{c}\bar{e},\mathcal{N}) \Longrightarrow g(b) \in \mathrm{acl}(g(\bar{a}\bar{c})g(\bar{e}),\mathcal{N}) \Longrightarrow b' \in \mathrm{acl}(\bar{a}'\bar{e},\mathcal{N}) \Longrightarrow b' \in \mathrm{acl}(\bar{a}',\mathcal{N}),$$

where the last implication follows from the fact that $\bar{e} \in \mathrm{acl}(\emptyset,\mathcal{N})$, by an argument similar to the one in the previous claim. ⊣

Using the above reformulation (3) of high algebraicity, we get the following dichotomy. See Examples 20 and 21 for simple examples of structures on each side of the dichotomy.

LEMMA 9. *If $\mathcal{N}$ is highly algebraic, then for every finite tuple $\bar{c}$ there exist $\bar{a}$ and b in* $\mathrm{dom}(\mathcal{N})$ *so that the sets $\{\bar{a}\}$, $\{b\}$, and $\{\bar{c}\}$ are pairwise disjoint and one of following holds*:
(1) *$\{\bar{a}\} = \{a\}$ is a singleton and both $b \in \mathrm{acl}(a,\mathcal{N})$ and $a \in \mathrm{acl}(b,\mathcal{N})$ hold; or*
(2) *$b \in \mathrm{acl}(\bar{a},\mathcal{N})$ and there exists an infinite sequence $(\bar{a}^i : i \in \mathbb{N})$ of pairwise disjoint tuples so that for every $i \in \mathbb{N}$ we have that $\mathrm{qft}(\bar{a}^i b,\mathcal{N}) = \mathrm{qft}(\bar{a}b,\mathcal{N})$*.

PROOF. By Lemma 7(3) there exist $\bar{a}$ and b in $\mathrm{dom}(\mathcal{N})$ so that the sets $\{\bar{a}\}$, $\{b\}$, and $\{\bar{c}\}$ are pairwise disjoint and $b \in \mathrm{acl}(\bar{a},\mathcal{N})$. Assume that there is no such $\bar{a}$ and b satisfies (2) above. Choose $\bar{a}$ and b as in Lemma 7(3) so that $\bar{a}$ is of minimum possible size. Since by assumption $\bar{a}, b$ do not satisfy (2) we have that $\bar{a} \neq \emptyset$.

Let now a be any entry of $\bar{a}$ so that $a \in \mathrm{acl}(b,\mathcal{N})$. Such entry exists since, otherwise, by Lemma 6 we would have that (2) holds for this $\bar{a}$ and b. But since $\{b\}$ is a singleton for which there is $a \in \mathrm{acl}(b,\mathcal{N})$, the minimality assumption on the length of $\bar{a}$ above implies that $\{\bar{a}\} = \{a\}$ and hence (1) above follows. ⊣

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3.2. Proof of Lemma 5. We may now conclude the proof of Lemma 5 which, in turn, establishes the $(\Rightarrow)$ direction of Theorem 2. Recall that throughout this section we assume that the structure $\mathcal{N} \in \mathrm{Str}(\mathcal{L}, \mathbb{N})$ is highly algebraic and R-ultrahomogeneous.

Proof of Lemma 5. Let $K \subseteq X$ be compact.

For every $s \in 2^{<\mathbb{N}}$ we will define a natural number ℓ_s and some permutation γ_s of the finite set $\ell_s = \{0, 1, \dots, \ell_s - 1\}$ so that for all $s \in 2^{<\mathbb{N}}$ we have that:

(i) $\ell_\emptyset = 0$ and $\gamma_\emptyset$ is the empty permutation;
(ii) $\ell_{s^\frown 0} = \ell_{s^\frown 1}$ and $\ell_{s^\frown 0}, \ell_{s^\frown 1} > \ell_s$, as well as $\gamma_{s^\frown 0} \restriction \ell_s = \gamma_s$ and $\gamma_{s^\frown 1} \restriction \ell_s = \gamma_s$;
(iii) $g_0 K \cap g_1 K = \emptyset$, for all $g_0, g_1 \in \mathrm{Sym}(\mathbb{N})$ with $g_0 \restriction \ell_{s^\frown 0} = \gamma_{s^\frown 0}$ and $g_1 \restriction \ell_{s^\frown 1} = \gamma_{s^\frown 1}$.

