THE RELATIVISTIC CLOCK PROBLEM

By F. I. MIKHAIL

Communicated by W. H. McCrea

Received 7 March 1952

ABSTRACT. The so-called 'clock paradox' is concerned with the difference in the time-intervals reckoned by two observers in relative motion for the lapse of time between two encounters. In this paper the problem is treated purely by general relativity by considering a particular example in which the two observers are attached to two test-particles moving freely in the field of a gravitating mass; one of these makes complete revolutions in a circular orbit while the other moves radially outwards and inwards. The time-interval between two successive encounters is shorter in the reckoning of the former than in that of the latter. The difference is found to agree qualitatively with a naïve application of special relativity.

1. *Introduction.* One of the interesting problems connected with the phenomenon of 'time dilatation' of the special theory of relativity that has often been referred to, both in scientific and popular writings, is the so-called 'clock paradox'. This is usually presented by two observers C, L, say, carrying two identically constructed clocks initially together and at rest*. Then while C remains at rest, L moves away from him with a relative velocity v for a considerable time, reverses his direction and subsequently rejoins C. The readings of the two clocks are then compared and the elapsed time-intervals, according to the special theory of relativity, are related by the equation

$$\Delta t_{\mathrm{C}} = \Delta t_{\mathrm{L}}(1 - v^2/c^2)^{-\frac{1}{2}} \doteqdot \Delta t_{\mathrm{L}}(1 + \tfrac{1}{2}v^2/c^2), \tag{1·1}$$

which shows that the duration of L's journey is shorter in L's reckoning than in C's. The apparent paradox arises from the consideration of the relative motion of the two observers only, in consequence of which L could equally regard himself as remaining at rest and C as moving away with the velocity $-v$ and returning with the velocity $+v$. In this case with C having the moving clock, it appears as if C should reckon a shorter duration of the journey. Since both conclusions cannot be correct, we have a paradox.

This paradox was first introduced into relativity theory by Einstein himself (1, 2) in 1905 and later in 1911, and has since attracted the attention of many writers (see, for example, (4, 5, 6, 7, 9)). It was generally assumed that the two observers are completely symmetric as far as their relative motion alone is considered. It was only very recently that McCrea (8) has shown, within the framework of special relativity, that this is not so, and hence he significantly pointed out that there is really no paradox at all in such a problem.

On the other hand, as the problem involves the slowing down of the motion of one or other of the two observers and even the reversing of the direction of their relative motion, it necessitates the existence of some type of force or field of force. For a strict treatment of the problem, this fact makes reference to the general theory of relativity essential, since special relativity is concerned only with uniform relative velocities of

* C, L will be associated later with the observer moving in a *circular* orbit and the one moving in a straight *line* respectively.

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

observers and does not consider the effects of relative accelerations. Tolman ((10), pp. 194–7) has shown that the problem is resolvable by considering the phenomenon of general relativity dealing with the 'effect of gravitational potential on the rate of an ideal clock'. The difficulty about discussions such as Tolman's is that they employ a somewhat mixed assortment of conceptions drawn from classical mechanics, special relativity and general relativity.

To state the problem purely in terms of general relativity, it is this: If space-time is such that it admits two distinct geodesics through a single pair of events, it is required to compare the time-intervals between these events as assigned by two observers having these geodesics as world-lines. The final result is to be discussed from the point of view of the light it throws upon the less rigorous treatments of the clock paradox.

The standard interpretation of general relativity theory leads immediately to the conclusion that the time-interval between two events A, B as measured by an observer who experiences them both is simply the 'interval' $\int_A^B ds$ along that observer's world-line. Mathematically, therefore, the problem, as stated above, reduces to evaluating the lengths of the arcs joining the two events along the distinct geodesics representing the world-lines of the two observers. There is no reason for the two arc-lengths to be in general the same, that is, for the two observers to assign equal time-intervals. It is seen, in particular, that the values of the time-intervals do not at all depend upon whether either observer chooses to regard himself or the other observer as being at rest.

The natural approach, therefore, is to seek an actual example which satisfies the primary requirement of admitting two geodesics which intersect in distinct events, and the two secondary requirements of allowing explicit evaluation of the intervals and of being capable of simple physical interpretation.

