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An algorithm for the construction of 3-manifolds from 2-complexes

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Introduction. In order to study 3-manifolds with some particular group as fundamental group, one likes to have examples. There is an absence in the literature of means of constructing such examples. In this paper, theorems are proved which give necessary and sufficient conditions for the canonical 2-complex which corresponds to a group presentation to be a spine (2-dimensional skeleton in a cell decomposition with one 3-cell) of a connected closed† orientable 3-manifold. By enumerating all presentations of a group given by a particular presentation, two of the theorems provide an effective algorithm which permits one to attempt in a systematic way to construct 3-manifolds with a given group as fundamental group. Since every closed 3-manifold has a spine which corresponds to a group presentation, if the group is the fundamental group of an orientable 3-manifold, then the algorithm will eventually yield a 3-manifold, if not, then one is out of luck. There is no way out of this since Stallings has shown (1) that *no* algorithm exists which can, for every finitely presented group G, answer the question: Is G the fundamental group of a 3-manifold?

The algorithm is effective in two other senses as well. In the first place, it is quite easy to apply. In the second place, it allows the construction of *every* orientable closed 3-manifold with a given 2-complex (corresponding to a group presentation) as spine.

In the final section some 3-manifolds are described which have been discovered by these methods. They appear not to be previously noted.

1. *Notation and conventions.* In studying cell-decompositions of a connected, closed, orientable 3-manifold M, we can restrict our attention, without loss of generality, to decompositions with one 0-cell, one 3-cell, equal numbers of 1 and 2-cells and 2-skeleton corresponding to a group presentation (see e.g. (2, 3), or (5)). Thus a 2-skeleton or spine has 1-vertex, n 1-cells and n 2-cells. We assume now that such a spinal 2-complex, which we denote by $\mathfrak{C}$, is given.

In order to avoid annoying technical complications we will henceforth also assume that every 1-cell of $\mathfrak{C}$ is incident with a 2-cell, and that every 2-cell is incident with a 1-cell. This is no loss of generality since it is easily seen that every 3-manifold has a cell decomposition with such a 2-complex as a 2-skeleton.

Corresponding to $\mathfrak{C}$ there is a group presentation $\mathfrak{P}$, presenting $\pi_1(\mathfrak{C})$ (and of course $\pi_1(M)$) with n generators in 1-1 correspondence with the 1-cells of $\mathfrak{C}$ and n relators in

† A manifold is called closed when it is compact and boundaryless.

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1-1 correspondence with the 2-cells. Conversely, given a group presentation $\mathfrak{P}$, we may construct in a canonical way (4), a 2-complex $\mathfrak{C}$ with a single vertex.

We shall determine certain properties satisfied by $\mathfrak{C}$ (and so $\mathfrak{P}$), and we shall show that these properties are sufficiently strong to imply the existence of M if $\mathfrak{C}$ alone is given.

2. *Main theorems.* Consider the spine $\mathfrak{C}$ of our generic, closed orientable 3-manifold M. Let N be a thickening of $\mathfrak{C}$ in M, i.e. a regular neighbourhood in some triangulation. Suppose $\mathfrak{C}$ has the single vertex, o, and as before that every 1-cell of $\mathfrak{C}$ is incident with a 2-cell of $\mathfrak{C}$, and conversely. Denote the 1-cells by $a_1, a_2, \ldots, a_n$, and the 2-cells by $d_1, d_2, \ldots, d_n$. The following paragraphs may be clarified by occasional reference to Figure 1.