Granted $(\gamma_s : s \in 2^{<\mathbb{N}})$ as above, we define the desired family $(g_\alpha : \alpha \in 2^{\mathbb{N}})$ by setting

$$g_\alpha := \bigcup_{n \in \mathbb{N}} \gamma_{\alpha|n}.$$

Then, (ii) implies that $g_\alpha \in \mathrm{Sym}(\mathbb{N})$ and (iii) implies that $g_\alpha K \cap g_\beta K = \emptyset$, whenever $\alpha \neq \beta$.

We proceed now to the definition of ℓ_s, γ_s by induction on s. Assume that ℓ_s, γ_s have been defined. We define $\ell_{s^\frown 0}, \ell_{s^\frown 1}$ and $\gamma_{s^\frown 0}, \gamma_{s^\frown 1}$ using the following construction.

Claim 10. *There exists a finite family $\{S_i : i \in I\}$ of ℓ_s-ary relations S_i so that*:

(I) *for every $\mathcal{M} \in K$ there is $i \in I$ so that $\mathcal{M} \models S_i(0, 1, \dots, \ell_s - 1)$;*
(II) *for every ℓ_s-ary tuple $\bar{e}$ with $\mathcal{N} \models S_i(\bar{e})$ we have that S_i captures $\bar{e}$ in $\mathcal{N}$.*

Proof of Claim. This follows from compactness of K since $\mathcal{N}$ is R-ultrahomogeneous. ⊣

For each $i \in I$ let $\bar{c}^i$ be a tuple in $\mathbb{N}$ so that $\mathcal{N} \models S_i(\bar{c}^i)$ and let $\bar{c}$ be the union of all these tuples. Apply Lemma 9 for this $\bar{c}$ to get $\bar{a}$, b so that $\{\bar{a}\}, \{b\}, \{\bar{c}\}$ are pairwise disjoint and for which either (1) or (2) of Lemma 9 holds. Since $\mathcal{N}$ is R-ultrahomogeneous, we may pick relations $P, Q, R \in \mathcal{L}$, so that: P captures $\mathrm{qft}(\bar{a}, \mathcal{N})$; Q captures $\mathrm{qft}(b, \mathcal{N})$; and R captures $\mathrm{qft}(\bar{a}b, \mathcal{N})$. Below we will abuse notation and use the letters $\bar{a}$ and b for potentially new tuples which satisfy P and Q, respectively, in any structure $\mathcal{M}$ with $\mathcal{M} \simeq_{\mathrm{iso}} \mathcal{N}$.

We define $\ell_{s^\frown 0}, \ell_{s^\frown 1}, \gamma_{s^\frown 0}, \gamma_{s^\frown 1}$ by considering separately cases (1) and (2) of Lemma 9.

Case (1). Assume that Lemma 9(1) holds for $\bar{a}, b, \bar{c}$. In particular, we have that both P, Q are unary relations and R is a binary relation.

Claim 11. *There exists $\ell > \ell_s$ and some set $Z \subseteq \mathbb{N} \setminus \{0, \dots, \ell - 1\}$ with $|Z| = \ell - \ell_s$, so that for every $\mathcal{M} \in K$ we have that*:

(a) *there exist $a, b \in \{\ell_s, \dots, \ell - 1\}$ so that $\mathcal{M} \models R(a, b)$;*
(b) *for every two $p, q \in Z$ we have that $\mathcal{M} \not\models R(p, q)$.*

Proof of Claim. By the choice of R and S_i, for every $i \in I$ we have that

$$\mathcal{N} \models \forall \bar{z} \left(S_i(\bar{z}) \implies \exists x \notin \bar{z} \ \exists y \notin \bar{z} \ \ R(x, y) \right).$$

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Hence, by Claim 10, for every $\mathcal{M} \simeq_{\text{iso}} \mathcal{N}$ there exists a least number $n(\mathcal{M}) \geq \ell_s$ so that $\mathcal{M} \models R(a,b)$ for some a,b with $\ell_s \leq a,b < n(\mathcal{M})$. Since $\mathcal{M} \mapsto n(\mathcal{M})$ is a continuous map, by compactness of K we can find $\ell := \sup_K n(\mathcal{M})$ satisfying property (a) above.