Such an example is not easy to find. One is supplied, however, by motion in the field of a gravitating mass (Schwarzschild 'exterior' metric). In this field it is possible for two test-particles to encounter each other more than once, i.e. for the geodesics representing their world-lines to intersect in more than one event.

In this paper the example is first formulated in general terms and then a special case is considered in detail. The result for this case is compared with the inference drawn from special relativity.

It has to be remarked that there is inevitably one difference between the problem as formulated here and the 'classical' problem. In the 'classical' statement of the problem, clock-synchronization was supposed to be achieved by the two observers being *at rest* relative to each other initially and finally. Orbits allowing this behaviour of the observers do not exist in general relativity for uncharged particles; clock-synchronization is here achieved by the universal interpretation of the element of interval ds.

2. *Analysis of the general problem.* To deal with the problem in its more general form, we choose as a frame of reference a set of spherical polar coordinates (r, θ, ϕ) as the space-like coordinates and the time t as the time-like coordinate in a four-dimensional Riemannian space. In terms of these coordinates the Schwarzschild exterior metric

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

describing the field outside the gravitating particle may be written in the form ((10), equation (82·9), p. 204)

$$ds^2 = (1-2m/r)\,dt^2 - (1-2m/r)^{-1}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2, \tag{2·1}$$

where m represents the mass of this particle when the units are chosen such that the velocity of light $c = 1$ and the Newtonian constant of gravitation $\gamma = 1$.

The equations of motion of a free particle in the plane $\theta = \frac{1}{2}\pi$ in the gravitational field (2·1) as obtained from the geodesic equations

$$\frac{d^2x^\alpha}{ds^2} + \{\mu, \nu, \alpha\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{2·2}$$

may be written in the form (cf. (10), pp. 205–7)

$$\frac{d^2r}{ds^2} - \frac{m/r^2}{1-2m/r}\left(\frac{dr}{ds}\right)^2 - r\left(1-\frac{2m}{r}\right)\left(\frac{d\phi}{ds}\right)^2 + \frac{m}{r^2}\left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 = 0, \tag{2·3}$$

$$\frac{d\phi}{ds} = \frac{h}{r^2}, \tag{2·4}$$

$$\frac{dt}{ds} = \frac{k}{(1-2m/r)}, \tag{2·5}$$

where h and k are two constants of integration characterizing the particular orbit under consideration.

The original expression (2·1) for the line-element itself provides a first integral of (2·3) in the form

$$\frac{1}{1-2m/r}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - \left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 + 1 = 0. \tag{2·6}$$

Substituting from (2·4), (2·5) into (2·6), and using a new variable $u = 1/r$, we get

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2m}{h^2}u + \frac{k^2-1}{h^2}, \tag{2·7}$$

giving the differential equation for the orbit of the free particle. This has been solved completely by Forsyth (3), but as we do not intend to carry out here the full calculation for this general case, we may simply refer to this solution by

$$r = r(\phi, \lambda), \tag{2·8}$$

where λ is a certain parameter indicating the particular orbit to be considered, and the initial line from which ϕ is measured has been suitably chosen. Unfortunately, the function $r(\phi, \lambda)$, as given by Forsyth (§ 3, p. 148), involves elliptic functions which do not make it practicable to obtain an explicit solution to the present problem. A result may only be obtained by expanding these elliptic functions and using successive approximations. Consequently there will be little advantage in using this general solution for the central orbit.

However, we can, in principle, obtain an answer to our problem in this general form up to any order of approximation by applying the following procedure. We consider the observer C to accompany a particle moving in an elliptic orbit of small eccentricity and the observer L to accompany a particle moving in a thin ellipse with a much larger major axis. We consider the case in which the two orbits have parallel axes and intersect in two points P, Q for which $\phi = \pm\alpha$, say.

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

We suppose C, L to pass through Q simultaneously. As it is possible to do, we further suppose the parameters of the orbits to be so related that after L has traversed the (longer) arc QP of his orbit he will pass through P simultaneously with C. In that case C must have made, say, $n(\geqslant 1)$ complete revolutions in his orbit and then again traversed the arc QP of that orbit, before he encounters L at P. The relativistic orbits are, of course, not strictly elliptic, but this is taken care of by the calculations.