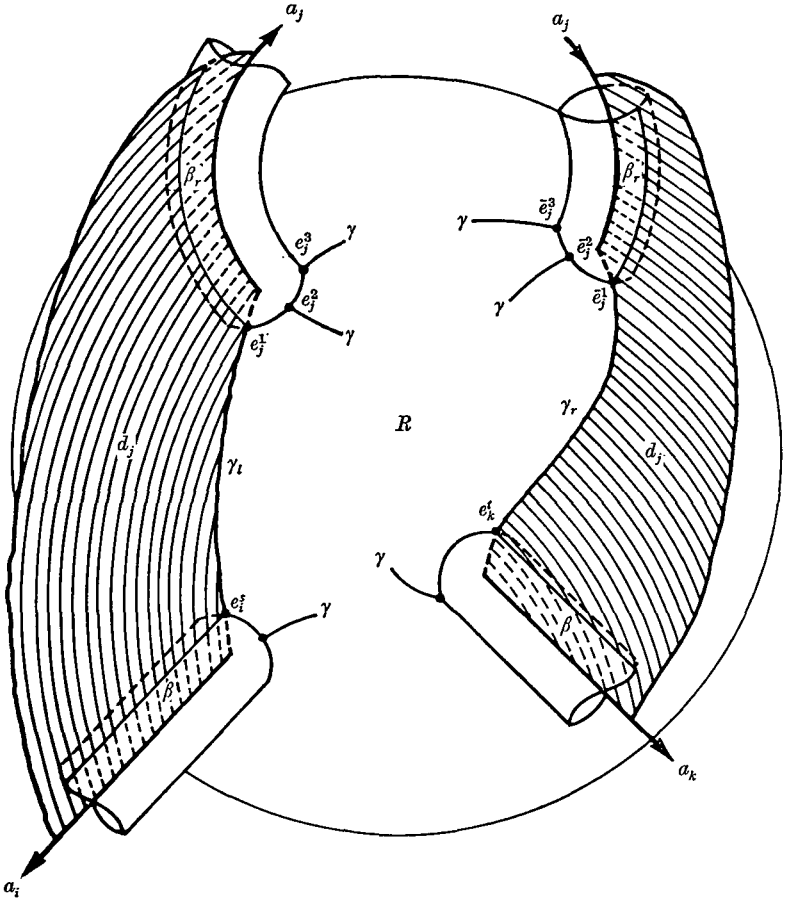


Fig. 1. $A = \ldots (e_i^s e_j^1)\,(\bar{e}_j^1 e_k^t)\ldots$; $\partial d_j = \ldots a_i^{-1} a_j a_k \ldots$.

Let $R \supset o$ be a regular neighbourhood of o so that $N \supset R$. Let $\alpha_i, \bar{\alpha}_i$ denote the points $a_i \cap \partial R$. Let $D_i, \bar{D}_i$ denote regular neighbourhoods of $\alpha_i, \bar{\alpha}_i$ on the 2-sphere ∂R. Orient D_i and $\bar{D}_i$ in opposite manners on ∂R and denote the points $\bigcup_j d_j \cap \partial D_i$ by $e_i^1, e_i^2, \ldots, e_i^{r(i)}$ and the points $\bigcup_j d_j \cap \partial \bar{D}_i$ by $\bar{e}_i^1, \ldots, \bar{e}_i^{r(i)}$ where these orderings are consistent

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with the orientations of $\partial D_i, \partial \bar{D}_i$ respectively. These e_i^j, $\bar{e}_i^j$ are the intersection of the boundary of a regular neighbourhood in M of the 1-skeleton of $\mathfrak{C}$, the boundary of a regular neighbourhood of the vertex of $\mathfrak{C}$ and the 2-skeleton of $\mathfrak{C}$. Since M is orientable a regular neighbourhood of any a_i is a solid torus T_i of genus 1. Notice

$$[\partial T_i - R] \cap [\bigcup_j d_j]$$

is a set of simple arcs, β_s, each joining a pair of points e_i^j and $\bar{e}_i^k$. Let us assume, without loss of generality, that we have ordered the $\bar{e}_i^k$ so that $j = k$ for each such arc. Notice that the intersection of the set of all d_j with the complement in ∂R of all cells D_i and $\bar{D}_i$ is also a set of simple arcs, γ_h, each joining a pair of points e_i^j, and e_i^l, either or both of which may be barred. We now orient each a_i so that the pairs (D_i, a_i), $(\bar{D}_i, a_i)$ are consistent with the orientation of M. This entire state of affairs is shown in Figure 1.

Let E denote the set consisting of all points e_i^j and $\bar{e}_i^j$.

Let A denote the permutation of E defined by interchanging the endpoints of the arcs γ_h.

Let B denote the permutation of E which interchanges e_i^j and $\bar{e}_i^j$, the endpoints of each arc β_s.

Let C denote the permutation of E defined by

$$C = \prod_{i=1}^{n} (e_i^1 e_i^2 \dots e_i^{r(i)}) (\bar{e}_i^{r(i)} \dots \bar{e}_i^2 \bar{e}_i^1).$$

If X is a permutation of E then denote by $|X|$ the number of cycles (transitivity sets) of X.

Recall that n is the number of 1-cells of $\mathfrak{C}$.

We shall prove:

THEOREM 1. *Suppose the group of permutations generated by A and C gives rise to L orbits (transitivity sets) in E. If $L = 1$, then*

(i) *The permutation group generated by AC and CB is transitive, and*

(ii) $|A| - |C| + 2 = |AC|$.

If $L > 1$, then the situation is more complicated, and we shall defer to a later section consideration of this case.

Suppose, on the other hand, a 2-complex $\mathfrak{C}'$ with one vertex, o, and equinumerous 1- and 2-cells is given corresponding to a group presentation, and we wish to know if $\mathfrak{C}'$ is the spine of a 3-manifold.