We next build Z inductively, adding one point at a time, as follows. Assume that some fragment Y of Z has been defined so that for all $\mathcal{M} \in K$ property (b) above holds for Y in place of Z. By compactness of K and the fact that for each $\mathcal{M} \simeq_{\text{iso}} \mathcal{N}$, and every fixed $p,q \in \mathbb{N}$, there exist at most finitely many solutions to the formulas $\varphi(y) \equiv R(p,y)$ and $\psi(x) \equiv R(x,q)$, we may find $\ell' > \max(Y), \ell$, so that for every $\mathcal{M} \in K$ we have that:

(i) for every $p \in Y$ and every $q \geq \ell'$ we have that $\mathcal{M} \not\models R(p,q)$;
(ii) for every $q \in Y$ and every $p \geq \ell'$ we have that $\mathcal{M} \not\models R(p,q)$.

It follows that property (b) above holds for $Y' := Y \cup \{\ell'\}$ in place of Z. Since $|Y'| > |Y|$, by repeating this process enough times we define Z, with $|Z| = \ell \setminus \ell_s$, satisfying (b) above. ⊣

Let now ℓ and Z as in the last claim and set $\ell_{s^\frown 0}, \ell_{s^\frown 1} := \max(Z) + 1$. Let $\gamma_{s^\frown 0}$ be the permutation on the set $\ell_{s^\frown 0}$, with $\gamma_{s^\frown 0} \restriction \ell_s = \gamma_s$, which is identity on $\ell_{s^\frown 0} \setminus \ell_s$. Let also $\gamma_{s^\frown 1}$ be any permutation on the set $\ell_{s^\frown 1}$, with $\gamma_{s^\frown 1} \restriction \ell_s = \gamma_s$, which induces a bijection between the sets $\ell \setminus \ell_s$ and Z. We are left to show that (iii) above holds.

Indeed, let $g_0, g_1 \in \text{Sym}(\mathbb{N})$ with $g_0 \restriction \ell_{s^\frown 0} = \gamma_{s^\frown 0}$ and $g_1 \restriction \ell_{s^\frown 1} = \gamma_{s^\frown 1}$ and assume towards a contradiction that $g_0 K \cap g_1 K \neq \emptyset$. Set $g := g_1^{-1} g_0$ and fix some $\mathcal{M} \in K$ so that $g_* \mathcal{M} \in K$. Since $\mathcal{M} \in K$, by (a) above we get $a,b < \ell$ so that $\mathcal{M} \models R(a,b)$. Hence, $g_* \mathcal{M} \models R(ga, gb)$. But since $g_* \mathcal{M} \in K$, by (b) above and the fact that $ga, gb \in Z$ we have $g_* \mathcal{M} \not\models R(ga, gb)$; a contradiction. If follows that $g_0 K \cap g_1 K = \emptyset$ as desired.

Case (2). Assume that Lemma 9(2) holds for $\bar{a}, b$, and $\bar{c}$.

Claim 12. *There exist $\ell, \ell', \ell'' \in \mathbb{N}$ with $\ell'' > \ell' > \ell > \ell_s$ so that for all $\mathcal{M} \in K$ we have*:

(a) *there exists $b \in \mathbb{N}$, with $\ell_s \leq b < \ell$, so that $\mathcal{M} \models Q(\bar{b})$;*
(b) *for all $b < \ell$ with $\mathcal{M} \models Q(\bar{b})$, there is $\bar{a}$ in $\ell' \setminus \ell$ so that $\mathcal{M} \models R(\bar{a}, b)$;*
(c) *for all $\bar{a}$ in $\ell' \setminus \ell$ with $\mathcal{M} \models P(\bar{a})$, there exists no $b \geq \ell''$ so that $\mathcal{M} \models R(\bar{a}, b)$.*

Proof of Claim. By compactness of K we can find some $\ell > \ell_s$ so that for all $\mathcal{M} \in K$ property (a) holds. Since each $\mathcal{M}$ with $\mathcal{M} \simeq_{\text{iso}} \mathcal{N}$ is R-ultrahomogeneous and P, Q, R capture the pertinent quantifier-free types, for any $\mathcal{M} \in K$ and any fixed $b \in \mathbb{N}$ with $\mathcal{M} \models Q(b)$ there exists $\bar{a}$ so that $\mathcal{M} \models R(\bar{a}, b)$. In fact, since we are under the assumption that Lemma 9(2) holds, for every $\mathcal{M} \in K$ we can always find such $\bar{a}$ outside of $\{0, 1, \ldots, \ell - 1\}$. Hence by a second application of compactness of K we may find some $\ell' > \ell$ so that property (b) holds as well. By a third and final application of compactness of K, and since for each $\mathcal{M} \in K$ and every $\bar{a} \in \mathbb{N}$ there are only finitely many solutions to the formula $\varphi(y) \equiv R(\bar{a}, y) \wedge P(\bar{a})$, we may find $\ell'' > \ell'$ for which property (c) holds. ⊣