The parameter λ in (2·8) depends upon the parameters h, k for the orbit concerned. Equating the values of r given by (2·8) applied to each of the orbits, we obtain an equation for α in terms of the values of h, k for the two orbits. Also we have from (2·4), (2·5)

$$\frac{dt}{d\phi} = \frac{k}{h}\frac{r^2}{1-2m/r}. \tag{2·9}$$

Substituting for r from (2·8) and integrating with respect to ϕ over the range $(-\alpha, \alpha)$ or $(-\alpha, \alpha + 2n\pi)$ we can then express the fact that L, C have the same values of the t-coordinate at Q and also at P. This gives a further relation between α and the values of h, k for the two orbits. Thence α can be obtained by successive approximations in terms of the other parameters. Substituting in the previous equation for α we get the relation between these parameters ensuring that the orbits are such that L, C do meet at the two points P, Q. Finally, substituting for r from (2·8) in (2·4) and integrating over the same ranges of ϕ as before, we get expressions for the proper-times S_L, S_C measured by the two observers between their two encounters.

We have in fact carried out the calculation for this general case and we have found that the work involves many parameters, and therefore necessary assumptions about their order of magnitude must be made in order to carry out the solution up to a definite order of approximation. This fact reduces to a great extent the value of starting with such a general form of the problem. We were led, therefore, to think it more significant to examine fully the special case considered in the following section.

3. *Proper times of two special observers.* For simplicity and to reduce the number of parameters to be considered in determining the order of approximation of the result, we assume that the projection of C's world-line on the plane $\theta = \frac{1}{2}\pi$ is a circle of radius a, say, and that of L is a straight line going up to the point whose spatial coordinates are $(R, \frac{1}{2}\pi, 0)$. Further we suppose that $R \gg a$; that a/R, m/a are of the first order; and that $m/R = (m/a)(a/R)$ is of the second order.

To express the problem in more physical terms, at $t = 0$ we suppose that L is projected by C radially in some type of projectile, with a very large velocity. Therefore, the motion of L is radial throughout. As the projectile makes its journey outwards and inwards through the distance R, C goes round in his circular orbit for n revolutions and returns again to the initial point of projection just in time to pick up the projectile with L inside. Then each calculates the lapse in his proper time between the two events of projection and picking up as well as his time-coordinate at the latter occasion.

C's calculation $[r = a]$. From the line-element (2·1), we get

$$\left(1-\frac{2m}{a}\right)\left(\frac{dt}{ds}\right)^2 - a^2\left(\frac{d\phi}{ds}\right)^2 = 1 \tag{3·1}$$

and (2·3) gives

$$\frac{m}{a^2}\left(\frac{dt}{ds}\right)^2 - a\left(\frac{d\phi}{ds}\right)^2 = 0. \tag{3·2}$$

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

Solving (3·1), (3·2) for dt/ds, $d\phi/ds$,

$$\frac{dt}{ds} = \left(1-\frac{3m}{a}\right)^{-\frac{1}{2}}, \tag{3·3}$$

$$\frac{d\phi}{ds} = \frac{m^{\frac{1}{2}}}{a^{\frac{3}{2}}}\left(1-\frac{3m}{a}\right)^{-\frac{1}{2}}. \tag{3·4}$$

These will in turn give

$$\frac{dt}{d\phi} = \frac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}}. \tag{3·5}$$

Then from (3·4) and (3·5) respectively

$$S_C = \frac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1-\frac{3m}{a}\right)^{\frac{1}{2}}\int_0^{2n\pi} d\phi = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1-\frac{3m}{a}\right)^{\frac{1}{2}} \doteqdot \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1-\frac{3m}{2a}-\frac{9m^2}{8a^2}-\ldots\right), \tag{3·6}$$

$$t_C = \frac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\int_0^{2n\pi} d\phi = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}. \tag{3·7}$$

L's calculation [$\phi = 0$]. From (2·6), we get

$$\frac{1}{1-2m/r}\left(\frac{dr}{ds}\right)^2 - \left(1-\frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 + 1 = 0. \tag{3·8}$$