Note first the set E can be defined independently of an embedding of $\mathfrak{C}$ in a 3-manifold as the intersection of the boundary of a neighbourhood of the 1-skeleton of $\mathfrak{C}'$ with the boundary of a neighbourhood of the vertex of $\mathfrak{C}'$. Furthermore, the possibilities for the permutations A, B, and C can be enumerated, independently of an embedding of $\mathfrak{C}'$ in a 3-manifold, as follows. As before, we denote the 1-cells and 2-cells of $\mathfrak{C}'$ by $a_1 \dots a_n$, and $d_1 \dots d_n$ respectively.

Let us denote by t a regular neighbourhood on $\mathfrak{C}'$ of the 1-skeleton of $\mathfrak{C}'$, and by b a regular neighbourhood on $\mathfrak{C}'$ of o.

The set E consists of the points $\partial t \cap \partial b$. These points we denote by e_i^j, $\bar{e}_i^j$, where $e_i^j, \bar{e}_i^j$ lie in a regular neighbourhood of a_i, and are joined by an arc in $\partial t - b$.

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The *numbering* of these points induces the permutation of them defined by

$$C = \prod_{i=1}^{n} (e_i^1 e_i^2 \dots e_i^{r(i)}) (\bar{e}_i^{r(i)} \dots \bar{e}_i^2 \bar{e}_i^1).$$

A simple curve near each ∂d_i intersects b in a set of simple arcs with endpoints members of the set E; interchanging the endpoints of those arcs defines the fixed point free involution A of E, while the permutation

$$B = \prod_{i=1}^{n} \prod_{j=1}^{r(i)} (e_i^j \bar{e}_i^j).$$

We may now check the necessary conditions of Theorem 1 to see if $\mathfrak{C}'$ is a candidate for a 3-manifold spine. In fact we can do more, since we can prove:

THEOREM 2. *Suppose the group of permutations generated by A and C gives rise to L transitivity sets (orbits) in E. If $L = 1$ and if,*

(i) *The group of permutations generated by AC and BC is transitive, and if,*

(ii) $|A| - |C| + 2 = |AC|$ *then there exists an orientable 3-manifold M', with spine $\mathfrak{C}'$ so that the points of intersection of the boundary of a regular neighbourhood of the 1-skeleton, the boundary of a regular neighbourhood of the vertex of $\mathfrak{C}'$, and the 2-skeleton of $\mathfrak{C}'$ are in 1-1 correspondence with the elements of E. Furthermore, the permutations A, B, C of these points are just those described in Theorem 1.*

Again, we must defer to a later section consideration of the case $L > 1$.

These theorems may be clarified by means of an example.

Consider (Figure 2) the 2-complex $\mathfrak{C}$ corresponding to the presentation

$$(a_1 a_2;\ a_1^3,\ a_1 a_2 a_1^{-1} a_2^{-1}).$$

The oriented 1-cells of $\mathfrak{C}$ we denote a_1, a_2, and the 2-cells d_1, d_2. Then ∂d_1 is attached by the word $a_1 a_1 a_1$, and ∂d_2 by $a_1 a_2 a_1^{-1} a_2^{-1}$.

The set E then consists of 14 elements, 2 for each occurrence of a generator in a relator. Suppose we denote these elements $e_1^1, e_1^2, e_1^3, e_1^4, e_1^5, \bar{e}_1^1, \bar{e}_1^2, \bar{e}_1^3, \bar{e}_1^4, \bar{e}_1^5, e_2^1, e_2^2, \bar{e}_2^1, \bar{e}_2^2$, assume this numbering chosen so that an appropriate closed curve parallel to and near ∂d_1 intersects $e_1^1, \bar{e}_1^1, e_1^2, \bar{e}_1^2, e_1^3, \bar{e}_1^3$ in that order, while a curve near ∂d_2 intersects $e_1^4, \bar{e}_1^4, e_2^1, \bar{e}_2^1, \bar{e}_1^5, e_1^5, \bar{e}_2^2, e_2^2$ in that order. Then

$$A = (\bar{e}_1^1 e_1^2)(\bar{e}_1^2 e_1^3)(\bar{e}_1^3 e_1^1)(\bar{e}_1^4 e_2^1)(\bar{e}_2^1 \bar{e}_1^5)(e_1^5 \bar{e}_2^2)(e_2^2 e_1^4),$$

$$B = (e_1^1 \bar{e}_1^1)(e_1^2 \bar{e}_1^2)(e_1^3 \bar{e}_1^3)(e_1^4 \bar{e}_1^4)(e_2^1 \bar{e}_2^1)(\bar{e}_1^5 e_1^5)(\bar{e}_2^2 e_2^2),$$

$$C = (e_1^1 e_1^2 e_1^3 e_1^4 e_1^5)(\bar{e}_1^5 \bar{e}_1^4 \bar{e}_1^3 \bar{e}_1^2 \bar{e}_1^1)(e_2^1 e_2^2)(\bar{e}_2^2 \bar{e}_2^1),$$