Let now ℓ, ℓ', ℓ'' as in the above claim and set $\ell_{s^\frown 0}, \ell_{s^\frown 1} := \ell'' + \ell$.

Let $\gamma_{s^\frown 0}$ be the permutation on the set $\ell_{s^\frown 0}$, with $\gamma_{s^\frown 0} \restriction \ell_s = \gamma_s$, which is identity on $\ell_{s^\frown 0} \setminus \ell_s$. Let also $\gamma_{s^\frown 1}$ be any permutation on the set $\ell_{s^\frown 1}$, with $\gamma_{s^\frown 1} \restriction \ell_s = \gamma_s$,

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which fixes every $k \in \{\ell, \dots, \ell'' - 1\}$, but which exchanges k and $k + \ell''$, for all k with $\ell_s \leq k < \ell$. We are left to show that (iii) above holds.

Let $g_0, g_1 \in \mathrm{Sym}(\mathbb{N})$ with $g_0 \restriction \ell_s{}^\frown 0 = \gamma_s{}^\frown 0$ and $g_1 \restriction \ell_s{}^\frown 1 = \gamma_s{}^\frown 1$ and assume towards a contradiction that $g_0 K \cap g_1 K \neq \emptyset$. Set $g := g_1^{-1} g_0$ and fix some $\mathcal{M} \in K$ so that $g_* \mathcal{M} \in K$. Since $\mathcal{M} \in K$, by (a) and (b) above we get b in $\ell \setminus \ell_s$ and $\bar{a}$ in $\ell' \setminus \ell$ so that $\mathcal{M} \models R(\bar{a}, b)$. Hence, $g_* \mathcal{M} \models R(g\bar{a}, gb)$, and since g fixes $\bar{a}$ we have that $g_* \mathcal{M} \models R(\bar{a}, gb)$. But then, by applying (c) to $g_* \mathcal{M} \in K$, we get a contradiction with the fact that $gb \geq \ell''$. ⊣

§4. Structures which are not highly algebraic are quasi-invariantly random. The following is the main result of this section.

THEOREM 13. *Let $\mathcal{M}$ be a $\mathcal{L}$-structure such that there is a finite $B \subset \mathbb{N}$ satisfying for all $\bar{a} \in \mathbb{N}$, $\mathrm{acl}(\bar{a}, \mathcal{M}) \subseteq \{\bar{a}\} \cup B$. Then there is a $\mathrm{Sym}(\mathbb{N})$-quasi-invariant probability measure on $\mathrm{Str}(\mathcal{L}, \mathbb{N})$ concentrated on structures isomorphic to $\mathcal{M}$.*

PROOF. We may assume without loss of generality that $\mathcal{L}$ only contains relation symbols. Indeed, if $\mathcal{L}$ contained function symbols, then we can always move a the language $\mathcal{L}'$ which contains all relational symbol of $\mathcal{L}$; and for each $f \in \mathcal{L}$ function symbol of arity r, $\mathcal{L}'$ contains a relation symbol R_f of arity $r + 1$. Similarly, from $\mathcal{M}$, move to the $\mathcal{L}'$-structure $\mathcal{M}'$, with: $\mathrm{dom}(\mathcal{M}) = \mathbb{N} = \mathrm{dom}(\mathcal{M}')$; relations from $\mathcal{L}$ are interpreted as in $\mathcal{M}$; and for each function symbol f from $\mathcal{L}$, set $R_f^{\mathcal{M}'}(\bar{x}, a) \Leftrightarrow f^{\mathcal{M}}(\bar{x}) = a$. Notice that $\mathcal{M}$ and $\mathcal{M}'$ have the same automorphism group (as permutation groups). In particular, $\mathcal{M}'$ is highly algebraic iff $\mathcal{M}$ is highly algebraic. Moreover this construction induces an S_∞-equivariant Borel isomorphism between $[\mathcal{M}]_{\mathrm{iso}} \subseteq \mathrm{Str}(\mathcal{L}, \mathbb{N})$ and $[\mathcal{M}']_{\mathrm{iso}} \subseteq \mathrm{Str}(\mathcal{L}', \mathbb{N})$, the existence of a quasi-invariant probability measure on $[\mathcal{M}']_{\mathrm{iso}}$ implies the same for $[\mathcal{M}]_{\mathrm{iso}}$.