But from (2·5)

$$\frac{dt}{ds} = \frac{k}{(1-2m/r)}. \tag{3·9}$$

Substituting into (3·8),

$$\left(\frac{dr}{ds}\right)^2 = k^2 - 1 + \frac{2m}{r}. \tag{3·10}$$

But if the projectile is to return after reaching an extreme distance R (i.e. $\frac{dr}{ds} = 0$ when $r = R$), we get

$$k^2 = 1 - \frac{2m}{R}. \tag{3·11}$$

Hence

$$\left(\frac{dr}{ds}\right)^2 = 2m\left(\frac{1}{r}-\frac{1}{R}\right). \tag{3·12}$$

We may note here that C has to project L with a velocity V given by

$$V = \left(\frac{2m}{a}-\frac{2m}{R}\right)^{\frac{1}{2}} \tag{3·13}$$

if L is to travel through the distance R and return back to join C.

From (3·12)

$$\frac{dr}{ds} = \pm\left(\frac{2m}{r}-\frac{2m}{R}\right)^{\frac{1}{2}}. \tag{3·14}$$

Hence

$$S_L = 2\left(\frac{R}{2m}\right)^{\frac{1}{2}}\int_a^R \frac{r^{\frac{1}{2}}}{(R-r)^{\frac{1}{2}}}dr = \left(\frac{2R}{m}\right)^{\frac{1}{2}}\left[a^{\frac{1}{2}}(R-a)^{\frac{1}{2}} + R\cos^{-1}\left(\frac{a}{R}\right)^{\frac{1}{2}}\right]$$

$$\doteqdot \left(\frac{2}{m}\right)^{\frac{1}{2}} R^{\frac{3}{2}}\left[\frac{\pi}{2} - \frac{2}{3}\left(\frac{a}{R}\right)^{\frac{3}{2}} - \frac{1}{5}\left(\frac{a}{R}\right)^{\frac{5}{2}} - \frac{3}{28}\left(\frac{a}{R}\right)^{\frac{7}{2}} - \ldots\right]. \tag{3·15}$$

From (3·9) and (3·14) we get, by using (3·11),

$$\frac{dt}{dr} = \pm\left(\frac{R-2m}{2m}\right)^{\frac{1}{2}}\frac{r^{\frac{3}{2}}}{(r-2m)(R-r)^{\frac{1}{2}}}.$$

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

Thus
$$t_{\mathrm{L}} = 2\left(\frac{R-2m}{2m}\right)^{\frac{1}{2}}\int_a^R \frac{r^{\frac{3}{2}}}{(r-2m)(R-r)^{\frac{1}{2}}}\,dr$$
$$= \left(\frac{2}{m}\right)^{\frac{1}{2}}(R-2m)^{\frac{1}{2}}\left[a^{\frac{1}{2}}(R-a)^{\frac{1}{2}} + R\cos^{-1}\left(\frac{a}{R}\right)^{\frac{1}{2}} + 4m\cos^{-1}\left(\frac{a}{R}\right)^{\frac{1}{2}}\right.$$
$$\left. + 4m\left(\frac{2m}{R-2m}\right)^{\frac{1}{2}}\tanh^{-1}\left\{\frac{2m(R-a)}{a(R-2m)}\right\}^{\frac{1}{2}}\right]. \quad (3{\cdot}16)$$

Recalling our original assumption that $\frac{m}{a}$, $\frac{a}{R}$ are of the first order while $\frac{m}{R}$ is of the second order and expanding the functions on the right-hand side of (3·16), we get

$$t_{\mathrm{L}} = \left(\frac{2}{m}\right)^{\frac{1}{2}}R^{\frac{3}{2}}\left[\frac{\pi}{2} - \frac{2}{3}\left(\frac{a}{R}\right)^{\frac{3}{2}} + \frac{3\pi}{2}\frac{m}{R} - \left(\frac{1}{5}\left(\frac{a}{R}\right)^{\frac{5}{2}} + 4\frac{m}{R}\left(\frac{a}{R}\right)^{\frac{1}{2}}\right)\right.$$
$$\left. - \left(\frac{3}{28}\left(\frac{a}{R}\right)^{\frac{7}{2}} - 8\left(\frac{m}{R}\right)^{\frac{3}{2}}\left(\frac{m}{a}\right)^{\frac{1}{2}}\right) - \frac{9\pi}{4}\left(\frac{m}{R}\right)^2 + \ldots\right]. \quad (3{\cdot}17)$$