$$AC = (e_1^1 \bar{e}_1^2 e_1^4 e_2^1 \bar{e}_1^3 e_1^2 \bar{e}_1^5 \bar{e}_2^2)(\bar{e}_1^1 e_1^3)(\bar{e}_1^4 e_2^2 e_1^5 \bar{e}_2^1)$$

so

$$|AC| = 3,$$

$$|A| = 7,$$

$$|C| = 4.$$

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A and C generate a transitive group and

$$|A| - |C| + 2 = 7 - 4 + 2 \neq 3 = |AC|.$$

Thus we conclude that this numbering is inconsistent with $\mathfrak{C}$ a spine of an orientable 3-manifold. Other orderings may be tried, for example:
We may renumber the points e_1^j, and the points e_2^j. Then B and C will be unchanged but suppose our numbering is such that a curve near ∂d_1 intersects points $e_i^j, \bar{e}_i^j$ in the

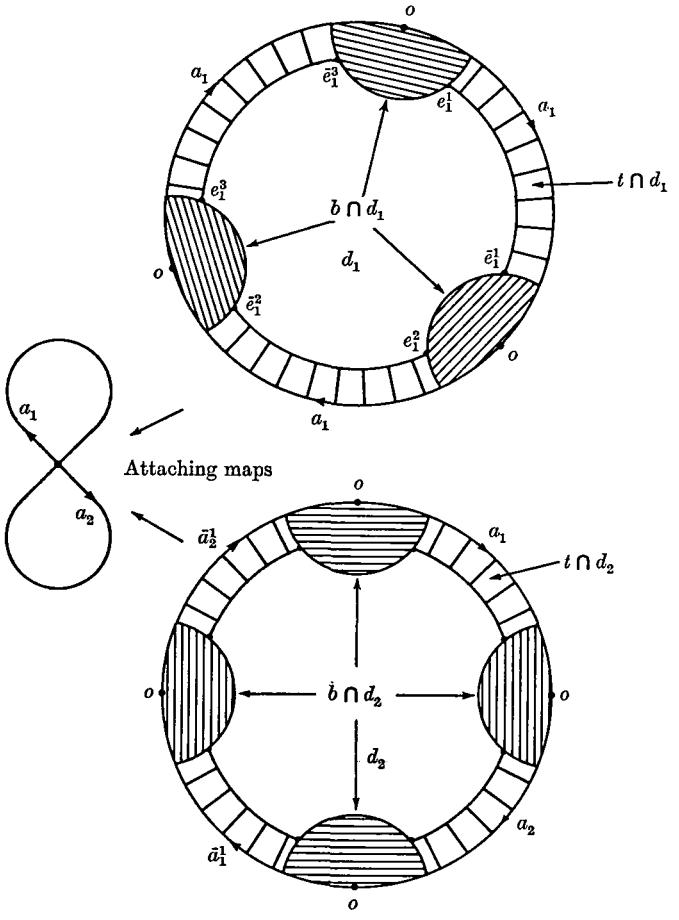


Fig. 2

order $e_1^4, \bar{e}_1^4, e_1^1, \bar{e}_1^1, e_1^3, \bar{e}_1^3$ and a curve near ∂d_2 intersects the points $e_1^2, \bar{e}_1^2, e_2^2, \bar{e}_2^2, \bar{e}_1^5, e_1^5, \bar{e}_2^1, e_2^1$ in that order. Then

$$A = (\bar{e}_1^4 e_1^1)(\bar{e}_1^1 e_1^3)(\bar{e}_1^3 e_1^4)(\bar{e}_1^2 e_2^2)(\bar{e}_2^2 \bar{e}_1^5)(e_1^5 \bar{e}_2^1)(e_1^1 e_1^2)$$

and

$$AC = (e_1^1 \bar{e}_1^3 e_1^5 \bar{e}_2^2 \bar{e}_1^4 e_1^2 e_2^2 \bar{e}_1^1 e_1^4 \bar{e}_1^2 e_2^1 e_1^3 \bar{e}_1^5 \bar{e}_2^1)$$

and A and C generate a transitive group; but,

$$|A| - |C| + 2 = 7 - 4 + 2 \neq |AC| = 1$$

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so that again this ordering is inconsistent with $\mathfrak{C}$ a 3-manifold spine. (In fact, one may check that no ordering will be consistent.)

From these computations it is clear that given a 2-complex $\mathfrak{C}'$ with the vertex, o, one may associate with $\mathfrak{C}'$ a set E, and permutations A, B, C in only a finite number of ways so that E, A, B, C can possibly correspond to E, A, B, C in Theorem 1. Specifically, if $\mathfrak{C}'$ is to be a spine E must correspond in a 1-1 manner to the points of intersection of the boundary of a regular neighbourhood in $\mathfrak{C}'$; of o with the boundary of a regular neighbourhood in $\mathfrak{C}'$ of the 1-skeleton of $\mathfrak{C}'$; B is then determined as is A, while C is fixed by a numbering of these points. While this was demonstrated in the preceding example in showing how a 2-complex is not a spine, we now give some examples in a positive direction.