Let now B as in the statement of Theorem 13 (for instance we can take B to be the algebraic closure of the empty set). We view the set B as enumerated by $\bar{b} = (b_0, \dots, b_{l-1}) \in \mathbb{N}^\ell$, with $|B| = \ell$, and we fix some new "atom" $z \notin \mathbb{N}$.

For each $\bar{c} \in \mathbb{N}^\ell$ we consider a new language $\mathcal{L}_{\bar{c}}$. This language contains, for each $R \in \mathcal{L}$ of arity n and each $f : \{0, \dots, n-1\} \to \{\bar{c}\} \cup \{z\}$, a symbol $R_f^{\bar{c}}$ of arity $|f^{-1}(z)|$. In addition, given any tuple $\bar{a} \in (\mathbb{N} \setminus \{\bar{c}\})^k$ and any map $f : \{0, \dots, n-1\} \to \{\bar{c}\} \cup \{z\}$ such that $|f^{-1}(z)| = k$, we define $\bar{a}_f \in \mathbb{N}^n$, as the unique tuple such that:

i) for each $i \in n$, with $f(i) \in \{\bar{c}\}$, the i-th coordinate of $\bar{a}_f$ is $f(i)$;
ii) removing all entries of $\bar{a}_f$ that are in $\{\bar{c}\}$ gives $\bar{a}$.

For any $\mathcal{N} \in \mathrm{Str}(\mathcal{L}, \mathbb{N})$ and any fixed $\bar{c} \in \mathbb{N}^\ell$, we define $\mathcal{N}_{\bar{c}} \in \mathrm{Str}(\mathcal{L}_{\bar{c}}, \mathbb{N} \setminus \{\bar{c}\})$ by setting

$$\mathcal{N}_{\bar{c}} \models R_f^{\bar{c}}(\bar{a}) \iff \mathcal{N} \models R(\bar{a}_f),$$

for every $R_f^{\bar{c}} \in \mathcal{L}_{\bar{c}}$. Notice that there is a natural isomorphism of the permutation groups $\mathrm{Aut}(\mathcal{N}_{\bar{c}})$ and $\mathrm{Aut}(\mathcal{N})_{\bar{c}}$, defined by extending each $g \in \mathrm{Aut}(\mathcal{N}_{\bar{c}})$ to the identity on $\{\bar{c}\}$.

CLAIM 14. *The $\mathcal{L}_{\bar{b}}$-structure $\mathcal{M}_{\bar{b}}$ has no algebraicity.*

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PROOF OF CLAIM. Assume that $d \in \mathbb{N} \setminus \{\bar{b}\}$ has finite orbit under $\mathrm{Aut}(\mathcal{M}_{\bar{b}})_{\bar{a}}$. Then d also has finite orbit under $\mathrm{Aut}(\mathcal{M})_{\bar{a}\bar{b}}$, since $\mathrm{Aut}(\mathcal{M}_{\bar{b}})_{\bar{a}}$ and $\mathrm{Aut}(\mathcal{M})_{\bar{a}\bar{b}}$ are the same permutation groups on $\mathbb{N} \setminus \{\bar{b}\}$. Therefore d must be in $\{\bar{a}\} \cup \{\bar{b}\}$. Since $d \notin \{\bar{b}\}$, we have $d \in \{\bar{a}\}$. ⊣