Hence in order to relate the two proper times S_{C}, S_{L} as given by (3·6), (3·15), we have to express the constant R appearing in (3·15) in terms of the other given constants, namely m, a and n. This is done by considering the fact that $t_{\mathrm{C}} = t_{\mathrm{L}}$ when C and L rejoin each other. Thus we get

$$\sqrt{2}\,n\pi a^{\frac{3}{2}} = R^{\frac{3}{2}}\left[\frac{\pi}{2} - \frac{2}{3}\left(\frac{a}{R}\right)^{\frac{3}{2}} + \frac{3\pi}{2}\frac{m}{R} - \left(\frac{1}{5}\left(\frac{a}{R}\right)^{\frac{5}{2}} + 4\frac{m}{R}\left(\frac{a}{R}\right)^{\frac{1}{2}}\right) - \ldots\right], \quad (3{\cdot}18)$$

from which we can obtain the necessary expression for R by successive approximations.

First approximation. Neglecting all terms of the first and higher orders inside the bracket on the right-hand side of (3·18), we get

$$R^{\frac{3}{2}} = 2\sqrt{2}\,na^{\frac{3}{2}}. \quad (3{\cdot}19)$$

Substituting in (3·15), we get to the same order considered above

$$S_{\mathrm{L}} = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}, \quad (3{\cdot}20)$$

and from (3·6)
$$S_{\mathrm{C}} = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}. \quad (3{\cdot}21)$$

Hence to this order
$$S_{\mathrm{L}} = S_{\mathrm{C}} = nT, \quad (3{\cdot}22)$$

where nT is the time of C's journey when the problem is treated by classical mechanics.

Second approximation. Considering the second term inside the bracket on the right of (3·18), i.e. of order $\frac{3}{2}$, and substituting from (3·19) for

$$\left(\frac{a}{R}\right)^{\frac{3}{2}} = \frac{1}{2^{\frac{3}{2}}n} \quad (3{\cdot}23)$$

we get
$$R^{\frac{3}{2}} = 2^{\frac{3}{2}}na^{\frac{3}{2}}\left(1 + \frac{\sqrt{2}}{3\pi}\frac{1}{n}\right). \quad (3{\cdot}24)$$

Substituting into (3·15) and noting that $1/n$, as seen from (3·23), may be considered of order $\frac{3}{2}$, we get

$$S_{\mathrm{L}} = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1 + \frac{\sqrt{2}}{3\pi}\frac{1}{n}\right)\left(1 - \frac{\sqrt{2}}{3\pi}\frac{1}{n}\right) \doteq \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}. \quad (3{\cdot}25)$$

Also from (3·6)
$$S_{\mathrm{C}} = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1 - \frac{3m}{2a}\right). \quad (3{\cdot}26)$$

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

Hence
$$S_L - S_C = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}} \frac{3m}{2a} = \frac{3m}{2a} nT. \tag{3·27}$$

This shows that in this particular example and under the given assumptions S_L is greater than S_C, and their difference to a second order of approximation is given by (3·27).

Third approximation. It may be worth stating the result of this calculation to a third order of approximation. In this case we get

$$R^{\frac{3}{2}} = 2\sqrt{2}\, na^{\frac{3}{2}}\left(1 + \frac{\sqrt{2}}{3\pi n} - \frac{3}{2n^{\frac{3}{2}}}\frac{m}{a}\right), \tag{3·28}$$

$$S_L = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1 - \frac{3}{2n^{\frac{3}{2}}}\frac{m}{a}\right), \tag{3·29}$$

$$S_C = \frac{2n\pi a^{\frac{3}{2}}}{m^{\frac{1}{2}}}\left(1 - \frac{3}{2}\frac{m}{a} - \frac{9}{8}\left(\frac{m}{a}\right)^2\right), \tag{3·30}$$

$$S_L - S_C = \frac{3m}{2a}\left(1 - \frac{1}{n^{\frac{3}{2}}} + \frac{3}{4}\frac{m}{a}\right) nT. \tag{3·31}$$

4. *Comparison with special relativity.* By so treating the clock problem solely in terms of general relativity, we have made clear the fact that, from the point of view of general relativity, there is no question of the two observers assigning equal times to the duration of their journeys, and consequently there is no way of a paradox arising. This is clearly seen, as pointed out above, from the fact that the mere passing of two different geodesics through a pair of common points of a space-time does not imply by any means that their parts joining the two points are equal in length. On the other hand, this makes comparison with special relativity rather difficult. In the previous less rigorous treatments this comparison was made possible by restricting the application of the general theory of relativity to negligible parts of the paths of the two observers and for negligible intervals of time. Unfortunately we cannot do so in the present treatment. Nevertheless we can use a naïve application of special relativity which will enable us to get a result comparable with our result (3·27).