Suppose $\mathfrak{C}'$ is given to us as the 2-complex corresponding to the group presentation

$$\mathfrak{P} = (x_1, x_2, x_3;\ x_1^{-1}x_2x_3,\ x_1x_2^{-1}x_3,\ x_1x_2x_3^{-1}).$$

Then we take as our set

$$E = \{e_1^1, e_2^1, e_3^1, e_1^2, e_2^2, e_3^2, e_1^3, e_2^3, e_3^3, \bar{e}_1^1, \bar{e}_2^1, \bar{e}_3^1, \bar{e}_1^2, \bar{e}_2^2, \bar{e}_3^2, \bar{e}_1^3, \bar{e}_2^3, \bar{e}_3^3\},$$

which is more conveniently dealt with by identifying this set with

$$E = \{1, 2, 3, 4, 5, 6, 7, 8, 9,$$
$$\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}\}.$$

Then
$$B = (1\bar{1})\,(2\bar{2})\,(3\bar{3})\,(4\bar{4})\,(5\bar{5})\,(6\bar{6})\,(7\bar{7})\,(8\bar{8})\,(9\bar{9}),$$

$$A = (1\bar{2})\,(23)\,(\bar{3}\bar{1})\,(\bar{4}\bar{5})\,(56)\,(\bar{6}4)\,(\bar{7}8)\,(\bar{8}\bar{9})\,(97),$$

$$C = (147)\,(258)\,(359)\,(\bar{7}\bar{4}\bar{1})\,(\bar{8}\bar{5}\bar{2})\,(\bar{9}\bar{6}\bar{3})$$

and
$$AC = (159)\,(24\bar{3}\bar{7})\,(3\bar{8}6\bar{7})\,(68\bar{4}\bar{2})\,(\bar{1}\bar{9}\bar{5})$$

it is easily checked that A and C generate a transitive group; hence

$$|A| - |C| + 2 = |AC|.$$

Furthermore, AC and BC generate a transitive group. Hence by Theorem 2, $\mathfrak{P}$ presents the fundamental group of a closed orientable 3-manifold with $\mathfrak{C}'$ as a spine. We remark that $\mathfrak{P}$ presents the quaternion group.

3. *Applications.* An application which may be pursued with the aid of a computer is the following. One can effectively enumerate all presentations with equal numbers of generators and relations, and to each of these one may apply Theorem 2, or 2'† after selecting a cyclic ordering of the occurrences of each generator in the set of relators (and an ordering of orbits and selection of pairs as required for Theorem 2'†). By applying the theorems to all possible cyclic orderings (and orbit orderings and pair selections) we may determine which presentations correspond to spines of 3-manifolds.

† Theorem 2' appears in section 5 and deals with the case $L \neq 1$.

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In this manner, we may start a catalogue *with possible repetitions* of all orientable closed 3-manifolds.†

During the course of trying these methods on various presentations, I stumbled across a countable family of 3-manifolds, M_n, which seem to be unnoted in the literature. Each of these manifolds has a spine which corresponds to a presentation of the following form:

$$\pi_1(M_n) \approx (x_1, x_2, \ldots, x_n; x_1^{-1}x_2 \ldots x_n, x_1x_2^{-1} \ldots x_n, \ldots, x_1x_2 \ldots x_n^{-1}).$$

For $n = 3$, the previous section described appropriate calculations. For other values of n the necessary computations are left to the diligent reader.

Notice that these groups may also be presented as follows:

$$(x_1, x_2, \ldots, x_n; x_1^2 = x_2^2 = \ldots = x_n^2, x_1x_2 \ldots x_{n-1} = x_n).$$

One proves without difficulty that

$$\underbrace{H_1(M_n) \approx Z_2 \oplus \ldots \oplus Z_2 \oplus Z_{2(n-2)}}_{n-1}.$$

so that the Heegard genus of $M_n \geqslant n$.

4. *Proofs. Proof of Theorem* 1. By hypothesis, in the spine $\mathfrak{C}$ every 2-cell is incident with a 1-cell, and every 1-cell is incident with a 2-cell. Let E, B, A, be defined by the presentation corresponding to $\mathfrak{C}$, and C by the imbedding of $\mathfrak{C}$, and an orientation of M.

(1) We assert that the number of transitivity sets of the group generated by A and C is equal to the number of components of the graph Γ which is the intersection in M of $\mathfrak{C}$ with ∂R, the boundary of a small ball neighbourhood, R, of the vertex, o, of $\mathfrak{C}$.