Let now W be the collection of all pairs $(\bar{c}, \mathcal{N})$, where $\bar{c} \in \mathbb{N}^{\ell}$ and $\mathcal{N} \in \mathrm{Str}(\mathcal{L}_{\bar{c}}, \mathbb{N} \setminus \{\bar{c}\})$. We define an analog of the logic action of $\mathrm{Sym}(\mathbb{N})$ on W as follows: for $g \in \mathrm{Sym}(\mathbb{N})$ and $(\bar{c}, \mathcal{N}) \in W$ we set $g \cdot (\bar{c}, \mathcal{N}) := (g(\bar{c}), g_*\mathcal{N})$, where $g_*\mathcal{N} \in \mathrm{Str}(\mathcal{L}_{g(\bar{c})}, \mathbb{N} \setminus \{g(\bar{c})\})$ is defined by naturally extending the formulas from Section 2.2. That is, by setting, for all $R^{\bar{c}}_f \in \mathcal{L}_{\bar{c}}$,

$$g_*\mathcal{N} \models R^{g(\bar{c})}_{g \circ f}(\bar{a}) \iff \mathcal{N} \models R^{\bar{c}}_f(g^{-1}(\bar{a})).$$

Let Y be the $\mathrm{Sym}(\mathbb{N})$-orbit of $(\bar{b}, \mathcal{M}_{\bar{b}})$ in W and consider the map $\beta \colon Y \to \mathrm{Str}(\mathcal{L}, \mathbb{N})$ which is defined as follows: for all $(\bar{c}, \mathcal{N}) \in Y$ and $R \in \mathcal{L}$, $\beta(\bar{c}, \mathcal{N}) \models R(\bar{a})$ holds if and only if $\mathcal{N} \models R^{\bar{c}}_f(\bar{e})$ holds, for the unique $f : n \to \{\bar{c}\} \cup \{z\}$ and $\bar{e} \in (\mathbb{N} \setminus \{\bar{c}\})^k$ such that $\bar{e}_f = \bar{a}$.

CLAIM 15. *β is S_∞-equivariant.*

PROOF OF CLAIM. Let $g \in \mathrm{Sym}(\mathbb{N})$ and $(\bar{c}, \mathcal{N}) \in Y$, we need to prove that

$$\beta(g(\bar{c}), g_*\mathcal{N}) = g_*\beta(\bar{c}, \mathcal{N}).$$

To see this, let $R \in \mathcal{L}$ and $\bar{a}$ in $\mathbb{N}$. We have:

$$\begin{aligned}
\beta(g(\bar{c}), g_*\mathcal{N}) \models R(\bar{a}) \;&\Leftrightarrow\; g_*\mathcal{N} \models R^{g(\bar{c})}_f(\bar{e}), \text{for the unique } f : n \to \{g(\bar{c})\} \cup \{z\} \\
&\quad \text{and } \bar{e} \in (\mathbb{N} \setminus \{g(\bar{c})\})^k \text{ such that } \bar{e}_f = \bar{a}. \\
&\Leftrightarrow\; \mathcal{N} \models R^{\bar{c}}_{g^{-1} \circ f}(g^{-1}(\bar{e})), \text{for the unique } f : n \to \{g(\bar{c})\} \cup \{z\} \\
&\quad \text{and } \bar{e} \in (\mathbb{N} \setminus \{g(\bar{c})\})^k \text{ such that } \bar{e}_f = \bar{a}. \\
&\Leftrightarrow\; \mathcal{N} \models R^{\bar{c}}_h(\bar{d}), \text{for the unique } h : n \to \{\bar{c}\} \cup \{z\} \\
&\quad \text{and } \bar{d} \in (\mathbb{N} \setminus \{\bar{c}\})^k \text{ such that } g(\bar{d}_h) = \bar{a}, \\
&\quad \text{namely, for } h := g^{-1} \circ f \text{ and } \bar{d} := g^{-1}(\bar{e}). \\
&\Leftrightarrow\; \mathcal{N} \models R^{\bar{c}}_f(\bar{e}), \text{for the unique } f : n \to \{\bar{c}\} \cup \{z\} \\
&\quad \text{and } \bar{e} \in (\mathbb{N} \setminus \{\bar{c}\})^k \text{ such that } \bar{e}_f = g^{-1}(\bar{a}). \\
&\Leftrightarrow\; \beta(\bar{c}, \mathcal{N}) \models R(g^{-1}(\bar{a})). \\
&\Leftrightarrow\; g_*\beta(\bar{c}, \mathcal{N}) \models R(\bar{a}).
\end{aligned}$$

⊣

CLAIM 16. *$\beta(Y) = [\mathcal{M}]_{\mathrm{iso}}$.*

PROOF. This follows from the previous claim, since $\mathcal{M} = \beta(\mathcal{M}_{\bar{b}}, \bar{b})$. ⊣