The first consideration in any application of special relativity is to select an inertial frame. In the gravitational field here considered, there is, of course, no such frame for the whole of the space-time. However, the coordinate system used in (2·1) is itself an inertial frame in the limit of infinite distance from the spatial origin; elsewhere it can be treated as an arbitrarily good approximation to an inertial frame by taking m/a sufficiently small.

Accepting this we can proceed as follows. Classically, the particle C describes a circle of radius a with period T and speed

$$V_C = \frac{2\pi a}{T} = \left(\frac{m}{a}\right)^{\frac{1}{2}} \tag{4·1}$$

as judged by an observer at rest in the frame. A naïve application of special relativity leads therefore to the conclusion that, taking account of time-dilatation, C will assign to his journey the time

$$T_C = nT\,(1 - V_C^2)^{\frac{1}{2}} \doteqdot nT\left(1 - \frac{m}{2a}\right). \tag{4·2}$$

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press

Treating the projectile L in a similar manner, we find that its path, to a first order of approximation, is given by

$$2(R-a) \doteqdot 4n^{\frac{1}{3}}a \tag{4·3}$$

according to (3·19), and its mean speed is given by

$$V_{\mathrm{L}} = \frac{4n^{\frac{1}{3}}a}{nT} \approx \frac{2}{\pi n^{\frac{1}{3}}}\left(\frac{m}{a}\right)^{\frac{1}{2}}. \tag{4·4}$$

Hence

$$T_{\mathrm{L}} = nT\,(1-V_{\mathrm{L}}^2)^{\frac{1}{2}} \doteqdot nT\left(1-\frac{2}{\pi^2 n^{\frac{2}{3}}}\frac{m}{a}\right) \doteqdot nT \tag{4·5}$$

to the same order of approximation considered in (4·2). To this order we have, therefore,

$$T_{\mathrm{L}} - T_{\mathrm{C}} = \tfrac{1}{2}(m/a)\,nT. \tag{4·6}$$

Since we have in fact only an approximation to an inertial frame and since in any case the particles are throughout in accelerated motion relative to that frame, we have no right to expect this crude attempt at applying special relativity to have much significance. But there may be some slight interest in noticing that it does give in (4·6) an effect of the same order of magnitude as the general relativity result (3·27), though it is out by a factor 3.

It is a pleasure to express my thanks to Professor W. H. McCrea, who first directed my attention to this problem and its method of approach, for many helpful discussions and suggestions.

REFERENCES

(1) Einstein, A. *Ann. Phys., Lpz.*, (4), 17 (1905), 891–921.
(2) Einstein, A. *Ann. Phys., Lpz.*, (4), 35 (1911), 898–908.
(3) Forsyth, A. R. *Proc. roy. Soc.* A, 97 (1920), 145–51.
(4) Hill, E. L. *Phys. Rev.* (2), 72 (1947), 236–40.
(5) Hoyle, F. Exploring the solar system. *The Times*, 16 October 1950.
(6) Ives, H. E. *Nature, Lond.*, 168 (1951), 246.
(7) Jellinek, K. *Weltsystem, Weltäther und die Relativitäts-theorie* (Basle, 1950), pp. 99 and 228.
(8) McCrea, W. H. *Nature, Lond.*, 167 (1951), 680.
(9) Milne, E. A. and Whitrow, G. J. *Phil. Mag.* (7), 40 (1949), 1244–9.
(10) Tolman, R. C. *Relativity, thermodynamics and cosmology* (Oxford, 1934).

Royal Holloway College
Englefield Green, Surrey

https://doi.org/10.1017/S0305004100076398 Published online by Cambridge University Press