This is seen as follows. ∂R intersects the 1-skeleton of $\mathfrak{C}$ in points $\alpha_1, \bar{\alpha}_1, \ldots, \alpha_n, \bar{\alpha}_n$ where the $\alpha_i, \bar{\alpha}_i$ lie on the 1-cell a_i in $\mathfrak{C}$. These $\alpha_i\bar{\alpha}_i$ are the vertices of Γ.

On ∂R we defined small disc neighbourhoods D_i of each α_i and $\bar{D}_i$ of each $\bar{\alpha}_i$, where $D_i, \bar{D}_i$ are disjoint. Denote ∂D_i by S_i, and $\partial \bar{D}_i$ by $\bar{S}_i$. Now each $S_i, \bar{S}_i$ intersects $\mathfrak{C}$ in the set of points we called $e_i^j, \bar{e}_i^j$ which correspond to the occurrences of the generator a_i in the relators of $\mathfrak{P}$. Recall $e_i^j \subset D_i$, $\bar{e}_i^j \subset \bar{D}_i$. Since each a_i intersects ∂R twice, each occurrence of an a_i in a relator corresponds to an element of E appearing unbarred and barred.

Now $(D_i \cup \bar{D}_i) \cap \mathfrak{C}$ consists of a number of disjoint spaces, each of which is the union of some arcs with a common end-point, and C cyclically permutes the other end-points (or what comes to the same thing elements in 1-1 correspondence with the end-points). On the other hand $\mathrm{cl}\,(\partial R - \bigcup_i (D_i \cup \bar{D}_i)) \cup \mathfrak{C}$ consists in some disjoint arcs γ_h. Each such arc runs from an e_i^j (possibly with a bar) to an e_k^l (possibly with a bar) and A interchanges the end-points of these arcs. Now our original contention should be clear. If Γ' is a connected component of Γ, then by judicious application of A and C an element

† If $L = 1$, it is easy to see that a presentation plus a numbering determines a unique manifold as follows: If M_1, M_2 have spines C_1, C_2 corresponding to the same presentation, then there is a homeomorphism from C_1 to C_2. This homeomorphism may be extended to a neighbourhood of the 1-skeleton of C_1 using the given numbering. The extension of this homeomorphism to M_1 is then obvious. The proof for $L > 1$ is equally simple.

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of E near a vertex of Γ' may be mapped to any other element of E near another vertex of Γ'. Conversely, if the group generated by A and C gives rise to L transitivity sets then of course each transitivity set contains all elements of E near a given vertex, and if elements of a transitivity set are near distinct vertices, then clearly these vertices may be connected by a sequence of arcs of Γ. This proves our assertion.

(2) Consider now a regular neighbourhood of the 1-skeleton of $\mathfrak{C}$, which we write as $R \cup_i T_i$, where the T_i are regular neighbourhoods of the cells a_i. Thus $T_i \cap \partial R = D_i \cup \bar{D}_i$.

Denote $\mathrm{cl}\,(\partial T_i - R)$ by F_i, so that $F_i \cap \partial R = S_i \cup \bar{S}_i$. Denote

$$\partial(R \cup_i T_i) = (\partial R \cup_i F_i) - \cup_i (\mathrm{int}\, D_i) \cup_i (\mathrm{int}\, \bar{D}_i)$$

by T. Observe $T - \mathfrak{C}$ must be connected, for the boundary of a regular neighbourhood of $\mathfrak{C}$ is a 2-sphere; and this boundary is made up of $2n$ discs along with the complement in T of a neighbourhood of $\mathfrak{C}$. But this latter is homeomorphic to $T - \mathfrak{C}$.

Now we assert that the transitivity of the group generated by AC and BC is equivalent to the connectedness of $T - \mathfrak{C}$.

This is seen as follows.

We have oriented $D_i, \bar{D}_i$ so that the orientation of the boundaries $S_i, \bar{S}_i$ which is thus induced agrees with the cyclic ordering of the points $(S_i \cup \bar{S}_i) \cap \mathfrak{C}$ given by C. Now a transitivity set of AC determines† a connected component of

$$\partial R - (\cup_i (D_i \cup_i \bar{D}_i) \cup \Gamma)$$

by virtue of the orientations of the $S_i, \bar{S}_i$ and the points in the transitivity set. On the other hand, by the same means, a transitivity set of BC defines a connected (4-sided) component of $F_i - \mathfrak{C}$. It is clear that the union of these components is $T - \mathfrak{C}$, and is connected if and only if AC and BC generate a transitive group.