CLAIM 17. *Y is a standard Borel space and β is a Borel map.*

PROOF. Notice first that W is a Polish space as it is the countable disjoint union of the Polish spaces $\mathrm{Str}(\mathcal{L}_{\bar{c}}, \mathbb{N} \setminus \{\bar{c}\})$, where the union ranges over all $\bar{c} \in \mathbb{N}^{\ell}$. Since Y

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is a single orbit of the continuous action of $\mathrm{Sym}(\mathbb{N})$ on W we have that Y is a Borel subset of W, and hence, a standard Borel space. It is straightforward to check that the map β is, in fact, continuous, as whether $\beta(\bar{c},\mathcal{N}) \models R(\bar{a})$ holds depends only on whether $\mathcal{N} \models R_f^{\bar{c}}(\bar{e})$ holds, for a single relation $R_f^{\bar{c}}$ and a single tuple $\bar{e}$. ⊣

Claim 18. *There is a measure λ on Y that is* $\mathrm{Sym}(\mathbb{N})$*-quasi-invariant.*

Proof. First, using [5, Theorem 1.1] and Claim 14, we may fix some $\mathrm{Sym}(\mathbb{N}\setminus\{\bar{b}\})$-invariant measure $\mu_{\bar{b}}$ on $\mathrm{Str}(\mathcal{L}_{\bar{b}},\mathbb{N}\setminus\{\bar{b}\})$, which concentrates on the orbit of $\mathcal{M}_{\bar{b}}$. We now define a measure on $\mathrm{Str}(\mathcal{L}_{\bar{c}},\mathbb{N}\setminus\{\bar{c}\})$ by taking any element g of $\mathrm{Sym}(\mathbb{N})$ sending $\bar{b}$ to $\bar{c}$ and setting $\mu_{\bar{c}} = g_*\mu_{\bar{b}}$. The obtained measure is $\mathrm{Sym}(\mathbb{N}\setminus\{\bar{c}\})$-invariant and does not depend on the choice of g by $\mathrm{Sym}(\mathbb{N}\setminus\{\bar{b}\})$-invariance of $\mu_{\bar{b}}$. Denote by $\delta_{\bar{c}}$ the measure on $\mathbb{N}^\ell$ such that for $A\subseteq\mathbb{N}^\ell$, $\delta_{\bar{c}}(A)=1$ if $\bar{c}\in A$ and 0 otherwise. Now take any fully supported measure ν on the countable set $\mathbb{N}^\ell$ and integrate $\delta_{\bar{c}}\otimes\mu_{\bar{c}}$ along ν, since $\mathbb{N}^\ell$ is countable this is always well-defined. We denote by λ the obtained measure on Y, as for every $\bar{c}$, $\delta_{\bar{c}}\otimes\mu_{\bar{c}}$ is a measure on Y.

To see that λ is $\mathrm{Sym}(\mathbb{N})$-quasi-invariant, let $A\subseteq Y$ with $\lambda(A)>0$, and pick $\bar{d}\in\mathbb{N}^\ell$ with

$$\lambda\big(\{(\bar{c},\mathcal{N})\in A\colon \bar{c}=\bar{d}\}\big)>0.$$

This implies that

$$\mu_{\bar{d}}\big(\{\mathcal{N}\in\mathrm{Str}(\mathcal{L}_{\bar{d}},\mathbb{N}\setminus\{\bar{d}\})\colon (\bar{d},\mathcal{N})\in A\}\big)>0.$$

Let now $g\in\mathrm{Sym}(\mathbb{N})$ and notice that, by the definition $\mu_{\bar{c}}$, we have $\mu_{g(\bar{d})}=g_*\mu_{\bar{d}}$. Hence,

$$\mu_{g(\bar{d})}\big(\{g_*\mathcal{N}\in\mathrm{Str}(\mathcal{L}_{g(\bar{d})},\mathbb{N}\setminus\{g(\bar{d})\})\colon (\bar{d},\mathcal{N})\in A\}\big)>0.$$

Since ν is fully supported, we have that $\lambda(g(A))\geq\lambda\big(\{(\bar{c},\mathcal{N})\in g(A)\colon \bar{c}=g(\bar{d})\}\big)>0$. ⊣

To conclude with the proof of Theorem 13, we simply push forward λ via β. By Claims 16 and 17 we have an $\mathrm{Sym}(\mathbb{N})$-quasi-invariant measure on $[\mathcal{M}]_{\mathrm{iso}}$ which can be normalized to a probability measure without losing $\mathrm{Sym}(\mathbb{N})$-quasi-invariance. ⊣

We would like to thanks the anonymous referee for suggesting the following alternative argument for establishing Theorem 13, which is based on [4, Proposition 6.14].