(3) Let us now calculate the Euler characteristic, $\chi(\partial R)$, of ∂R, by means of the cellular decomposition of ∂R induced by adjoining to Γ, $L-1$ disjoint simple arcs between vertices of Γ and lying on ∂R so that the resulting graph Γ' is connected. Of course when Γ is connected this is superfluous, however, we will need this computation later.

$$\#\text{ vertices of } \Gamma = \#\text{ vertices of } \Gamma' = 2n$$
$$= |C|$$

(recall n = number of generators in $\mathfrak{P}$ = number of 1-cells of $\mathfrak{C} = \frac{1}{2}|C|$);

$$\#\text{ edges of } \Gamma' = \#\text{ edges of } \Gamma + L - 1.$$

(The presence of L is justified by section 1)

$$= \sum_{i=1}^{n} r(i) + L - 1$$

($\Sigma r(i)$ = number $i = 1$ of letters in the relators of $\mathfrak{P} = |B| = |A|$)

$$= |A| + L - 1,$$

$$\#\text{ discs} = \#\text{ discs of } \partial R - \Gamma' = |AC| - L + 1.$$

† This correspondence is of course not unique if Γ is not connected.

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(This follows from sections 1 and 2 and the fact that L arcs have been adjoined to Γ.) Thus $2\cdot\frac{1}{2}|C|-|A|-L+1+|AC|-L+1=\chi(\partial R)=2$ or $|A|-|C|+2L=|AC|$.

Thus, putting (1), (2), and (3) together we have Theorem 1.

Proof of Theorem 2. *Suppose now we are given (with the notation as before) a 2-complex* $\mathfrak{C}'$, *by* $\mathfrak{P}$, *a presentation with n generators and n relations. We may define E in the manner described in section* 2, *just after the statement of Theorem* 1. *Suppose we select some cyclic ordering of the 2-cells incident with each 1-cell. This selection, along with* $\mathfrak{P}$, *will fix (modulo trivial renaming) the permutations A, B, C.*

A graph Γ is the boundary of a nice small (regular) neighbourhood of the vertex of $\mathfrak{C}'$, and our ordering of the 2-cells induces an ordering of the edges about each vertex of Γ. Recalling the proof of Theorem 1, Γ has L components if and only if A and C generate a group giving rise to L transitivity sets of E. Now as in (10) and (9) the ordering of the edges of Γ about each vertex induces an imbedding of Γ on an orientable 2-manifold (with L components). The Euler characteristic of this 2-manifold can be calculated as we have done in (3) and is $|C|-|A|+|AC|$. If this number is $2L$, each component of Γ is imbedded on a 2-sphere, and we may imbed Γ on the boundary, ∂R of a 3-cell R. The join of Γ with an interior point of R imbeds a piece of $\mathfrak{C}'$ in a piece of a 3-manifold. Arcs $a_1, a_2, \ldots, a_n$ may be attached to ∂R, and thickened slightly so that a neighbourhood of the 1-skeleton of $\mathfrak{C}'$ is imbedded in an (orientable) handle body of genus n.

Notice that this is possible because the orderings of the edges of Γ are consistent with the orderings of the 2-cells of $\mathfrak{C}'$ about each arc a_i, and we are assuming orientable handles are being added.

We have now imbedded $\mathfrak{C}'-\bigcup_{i=1}^{n} H_i$ in a handle body, where the H_i are discs in the complement of a neighbourhood of the 1-skeleton of $\mathfrak{C}'$. Of course these n 2-discs may be attached to our handle body and thickened slightly to imbed $\mathfrak{C}'$ in a 3-manifold. This 3-manifold will have a 2-sphere boundary if and only if the complement of the intersection of $\mathfrak{C}'$ with the boundary of our handle body is connected.† Referring again to the previous proof we see that this latter condition is satisfied when $L=1$ if and only if AC and BC generate a transitive group. If this condition is satisfied we attach the H_i, thicken them, and obtain a 3-manifold M with a 2-sphere boundary to which a single 3-cell may be attached. Clearly this attaching map may be modified so that $\mathfrak{C}'$ is a spine of M. This completes the proof of Theorem 2.

5. *The case* $L\neq 1$. We retain all previous notation in this section.

The situation when $L\neq 1$, i.e. when Γ is not connected is sufficiently unpleasant to be treated as a special case, and modifications in the preceding arguments must be made along the following lines: Assume first $\mathfrak{C}$ is a spine of M:

One equates the transitivity of a certain permutation group to the connectivity of the boundary of a regular neighbourhood of $\mathfrak{C}$. In the case $L=1$, this was the group generated by AC and BC. If $L>1$, one must enlarge this group by permutations

† If this complement is not connected, then the boundary of the 3-manifold will not be connected, if the complement is connected then each attached disc raises the Euler characteristic by 2, so adding n discs yields a 2-sphere.

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defined below. These permutations are necessary for one needs to specify more precisely how the components of Γ are imbedded on a 2-sphere, provided (as in Theorem 2) that $|C|-|A|+|AC| = 2L$. This equation assures that each component of Γ is imbedded on a 2-sphere.

Once the embedding of Γ on a 2-sphere is done, we must assume the transitivity of a certain group so that the boundary of a tentative neighbourhood of $\mathfrak{C}$ is connected.

To make these modifications, we need to define an extra permutation, which we shall call F, that takes into account the fact that components of the complement of Γ on ∂R may coincide for distinct components of Γ.

We are assuming the 3-manifold M and a spine $\mathfrak{C}$ are given, so that A, B, C are constructed as before and let Q_i denote a component of $\partial R - \Gamma$. Q_i has on its boundary, points $p_1^i, p_2^i, \ldots, p_{2w}^i$ from the set E; half of these points constitute an orbit of the permutation AC and correspond to the region Q_i. Select one such point p_j^i in each component of Γ which lies on the boundary of the component Q_i of $\partial R - \Gamma$. Now form a cycle of the permutation F by selecting any ordering of the selected p_j^i on the boundary of the component Q_i of $\partial R - \Gamma$. Having done this for each component Q_i of $\partial R - \Gamma$, multiply these (disjoint) cycles together to form F.

Thus, two points p_j^i, p_l^k lie in the same orbit of F if and only if they belong to orbits of AC which correspond to the same region of $\partial R - \Gamma$. We may now state,

THEOREM 1′. *Suppose the group of permutations generated by A and C gives rise to L orbits in E. Then,*

(i) *The permutation group generated by AC, BC and F is transitive, and*

(ii) $|A|-|C|+2L = |AC|$.

Now if, on the other hand, a presentation is given and we denote as before the canonical 2-complex corresponding to this presentation by $\mathfrak{C}'$, then we may construct permutations A, B, C, F. A, B, and C are constructed as before while F is defined in this context as follows: Select in each transitivity set of AC a representative point p_j. Order the orbits of the group of permutations generated by A and C, denoting them $0_1, 0_2, \ldots, 0_M$, select in 0_1 a p_j and in 0_2 a p_j, select in $0_1 \cup 0_2$ a p_j and in 0_3 a p_j, select in $0_1 \cup 0_2 \cup 0_3$ a p_j and in 0_4 a p_j, etc. (This defines a specific embedding of Γ on a 2-manifold. Though A and C define an embedding of each component of Γ on a 2-manifold, Γ itself is not explicitly embedded on a connected 2-manifold until the pieces are put together. The preceding paragraph describes in general how the pieces are put together.) Now this construction gives rise to a set of involutions; namely, every time a selection of a p_j in a union of 0_j's and a p_j in a new 0_l is made this pair defines the involution which interchanges them and leaves everything else alone. This set of involutions generates a group and F may be defined as a permutation which cyclically permutes the elements of each transitivity set of this group.

Now we may state,

THEOREM 2′. *Suppose the group of permutations generated by A and C gives rise to L transitivity sets (orbits) in E. If,*

(i) *The group of permutations generated by AC, CB, and F is transitive, and if,*

(ii) $|A|-|C|+2L = |AC|$.

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Then there exists an orientable 3-manifold M' with spine $\mathfrak{C}'$ so that the points of intersection of the boundary of a regular neighbourhood of the 1-skeleton, the boundary of a regular neighbourhood of the vertex of $\mathfrak{C}'$, and the 2-skeleton of $\mathfrak{C}$ are in 1-1 correspondence with the elements of E. Furthermore, the permutations A, B, C of these points are just those described in Theorem 1'.

The proofs of these theorems are almost identical to the proofs of Theorems 1 and 2.

Theorem 1', part (i) is simply a consequence of the fact that $T - \mathfrak{C}$ must be connected and this occurs if and only if the group defined by AC, BC and F is transitive. Part (ii) has already been verified in the proof of Theorem 1 (Part 3).

Theorem 2' is proved by the same construction as the proof of Theorem 2, the transitivity of the group AC, BC and F generate now assuring the connectivity of the complement of the simple closed curves on the boundary of the handlebody along which discs must be added (see the last paragraph of the proof of Theorem 2). The preceding is the only place $L > 1$ plays a role in the proof of the theorem.

6. *Some remarks.* (1) The absence of a theorem using the algebraic statements in this paper ought to be remedied. One should be able to prove a theorem which restricts the form of the relators of a presentation of a 3-manifold group (when this presentation corresponds to a spine). If this statement is sufficiently strong, then one may have a handle to characterize 3-manifold groups.

(2) Previously mentioned is a cataloguing procedure for 3-manifolds which will be begun with the aid of a computer.

(3) One should be able to relate Theorems 1, 2, 1' and 2' to the characterization problem for non-separating simple closed curves on a 2-manifold, since such curves may be used in the construction of 3-manifolds from a Heegard diagram.

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