Remark 19. Recall from the discussion in the introduction, following Theorem 2, that the set B in the statement of Theorem 13 can be taken to be the algebraic closure $\mathrm{acl}(\emptyset,\mathcal{M})$ of the empty set. Let $\mathcal{L}':=\mathcal{L}\cup\{U\}$, where U is a new unary predicate and set $\mathcal{M}'\in\mathrm{Str}(\mathcal{L}',\mathbb{N})$ be the expansion of $\mathcal{M}$, where U holds only for the elements of B. For each $F\subseteq\mathbb{N}$ with $|F|=|B|$, let $\mathcal{N}_F$ be the $\{U\}$-structure on $\mathbb{N}$, where U holds exactly on F. By [4, Proposition 6.14], for each such F we obtain an $\mathrm{Aut}(\mathcal{N}_F)$-invariant probability measure μ_F on $\mathrm{Str}(\mathcal{L}',\mathbb{N})$ which is supported on the set of those $\mathcal{L}'$-structures which are isomorphic to $\mathcal{M}'$ and whose $\{U\}$-reduct is $\mathcal{N}_F$. Then, the desired $\mathrm{Sym}(\mathbb{N})$-quasi-invariant measure λ on $[\mathcal{M}]_{\mathrm{iso}}$ is just the ν-mixture of the

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μ_F-measures, where ν is any probability measure on the set of $|B|$-sized subsets of natural numbers with full support.

§5. Examples. We start by providing two examples of simple, highly algebraic structures. Each example realizes the respective side of the dichotomy that we established in Lemma 9.

EXAMPLE 20. Let $\mathcal{L} := \{R\}$, where R is a binary relation, and let $\mathcal{N}$ be a graph that comprises of the disjoint union of countably infinite many edges. For example, set $\mathcal{N} = (\mathbb{N}, R^{\mathcal{N}})$, where for all $n, m \in \mathbb{N}$ we have that $(n, m) \in R^{\mathcal{N}}$ if and only if there is $k \in \mathbb{N}$ so that $\{n, m\} = \{2k, 2k+1\}$. Clearly $\mathcal{N}$ is highly algebraic and satisfies (1) of Lemma 9.

EXAMPLE 21. Let $\mathcal{L} := \{R\}$, where R is a binary relation, and let $\mathcal{N}$ be a graph that comprises of the disjoint union of countably many copies of a bipartite graph whose one part is countably infinite and the other is a singleton. For example, set $\mathcal{N} = (\mathbb{N}^2, R^{\mathcal{N}})$, where $\big((n_0, n_1), (m_0, m_1)\big) \in R^{\mathcal{N}}$ if and only if $m_0 = n_0$ and either of the following two hold: $n_1 = 0$ and $m_1 > 0$; or $m_1 = 0$ and $n_1 > 0$. Clearly $\mathcal{N}$ is highly algebraic and (its R-homogeneous expansion from Lemma 4) satisfies (2) of Lemma 9.

We close with some examples of structures which have algebraicity but are not highly algebraic. In view of [5] and Theorem 2, these are quasi-invariantly random structures which are not invariantly random.

EXAMPLE 22 (Finite inclusions). Let $\mathcal{N}$ be any ultrahomogeneous structure without algebraicity and let F be a finite substructure of it. Then the structure $(\mathcal{N}, U_F)$, which endows $\mathcal{N}$ with a unary predicate interpreted on F, is not highly algebraic and hence, by Theorem 2, is quasi-invariantly random.

EXAMPLE 23 (Generic finite quotients). Let $\mathcal{N}$ be any ultrahomogeneous structure without algebraicity and let $\pi\colon \mathrm{dom}(N) \to B$ be a generic function with B being a finite set. Consider the structure $\mathcal{N}_B$ given by setting $\mathrm{dom}(\mathcal{N}_B) := \mathrm{dom}(\mathcal{N}) \cup B$ and adding the map π, on top of the existing on $\mathcal{N}$, as the interpretation of a new function symbol. It follows that $\mathcal{N}_B$ is not highly algebraic and hence, by Theorem 2, quasi-invariantly random.

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